Almost structural completeness and structural completeness of nilpotent minimum logics.

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Admissibility Theory

Given a logic L, an L-**unifier** of a formula φ is a substitution σ such that $\vdash_L \sigma\varphi$.

A **single-conclusion rule** is an expression of the form Γ/φ where φ is a formula and Γ is a finite set of formulas.

Γ/φ is L-**derivable** in L iff $\Gamma \vdash_L \varphi$.

Γ/φ is L-**admissible** in L iff every common L-unifier of Γ is also an L-unifier of φ.

Γ/φ is **passive** L-**admissible** in L iff Γ has no common L-unifier.

Admissibility Theory

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A logic is **structurally complete** iff every admissible rule is a derivable rule.

A logic is **almost structurally complete** iff every admissible rule is either derivable rule or a passive admissible.

Nilpotent Minimum Logic

Nilpotent Minimum Logic (**NML**) is the axiomatic extension of the Monoidal t-norm logic (MTL) given by the axioms

Inv $\neg\neg\varphi \to \varphi$

WNM $(\psi * \varphi \to \bot) \vee (\psi \wedge \varphi \to \psi * \varphi)$

t-norm semantics

$[0,1]_{NM} = \langle \{a \in \mathbb{R} : 0 \leq a \leq 1\}; *, \rightarrow, \wedge, \vee, \neg, 0, 1 \rangle$ where $\wedge$ and $\vee$ are the meet and join with the usual order and for every $a, b \in [0,1]$,

$$a * b = \begin{cases} min\{a,b\}, & \text{if } b > 1-a; \\ 0, & \text{otherwise.} \end{cases}$$

$$a \rightarrow b = \begin{cases} 1, & \text{if } a \leq b; \\ max\{1-a, b\} & \text{otherwise.} \end{cases}$$

$$\neg a := a \rightarrow 0 = 1 - a$$

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Let $\Gamma \cup \{\varphi\} \subseteq Prop(X)$, then

$\Gamma \models_{[0,1]_{NM}} \varphi$ iff
for every $h : Prop(x) \rightarrow [0,1]$, $h(\varphi) = 1$ whenever $h\Gamma = \{1\}$

Completeness Theorem

Theorem (Esteva Godo 2001, Noguera et al 2008)

$$\Sigma \vdash_{NML} \varphi \textit{ iff } \Sigma \models_{[0,1]_{NM}} \varphi$$

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Algebraic logic

The Nilpotent Minimum Logic NML is algebraizable with $\mathbb{NM}$ the class of all NM-algebras as its equivalent quasivariety semantics.

Algebraic logic

Finitary Extensions of *NML*	$\longleftrightarrow$	Quasivarieties of $\mathbb{NM}$
Axiomatic Extensions	$\longleftrightarrow$	Varieties
(Finite) Axiomatization	$\longleftrightarrow$	(Finite) Axiomatization
Deduction Theorem	$\longleftrightarrow$	EDPCR
Local Deduction Theorem	$\longleftrightarrow$	RCEP
Interpolation Theorem	$\longleftrightarrow$	Amalgamation Property

Algebraic Admissibility Theory

Given a quasivariety $\mathbb{K}$, we say that a quasiequation

$$\alpha_1 \approx \gamma_1 \& \cdots \& \alpha_n \approx \gamma_n \Rightarrow \epsilon \approx \eta$$

is $\mathbb{K}$-**admissible** iff for every term substitution σ if $\mathbb{K} \models \sigma(\alpha_i) \approx \sigma(\gamma_i)$ for $i = 1 \div n$, then $\mathbb{K} \models \sigma(\epsilon) \approx \sigma(\eta)$.

is **passive** in $\mathbb{K}$ iff there is no term substitution σ such that $\mathbb{K} \models \sigma(\alpha_i) \approx \sigma(\gamma_i)$ for $i = 1 \div n$.

$\mathbb{K}$ is **structurally complete** iff every $\mathbb{K}$-admissible quasiequation is valid in $\mathbb{K}$.

$\mathbb{K}$ is **almost structurally complete** iff every admissible quasiequation is either valid in $\mathbb{K}$ or passive in $\mathbb{K}$.

Algebraic logic

Finitary Extensions of NML $\longleftrightarrow$ Quasivarieties of $\mathbb{NM}$

L $\longleftrightarrow$ $\mathbb{K}$

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Finitary Extensions of *NML* $\longleftrightarrow$ Quasivarieties of $\mathbb{NM}$

$$L \longleftrightarrow \mathbb{K}$$

$$\{\gamma_1, \ldots, \gamma_n\}/\varphi \longrightarrow \gamma_1 \approx 1 \& \cdots \& \gamma_n \approx 1 \Longrightarrow \varphi \approx 1$$

$$\{\alpha_1 \leftrightarrow \beta_1, \ldots, \alpha_n \leftrightarrow \beta_n\}/\epsilon \leftrightarrow \eta \longleftarrow \alpha_1 \approx \beta_1 \& \cdots \& \alpha_n \approx \beta_n \Longrightarrow \epsilon \approx \eta$$

Algebraic logic

Finitary Extensions of *NML*	$\longleftrightarrow$	Quasivarieties of $\mathbb{NM}$
L	$\longleftrightarrow$	$\mathbb{K}$
$\{\gamma_1, \ldots, \gamma_n\}/\varphi$	$\longrightarrow$	$\gamma_1 \approx 1 \& \cdots \& \gamma_n \approx 1 \Longrightarrow \varphi \approx 1$
$\{\alpha_1 \leftrightarrow \beta_1, \ldots, \alpha_n \leftrightarrow \beta_n\}/\epsilon \leftrightarrow \eta$	$\longleftarrow$	$\alpha_1 \approx \beta_1 \& \cdots \& \alpha_n \approx \beta_n \Longrightarrow \epsilon \approx \eta$
derivable in L	$\longleftrightarrow$	valid in $\mathbb{K}$
L-admissible	$\longleftrightarrow$	$\mathbb{K}$-admissible
passive L-admissible	$\longleftrightarrow$	passive in $\mathbb{K}$

Algebraic logic

Theorem (Rybakov 1997, Olson et al. 2008)

Let L be an algebraizable logic and $\mathbb{K}$ its quasivariety semantics, then L is (almost) structurally complete iff $\mathbb{K}$ is (almost) structurally complete.

Goal

To study (almost) structural completeness of all axiomatic extensions of NML

Goal

To study (almost) structural completeness of all subvarieties of $\mathbb{NM}$

Structural completeness and free algebras

Theorem (Bergman 1991)

Let $\mathbb{K}$ be a quasivariety, then the following properties are equivalent.

1. *$\mathbb{K}$ is structurally complete.*
2. *Each proper subquasivariety of $\mathbb{K}$ generates a proper subvariety of $\mathcal{V}(\mathbb{K})$*
3. *$\mathbb{K} = \mathcal{Q}(\mathbf{Free}_{\mathbb{K}}(\omega))$.*

Almost Structural completeness and free algebras

Theorem (Metcalfe-Röthlisberger 2013)

Let $\mathbb{K}$ *be a quasivariety. The following are equivalent for any* $\mathbf{B} \in \mathcal{S}(\mathbf{Free}_{\mathbb{K}}(\omega))$

1. $\mathbb{K}$ *is almost structurally complete.*
2. $\mathcal{Q}(\{\mathbf{A} \times \mathbf{B} : \mathbf{A} \in \mathbb{K}\}) = \mathcal{Q}(\mathbf{Free}_{\mathbb{K}}(\omega))$.
3. $\{\mathbf{A} \times \mathbf{B} : \mathbf{A} \in \mathbb{K}\} \subseteq \mathcal{Q}(\mathbf{Free}_{\mathbb{K}}(\omega))$.

NML and Gödel logic

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Gödel logic is a based t-norm logic where the t-norm is the minimum. It's associated negation is not involutive.

NML is a based t-norm logic where the the t-norm is the nilpotent minimum. It's associated negation is involutive. It's the "closest" to the minimum t-norm if you want the negation to be involutive. That is, an involutive version of the minimum t-norm.

Theorem (Dzik-Wronski 1973)

Gödel logic is structurally complete.

For every $n > 2$, $\mathbf{G}_n$ is embeddable into $\mathbf{Free}_{\mathbb{G}}(\omega)$.

NM-algebras

A **NM-algebra** is a bounded integral residuated lattice satisfying the following equations:

$$(x \to y) \vee (y \to x) \approx \bar{1} \quad \text{(L)}$$

$$\neg\neg x \approx x \quad \text{(I)}$$

$$\neg(x * y) \vee (x \wedge y \to x * y) \approx \bar{1} \quad \text{(WNM)}$$

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Example: $[0,1]_{NM}$ is a NM-algebra.

NM-chains

We say that a NM-algebra is a **NM-chain**, provided that it is totally ordered.

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Since the class of all NM-algebras, denoted by $\mathbb{NM}$, is a proper subvariety of MTL-algebras the decomposition theorem is also valid.

Proposition

Each NM-algebra is representable as a subdirect product of NM-chains

NM-chains

Let $\langle A, \leq, \bar{0}, \bar{1}\rangle$ a totally ordered bounded set equipped with an involutive negation $\neg$,

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$$a * b = \begin{cases} \bar{0}, & \text{if } b \leq \neg a; \\ a \wedge b, & \text{otherwise.} \end{cases} \qquad a \to b = \begin{cases} \bar{1}, & \text{if } a \leq b; \\ \neg a \vee b, & \text{otherwise.} \end{cases},$$

$$a \wedge b = min\{a, b\} \qquad a \vee b = max\{a, b\},$$

then $\mathbf{A} = \langle A, *, \to, \wedge, \vee, \bar{0}, \bar{1}\rangle$ is a NM-chain.

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then $\mathbf{A} = \langle A, *, \rightarrow, \wedge, \vee, \bar{0}, \bar{1}\rangle$ is a NM-chain.

Every NM-chain is of this form.

Finite NM-chains

Therefore up to isomorphism for each finite $n \in \mathbb{N}$, there is only one NM-chain $\mathbf{A}_n$ with exactly n elements.

$$\mathbf{A}_{2n+1} = \langle [-n, n] \cap \mathbb{Z}, *, \rightarrow, \wedge, \vee, -n, n \rangle.$$

$$\mathbf{A}_{2n} = \langle A_{2n+1} \setminus \{0\}, *, \rightarrow, \wedge, \vee, -n, n \rangle.$$

Notice that $\mathbf{A}_1$ is the trivial algebra, $\mathbf{A}_2$ the 2-element Boolean algebra and $\mathbf{A}_3$ the 3-element MV-algebra.

Let $\mathbf{A}$ be an NM-algebra,

$A_+ = \{a \in A : a > \neg a\}$
$A_- = \{a \in A : a < \neg a\}$.
$a \in A$ is a **negation fixpoint** (or just **fixpoint**, for short) iff $\neg a = a$.

Let $\mathbf{C}$ *be an NM-chain. Then*

- $C = C_+ \cup C_-$ *if* C *has no fixpoint.*
- $C = C_+ \cup C_- \cup \{c\}$ *if* c *is the fixpoint of* C*.* *Moreover* $C \setminus \{c\}$ *is the universe of a subalgebra of* $\mathbf{C}$ *which we denote by* $\mathbf{C}^-$*.*

$\mathbf{A_{2n}} = \mathbf{A_{2n+1}}^-$

NM-chains, Gödel chains

Let **A** *be an NM-chain. Then*

- *If* **A** *has negation fix point then* **A** *is the connected rotation of a Gödel chain.*
- *If* **A** *has no negation fixpoint then* **A** *is the disconnected rotation of the* $\overline{0}$*-free subreduct of a Gödel chain.*

NM-chains, Gödel chains

- $[\mathbf{0},\mathbf{1}]_{\mathbf{NM}} \cong ConRot([\mathbf{0},\mathbf{1}]_{\mathbf{G}})$
- $\mathbf{A_{2n+1}} \cong ConRot(\mathbf{G_{n+1}})$
- $\mathbf{A_{2n}} \cong DiscRot(\mathbf{G_n^+})$
- $[\mathbf{0},\mathbf{1}]_{\mathbf{NM}}^- \cong DiscRot((\mathbf{0},\mathbf{1}]_{\mathbf{G}}^+)$

Let $\nabla(x) = \neg(\neg x^2)^2$ and $\Delta(x) = (\neg(\neg x)^2)^2$ where x^2 is an abbreviation of $x * x$.

Lemma

Let $\mathbf{A}$ *be an NM-chain and let* $a \in A$. *Then we have*

$$\nabla(a) = \begin{cases} \bar{1}, & \text{if } a > \neg a; \\ \bar{0}, & \text{if } a \leq \neg a. \end{cases}$$

and

$$\Delta(a) = \begin{cases} \bar{1}, & \text{if } a \geq \neg a; \\ \bar{0}, & \text{if } a < \neg a. \end{cases}$$

Therefore, $\mathbf{A}$ *does not have a fixpoint iff* $\nabla(a) = \Delta(a)$ *for every* $a \in A$

NM-varieties

$\mathbb{NM}$ *is a locally finite variety.*

$$\mathbb{NM} = \mathcal{V}(\{\mathbf{A}_n : n > 1\})$$

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$\mathbb{NM}- = \mathbb{NM} + \nabla(x) \approx \Delta(x)$

$\mathbb{NM}- = \mathcal{V}(\{\mathbf{A}_{2n} : n > 0\})$

NM-varieties

Theorem (Gispert 03)

Every nontrivial variety of NM-algebras is of one of the following types:

1. $\mathbb{NM} = \mathcal{V}([\mathbf{0}, \mathbf{1}]) = \mathcal{V}(\{\mathbf{A}_n : n > 1\})$
2. $\mathbb{NM}- = \mathcal{V}([\mathbf{0}, \mathbf{1}]^-) = \mathcal{V}(\{\mathbf{A}_{2n} : n > 0\})$
3. $\mathbb{NM}_{2m+1} = \mathcal{V}(\mathbf{A}_{2m+1})$ *for some* $m > 0$
4. $\mathbb{NM}_{2n} = \mathcal{V}(\mathbf{A}_{2n})$ *for some* $n > 0$
5. $\mathbb{NM}_{2n2m+1} = \mathcal{V}(\{\mathbf{A}_{2n}, \mathbf{A}_{2m+1}\})$ *for some* $n > m > 0$
6. $\mathbb{NM}-_{2m+1} = \mathcal{V}(\{[\mathbf{0}, \mathbf{1}]^-, \mathbf{A}_{2m+1}\}) = \mathcal{V}(\{\mathbf{A}_{2n} : n > 0\} \cup \{\mathbf{A}_{2m+1}\})$

NM-varieties as quasivarieties

Theorem (Noguera et al. 08)

Every nontrivial variety of NM-algebras is of one of the following types:

1. $\mathbb{NM} = \mathcal{Q}([\mathbf{0},\mathbf{1}]) = \mathcal{Q}(\{\mathbf{A}_n : n > 1\})$
2. $\mathbb{NM}- = \mathcal{Q}([\mathbf{0},\mathbf{1}]^-) = \mathcal{Q}(\{\mathbf{A}_{2n} : n > 0\})$
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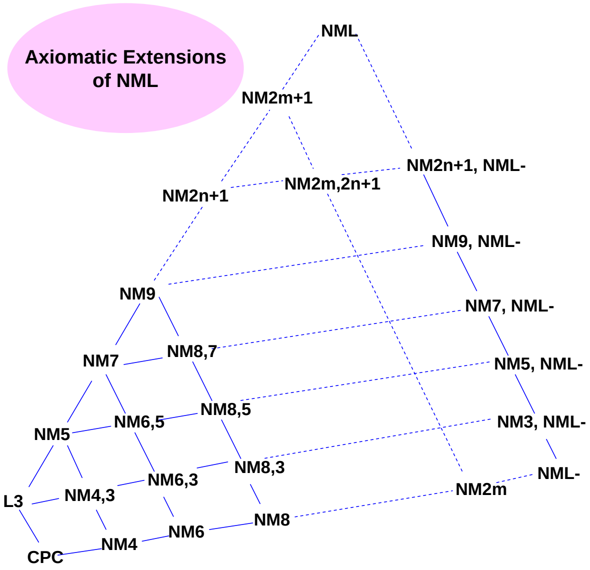
Axiomatic extensions of NML



Proposition

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Proof:

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If $h : Prop \to [0,1]$ is such that $h(p) = \frac{1}{2}$ then $h(\neg p \leftrightarrow p) = 1$ while $h(\bot) = 0 \neq 1$, hence $\neg p \leftrightarrow p \not\models_{[0,1]_{NM}} \bot$.

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$\neg p \leftrightarrow p/\bot$ is not NML-derivable and NML-admissible.

If $h : Prop \to [0,1]$ is such that $h(p) = \frac{1}{2}$ then $h(\neg p \leftrightarrow p) = 1$ while $h(\bot) = 0 \neq 1$, hence $\neg p \leftrightarrow p \not\models_{[0,1]_{NM}} \bot$.

Since $\mathbf{Free}_{\mathbb{NM}}$ has no negation fixpoint, then $\neg p \leftrightarrow p$ has no unifier, therefore $\neg p \leftrightarrow p/\bot$ is passive NML-admissible.

Theorem (Dzik-Wronski 1973)

Gödel logic is structurally complete

For every $n > 2$, $\mathbf{G}_n$ *is embeddable into* $\mathbf{Free}_{\mathbb{G}}(\omega)$

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Theorem (Cintula-Metcalfe 2009)

The positive fragment of the Gödel logic is structurally complete

For every $n > 2$, $\mathbf{G}_n^+$ *is embeddable into* $\mathbf{Free}_{\mathbb{G}^+}(\omega)$

Proposition

For every $n > 0$, $\mathbf{A}_{2n}$ *is embeddable into* $\mathbf{Free}_{\mathbb{NM}-}(\omega)$.

Proposition

For every $n > 0$, $\mathbf{A}_{2n}$ is embeddable into $\mathbf{Free}_{\mathbb{NM}-}(\omega)$.

Proof: Let $p_1, \ldots, p_{n-1} \in X$ be distinct variables, we define

$\varphi_1 = p_1 \vee \neg p_1$

$\varphi_i = ((((p_i \vee \neg p_i) \to \varphi_{i-1}) \to (p_i \vee \neg p_i)) \to (p_i \vee \neg p_i)) \quad i = 2 \div n-1$

$\varphi_n = \top$

then $f : A_{2n} \to Free_{\mathbb{NM}-}(\omega)$ defined by $f(i) = \begin{cases} \overline{\varphi_i}, & \text{if } i > 0; \\ \overline{\neg\varphi_i}, & \text{if } i < 0. \end{cases}$

is an embedding. □

Structural complete NM logics

$\mathcal{Q}(\mathbf{Free}_{\mathbb{NM}^-}) = \mathcal{Q}(\{\mathbf{A}_{2n} : n > 0\}) = \mathbb{NM}$

For every $n > 0$, $\mathcal{Q}(\mathbf{Free}_{\mathbb{NM}_{2n}}) = \mathcal{Q}(\mathbf{A}_{2n}) = \mathbb{NM}_{2n}$

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For every $n > 0$, $\mathcal{Q}(\mathbf{Free}_{\mathbb{NM}_{2n}}) = \mathcal{Q}(\mathbf{A}_{2n}) = \mathbb{NM}_{2n}$

Theorem

NML^- is hereditarily structurally complete.

Proposition

Let $\mathbb{M}$ be a non trivial variety of NM-algebras not satisfying the identity $\nabla(x) \approx \Delta(x)$. Then for every $k > 1$, $\mathbf{A}_2 \times \mathbf{A}_k$ is embeddable into $\mathbf{Free}_{\mathbb{M}}(\omega)$ if and only if $\mathbf{A}_k \in \mathbb{M}$

Proof:

Let $p_0, \ldots, p_{n-1} \in X$ be distinct variables if we define $\phi_n = \gamma_n = \top$, $\gamma_i = \varphi_i(p_1, \ldots, p_i)$ and $\phi_i = \varphi_{i+1}(p_0, \ldots, p_i)$ for $0 \leq i \leq n-1$, then

$$h : A_2 \times A_{2n+1} \to Free_{\mathbb{M}}(\omega)$$

$$h((1,m)) = \overline{\nabla(\phi_0) \vee \phi_m} \qquad h((-1,m)) = \overline{\neg\nabla(\phi_0) \wedge \phi_m}$$

$$h((1,0)) = \overline{\nabla(\phi_0) \vee \phi_0} \qquad h((-1,0)) = \overline{\neg\nabla(\phi_0) \wedge \phi_0}$$

$$h((1,-m)) = \overline{\nabla(\phi_0) \vee \neg\phi_m} \qquad h((-1,-m)) = \overline{\neg\nabla(\phi_0) \wedge \neg\phi_m}$$

$$g : A_2 \times A_{2n} \to Free_{\mathbb{M}}(\omega)$$

$$g((1,m)) = \overline{\neg\nabla(\gamma_1) \vee \gamma_m} \qquad g((-1,m)) = \overline{\nabla(\gamma_0) \wedge \gamma_m}$$

$$g((1,-m)) = \overline{\neg\nabla(\gamma_1) \vee \neg\gamma_m} \qquad g((-1,-m)) = \overline{\nabla(\gamma_1) \wedge \neg\gamma_m}$$

give the desired embeddings. □

Almost structural completeness of NM logics

If $\mathbb{M} \not\subseteq \mathbb{NM}-$, then

Almost structural completeness of NM logics

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$$\mathcal{Q}(\mathbf{Free}_{\mathbb{M}}) = \mathcal{Q}(\{\mathbf{A}_2 \times \mathbf{A}_k : \mathbf{A}_k \in \mathbb{M}\})$$

Almost structural completeness of NM logics

If $\mathbb{M} \not\subseteq \mathbb{NM}-$, then

$$\mathcal{Q}(\mathbf{Free}_{\mathbb{M}}) = \mathcal{Q}(\{\mathbf{A}_2 \times \mathbf{A}_k : \mathbf{A}_k \in \mathbb{M}\})$$

Theorem

$\mathbb{M}$ *is almost structurally complete*

Almost structural completeness of NM logics

Theorem

NML is almost structurally complete and all their consistent axiomatic extensions are almost structurally complete.

Theorem

For every axiomatic extension of NML the rule $\neg p \leftrightarrow p/\bot$ axiomatizes all passive admissible rules.

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For every axiomatic extension of NML the rule $\neg p \leftrightarrow p/\bot$ axiomatizes all passive admissible rules.

Proof:

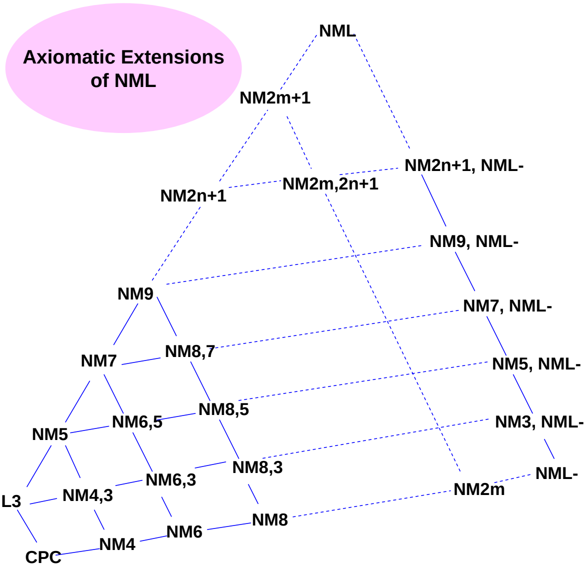
(Jeřábek 2010)

The rule $\neg(p \vee \neg p)^n/\bot$ axiomatizes all passive rules for every n-contractive axiomatic extension of MTL

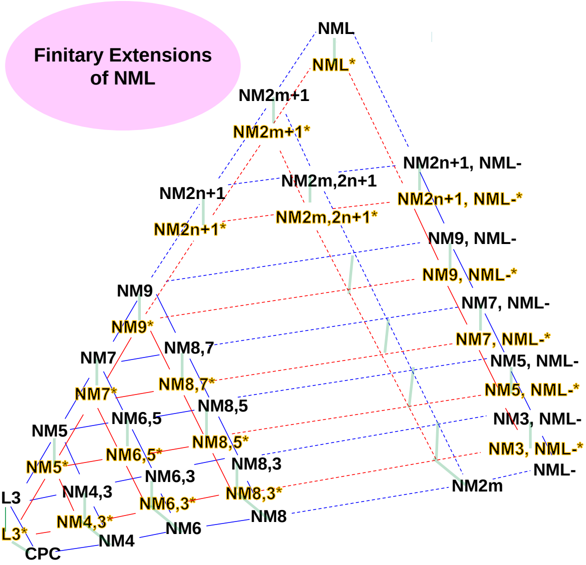
$$\neg p \leftrightarrow p \dashv\vdash_{NML} \neg(p \vee \neg p)^2$$

THANK YOU FOR YOUR ATTENTION

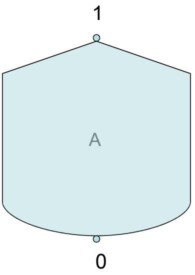
Axiomatic extensions of NML



Finitary extensions of NML



Connected Rotation

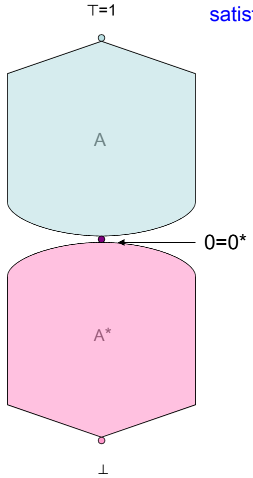


A= ⟨ A,∨,∧ ,⊗, ⇒, 0,1 ⟩ a MTL-algebra satisfying 1 or 2

1. A has no 0 divisors
2. For all 0 divisors a,b,
   a⇒0 = b⇒0

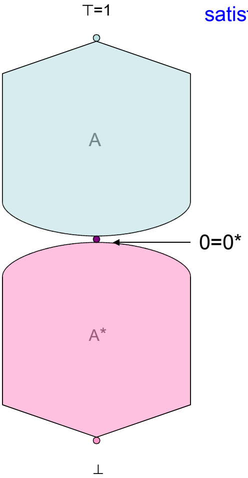
Connected Rotation

A= 〈 A,∨,∧ ,⊗, ⇒, 0,1 〉 a MTL-algebra satisfying 1 or 2.



For all a,b ∈ A,
a* ≤ b
a* ≤ b* iff b ≤ a

Connected Rotation



A= 〈 A,∨,∧ ,⊗, ⇒, 0,1 〉 a MTL-algebra satisfying 1 or 2.

B= 〈 A∪A*,∨,∧,&,→,¬, ⊤ 〉

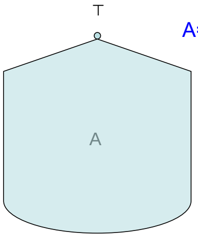
For all a,b ∈ A,
a* ≤ b
a* ≤ b* iff b ≤ a

¬a = a*
¬a* = a

a&b = a⊗b
a*&b* = (a⊗b)*
a&b* = (a⇒b)*

a→b = ¬(a&¬b)

Disconnected Rotation



A= ⟨A,∧,∨, ⊗, ⇒, ⊤⟩ basic semihoop

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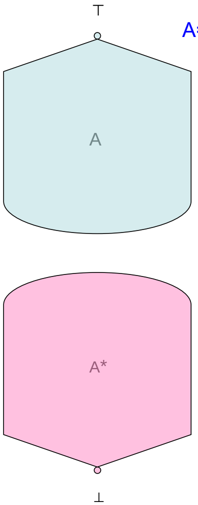
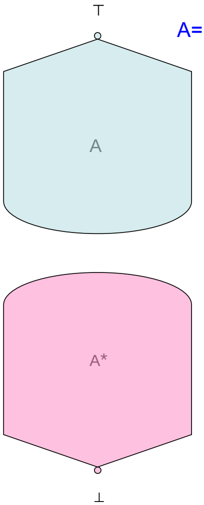


A= ⟨A,∧,∨, ⊗, ⇒, ⊤⟩ basic semihoop

For all a,b ∈ A,
a*< b
a*≤b* iff b≤a

Disconnected Rotation



$A = \langle A, \wedge, \vee, \otimes, \Rightarrow, \top \rangle$ basic semihoop

$B = \langle A \cup A^*, \wedge, \vee, \&, \rightarrow, \neg, \top \rangle$

For all $a, b \in A$,
$a^* < b$
$a^* \leq b^*$ iff $b \leq a$

$\neg a = a^*$
$\neg a^* = a$

$a \& b = a \otimes b$
$a^* \& b^* = (a \otimes b)^*$
$a \& b^* = (a \Rightarrow b)^*$

$a \rightarrow b = \neg(a \& \neg b)$