Hyperstates on strongly perfect MTL-algebras (with cancellative radical)

Tommaso Flaminio

Dipartimento di Scienze Teoriche e Applicate, Università dell’Insubria.
Italy

tommaso.flaminio@uninsubria.it
sites.google.com/site/tomflaminio

(with Lluis Godo and Sara Ugolini)
A state of an abelian \( \ell \)-group \( G = (G, +, - , \wedge, \vee, 0) \) is a map \( \sigma : G \to \mathbb{R} \) such that:

- \( \sigma(x + y) = \sigma(x) + \sigma(y) \),
- if \( x \geq 0 \) (in \( G \)) then \( \sigma(x) \geq 0 \) (in \( \mathbb{R} \)).
A state of an abelian \( \ell \)-group \( G = (G, +, -, \wedge, \vee, 0) \) is a map \( \sigma : G \to \mathbb{R} \) such that:

- \( \sigma(x + y) = \sigma(x) + \sigma(y) \),
- if \( x \geq 0 \) (in \( G \)) then \( \sigma(x) \geq 0 \) (in \( \mathbb{R} \)).

If \( u \in G \) is a strong unit for \( G \), then \( \sigma \) is a state of \( (G, u) \) if it satisfies

- \( \sigma(u) = 1 \).
A *state* of an abelian $\ell$-group $G = (G, +, -, \land, \lor, 0)$ is a map $\sigma: G \to \mathbb{R}$ such that:

- $\sigma(x + y) = \sigma(x) + \sigma(y)$,
- if $x \geq 0$ (in $G$) then $\sigma(x) \geq 0$ (in $\mathbb{R}$).

If $u \in G$ is a *strong unit* for $G$, then $\sigma$ is a state of $(G, u)$ if it satisfies

- $\sigma(u) = 1$.

Given an abelian ℓ-group with (strong) unit \((G, u)\), define on the interval \([0, u]\) the operations

\[
x \oplus y = (x + y) \land u \quad \text{and} \quad \neg x = u - x.
\]

Then, \(([0, u], \oplus, \neg, 0)\) is an MV-algebra and every MV-algebra is of this form.


T. Flaminio (DiSTA-Varese) Hyperstates
Given an abelian $\ell$-group with (strong) unit $(G, u)$, define on the interval $[0, u]$ the operations
\[ x \oplus y = (x + y) \land u \quad \text{and} \quad \neg x = u - x. \]
Then, $([0, u], \oplus, \neg, 0)$ is an MV-algebra and every MV-algebra is of this form.

States of MV-algebras can hence be introduced from states of abelian $\ell$-groups. Given any MV-algebra $A = (A, \oplus, \neg, 0)$, a state of $A$ is a map $s : A \to [0, 1]$ such that:

- $s(1) = 1$ (where $1 = \neg 0$),
- if $a \odot b = 0$, then $s(a \oplus b) = s(a) + s(b)$ (where $a \odot b = \neg(\neg a \oplus \neg b)$).
Given an abelian $\ell$-group with (strong) unit $(G, u)$, define on the interval $[0, u]$ the operations
\[
x \oplus y = (x + y) \land u \quad \text{and} \quad \neg x = u - x.
\]
Then, $([0, u], \oplus, \neg, 0)$ is an MV-algebra and every MV-algebra is of this form.

States of MV-algebras can hence be introduced from states of abelian $\ell$-groups. Given any MV-algebra $A = (A, \oplus, \neg, 0)$, a state of $A$ is a map $s : A \to [0, 1]$ such that:

- $s(1) = 1$ (where $1 = \neg 0$),
- if $a \odot b = 0$, then $s(a \oplus b) = s(a) + s(b)$ (where $a \odot b = \neg(\neg a \oplus \neg b)$).

The previous definition applies to any MV-algebra. Let us consider the case of *perfect MV-algebras*.
The previous definition applies to any MV-algebra. Let us consider the case of perfect MV-algebras.

In any MV-algebra $A$, let $Max(A)$ the set of its maximal filters and let $Rad(A) = \bigcap \{m \mid m \in Max(A)\}$ and $coRad(A) = \{-x \mid x \in Rad(A)\}$. 

An MV-algebra $A$ is perfect iff $A = Rad(A) \cup coRad(A)$. 

Every perfect MV-algebra has only one trivial state: $s(x) = 1$ if $x \in Rad(A)$ and $s(x) = 0$ if $x \in coRad(A)$. 

The previous definition applies to any MV-algebra. Let us consider the case of perfect MV-algebras.

In any MV-algebra $A$, let $\text{Max}(A)$ the set of its maximal filters and let

$$\text{Rad}(A) = \bigcap \{m \mid m \in \text{Max}(A)\} \text{ and } \text{coRad}(A) = \{\neg x \mid x \in \text{Rad}(A)\}.$$ 

An MV-algebra $A$ is perfect iff $A = \text{Rad}(A) \cup \text{coRad}(A)$. 


The previous definition applies to any MV-algebra. Let us consider the case of perfect MV-algebras.

In any MV-algebra $A$, let $\text{Max}(A)$ the set of its maximal filters and let

$$\text{Rad}(A) = \bigcap \{ m \mid m \in \text{Max}(A) \} \text{ and } \text{coRad}(A) = \{ \neg x \mid x \in \text{Rad}(A) \}.$$ 

An MV-algebra $A$ is perfect iff $A = \text{Rad}(A) \cup \text{coRad}(A)$.

Every perfect MV-algebra has only one trivial state: $s(x) = 1$ if $x \in \text{Rad}(A)$ and $s(x) = 0$ if $x \in \text{coRad}(A)$. 


T. Flaminio (DiSTA-Varese)
The previous definition applies to any MV-algebra. Let us consider the case of perfect MV-algebras.

In any MV-algebra $A$, let $\text{Max}(A)$ the set of its maximal filters and let

$$\text{Rad}(A) = \bigcap \{ m \mid m \in \text{Max}(A) \} \text{ and } \text{coRad}(A) = \{ \neg x \mid x \in \text{Rad}(A) \}.$$ 

An MV-algebra $A$ is perfect iff $A = \text{Rad}(A) \cup \text{coRad}(A)$.

Every perfect MV-algebra has only one trivial state: $s(x) = 1$ if $x \in \text{Rad}(A)$ and $s(x) = 0$ if $x \in \text{coRad}(A)$.

Perfect MV-algebras do not form a variety, but if we pick $A \in \text{DLMV}$, states are far from being less trivial since they can be regarded just as probability functions on the boolean skeleton $\mathcal{B}(A)$ of $A$. 
Perfect MV-algebras do not form a variety, but if we pick $A \in \mathbb{DLMV}$, states are far from being less trivial since they can be regarded just as probability functions on the boolean skeleton $\mathcal{B}(A)$ of $A$. 
Let $A$ be any algebra in $\mathcal{DLMV}$. The radical of $A$, with operations inherited by $A$, is a cancellative hoop.

$$(\text{Rad}(A), \circ, \rightarrow, \wedge, 1)$$

Hence it embeds (as an $\ell$-monoid) into the negative cone of an abelian $\ell$-group.
Let $A$ be any algebra in $\text{DLMV}$. The radical of $A$, with operations inherited by $A$, is a cancellative hoop.

$$(\text{Rad}(A), \circ, \to, \land, 1)$$

Hence it embeds (as an $\ell$-monoid) into the negative cone of an abelian $\ell$-group.
Let $A$ be any algebra in $\mathcal{DLMV}$. The radical of $A$, with operations inherited by $A$, is a cancellative hoop.

$$(\text{Rad}(A), \odot, \to, \land, 1)$$

Hence it embeds (as an $\ell$-monoid) into the negative cone of an abelian $\ell$-group.

Let us hence define a state of a cancellative hoop $(H, \odot, \to, \land, 1)$ as a map $w : H \to \mathbb{R}^-$ such that

$$w(1) = 0 \text{ and } w(a \odot b) = w(a) + w(b).$$
Let $B$ be a boolean algebra and let $H$ be a cancellative hoop. If $\vee_e : B \times H \to H$ is a map satisfying some suitable conditions, the system $(B, H, \vee_e)$ is called a cancellative hoop triple. If $(B, H, \vee_e)$ and $(B', H', \vee'_e)$ are cancellative hoop triples, a good morphism between them is a pair $(h, k)$ such that

- $h : B \to B'$ is a boolean homomorphism,
- $k : H \to H$ is a hoop homomorphism,
- $k(b \vee_e c) = h(b) \vee'_e k(c)$ for every $b \in B$ and $c \in H$. 

Cancellative hoop triples and good morphisms form a category (denoted $\mathcal{CH}$) which is equivalent to a class of MTL-algebras called strongly perfect with cancellative radical. This class includes DLMV and the category $\mathcal{P}$ of product algebras.


T. Flaminio (DiSTA-Varese)
Let $B$ be a boolean algebra and let $H$ be a cancellative hoop. If $\vee_e : B \times H \to H$ is a map satisfying some suitable conditions, the system $(B, H, \vee_e)$ is called a cancellative hoop triple. If $(B, H, \vee_e)$ and $(B', H', \vee'_e)$ are cancellative hoop triples, a good morphism between them is a pair $(h, k)$ such that

- $h : B \to B'$ is a boolean homomorphism,
- $k : H \to H$ is a hoop homomorphism,
- $k(b \vee_e c) = h(b) \vee'_e k(c)$ for every $b \in B$ and $c \in H$.

Cancellative hoop triples and good morphisms form a category (denoted $\mathcal{T}_{\text{CH}}$) which is equivalent to a class of MTL-algebras called strongly perfect with cancellative radical. This class includes $\text{DLMV}$ and the category $\mathbb{P}$ of product algebras.
Let $B$ be a boolean algebra and let $H$ be a cancellative hoop. If $\vee_e : B \times H \to H$ is a map satisfying some suitable conditions, the system $(B, H, \vee_e)$ is called a cancellative hoop triple. If $(B, H, \vee_e)$ and $(B', H', \vee'_e)$ are cancellative hoop triples, a good morphism between them is a pair $(h, k)$ such that

- $h : B \to B'$ is a boolean homomorphism,
- $k : H \to H$ is a hoop homomorphism,
- $k(b \vee_e c) = h(b) \vee'_e k(c)$ for every $b \in B$ and $c \in H$.

Cancellative hoop triples and good morphisms form a category (denoted $\mathcal{CH}$) which is equivalent to a class of MTL-algebras called strongly perfect with cancellative radical. This class includes $\mathbb{DLMV}$ and the category $\mathbb{P}$ of product algebras.

Given any strongly perfect MTL-algebra $A$ with cancellative radical, corresponding to a triple $(B, H, \lor_e)$, a state of $A$ should be made of

- A probability function $p$ on the boolean skeleton $\mathcal{B}(A)$ of $A$,
- A state $w$ of the radical $\mathcal{H}(A)$ of $A$.
Given any strongly perfect MTL-algebra $A$ with cancellative radical, corresponding to a triple $(B, H, \lor_e)$, a state of $A$ should be made of

- A probability function $p$ on the boolean skeleton $B(A)$ of $A$,
- A state $w$ of the radical $H(A)$ of $A$. 

\[ \text{Diagram} \]
Let \(A = (A, \odot, \rightarrow, \land, \lor, 0, 1)\) be a strongly perfect MTL-algebra with cancellative radical. A \textit{hyperstate} of \(A\) is a map

\[ s : A \to \mathcal{L}(\mathbb{R}) \]

where \(\mathcal{L}(\mathbb{R}) = \Gamma(\mathbb{R} \times_{\text{lex}} \mathbb{R}, (1, 0))\) and such that

- \(s(1) = 1\),
- \(s(a \oplus b) + s(a \odot b) = s(a) + s(b)\) (where \(a \oplus b = \neg(\neg a \odot \neg b)\)),
- if \(a \lor \neg a = 1\), then either \(s(a) = 0\), or there is \(n \in \mathbb{N}\) such that \(n.s(a) = 1\) (where \(n.x = x \oplus \ldots \oplus x\ \text{n-times}\)).
Any hyperstate of a strongly perfect MTL-algebra with cancellative radical satisfies:

(I) $s(\neg x) = 1 - s(x)$, and hence $s(0) = 0$,

(II) if $a \leq b$, then $s(a) \leq s(b)$,

(III) if $a \odot b = 0$, $s(a \oplus b) = s(a) + s(b)$,

(IV) if $a \oplus b = 1$, $s(a \odot b) = s(a) \odot s(b)$,

(V) $s(a \wedge b) + s(a \vee b) = s(a) + s(b)$,

(VI) The restriction $p$ of $s$ to $\mathcal{B}(A)$ is a $[0, 1]$-valued and finitely additive probability measure.
More interestingly we can prove the following:

**Theorem**

For every strongly perfect MTL-algebra $A$ with cancellative radical and for every hyperstate $s$ of $A$, there is a probability measure $p$ on $\mathcal{B}(A)$ and a state $w$ of $\mathcal{H}(A)$ such that, for every $a \in A$,

$$s(a) = p(b_a) + \varepsilon w(c).$$

where $c = [b_a \lor e \delta(c_a), \neg b_a \lor e c_a]$ is an element of the abelian $\ell$-group generated by the radical $\mathcal{H}(A)$. 
**Strongly perfect MTL-algebras** form a category which is equivalent to a category of triples \((B, H, \lor_e)\) where \(H\) is a **prelinear semihoop**.
Strongly perfect MTL-algebras form a category which is equivalent to a category of triples $(B, H, \vee_e)$ where $H$ is a prelinear semihoop.
Strongly perfect MTL-algebras form a category which is equivalent to a category of triples \((B, H, \vee_e)\) where \(H\) is a prelinear semihoop.
Strongly perfect MTL-algebras form a category which is equivalent to a category of triples \((B, H, \vee_e)\) where \(H\) is a prelinear semihoop.
Hence, defining states for these structures boils down in finding a reasonable (suitable) definition of *states of prelinear semihoops*. 
Hence, defining states for these structures boils down in finding a reasonable (suitable) definition of states of prelinear semihoops.

*Warning: The following is a work in progress!*
Hence, defining states for these structures boils down in finding a reasonable (suitable) definition of states of prelinear semihoops.

Warning: The following is a work in progress!

**Theorem**

Let $M = (M, +, 0)$ be a monoid. Then, there is an abelian group $K(M)$ and a monoid homomorphism $h : M \to K(M)$ which is injective iff $M$ is cancellative.
Hence, defining states for these structures boils down in finding a reasonable (suitable) definition of states of prelinear semihoops.

Warning: The following is a work in progress!

**Theorem**

Let $M = (M, +, 0)$ be a monoid. Then, there is an abelian group $K(M)$ and a monoid homomorphism $h : M \rightarrow K(M)$ which is injective iff $M$ is cancellative.

**Theorem**

Let $M = (M, +, \wedge, \vee, 0)$ be a lattice-ordered monoid. Then, there is an abelian $\ell$-group $K(M)$ and a $\ell$-monoid homomorphism $h : M \rightarrow K(M)$ which is injective iff $M$ is cancellative.
Let $H = (H, \odot, \rightarrow, \wedge, 1)$ be a prelinear semihoop and consider the $\ell$-monoid

$$M_H = (H, \odot, \wedge, \vee, 1)$$

Then, there is an $\ell$-monoid homomorphism $h : H \rightarrow K(M_H)$. Now, if $\sigma : K(M_H) \rightarrow \mathbb{R}$, states of prelinear semihoop should correspond to the composition maps

$$\sigma \circ h : H \rightarrow \mathbb{R}^-.$$

We can hence define a state of a prelinear semihoop $H$ as a map $w : H \rightarrow \mathbb{R}^-$ such that

- $w(1) = 0$,
- $w(a \odot b) = w(a) + w(b)$,
- if $a \leq b$, then $w(a) \leq w(b)$ (this condition is redundant if $H$ is cancellative!)
Thank you.