Hyperstates on strongly perfect MTL-algebras (with cancellative radical)

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(with Lluis Godo and Sara Ugolini)

A *state* of an abelian ℓ-group $G = (G, +, -, \wedge, \vee, 0)$ is a map $\sigma : G \to \mathbb{R}$ such that:

- $\sigma(x + y) = \sigma(x) + \sigma(y)$,
- if $x \geq 0$ (in G) then $\sigma(x) \geq 0$ (in $\mathbb{R}$).

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[K. R. Goodearl, *Partially Ordered Abelian Group with Interpolation*. AMS Math. Survey and Monographs, Vol. 20, 1986.]


σ
0


v
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σ
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0

Given an abelian ℓ-group with (strong) unit (G, u), define on the interval $[0, u]$ the operations

$$x \oplus y = (x + y) \wedge u \text{ and } \neg x = u - x.$$

Then, $([0, u], \oplus, \neg, 0)$ is an MV-algebra and every MV-algebra is of this form.

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States of MV-algebras can hence be introduced from states of abelian ℓ-groups. Given any MV-algebra $A = (A, \oplus, \neg, 0)$, a state of A is a map $s : A \to [0, 1]$ such that:

- $s(1) = 1$ (where $1 = \neg 0$),
- if $a \odot b = 0$, then $s(a \oplus b) = s(a) + s(b)$ (where $a \odot b = \neg(\neg a \oplus \neg b)$).

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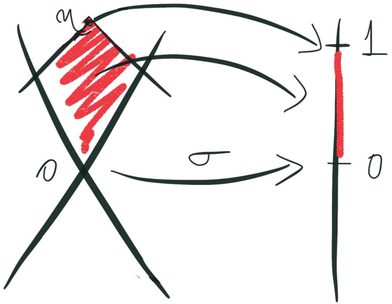
[D. Mundici, Averaging the Truth-value in Łukasiewicz Logic. *Studia Logica* 55(1), 113–127, 1995.]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The previous definition applies to any MV-algebra. Let us consider the case of *perfect MV-algebras*.

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$$Rad(A) = \bigcap\{\mathfrak{m} \mid \mathfrak{m} \in Max(A)\} \text{ and } coRad(A) = \{\neg x \mid x \in Rad(A)\}.$$

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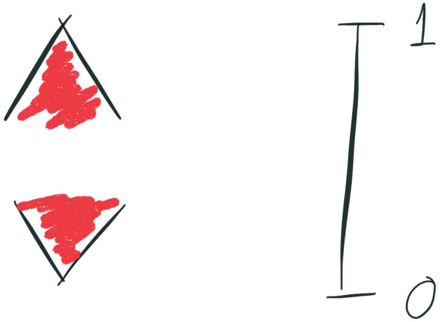
[A. Di Nola, G. Georgescu, I. Leuștean, States on perfect MV-algebras, in: V. Novak, I. Perfilieva (Eds.), *Discovering the World With Fuzzy Logic*, in: Stud. Fuzziness Soft Comput., vol. 57, Physica, Heidelberg, 105–125, 2000.]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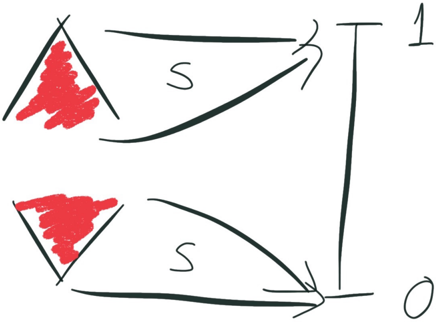
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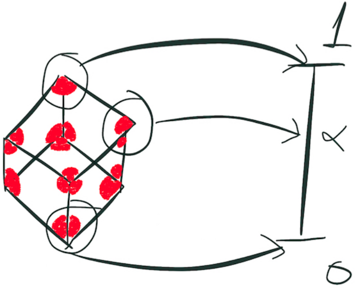

Let A be any algebra in $\mathbb{DLMV}$. The radical of A, with operations inherited by A, is a cancellative hoop.

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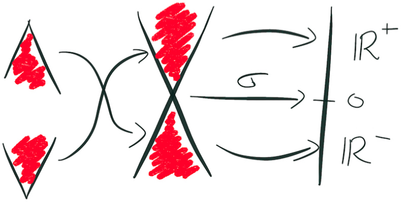

Let us hence define a *state of a cancellative hoop* $(H, \odot, \to, \wedge, 1)$ as a map $w : H \to \mathbb{R}^-$ such that

$$w(1) = 0 \text{ and } w(a \odot b) = w(a) + w(b).$$

Let B be a boolean algebra and let H be a cancellative hoop. If $\vee_e : B \times H \to H$ is a map satisfying some suitable conditions, the system $(B, H, \vee_e)$ is called a *cancellative hoop triple*. If $(B, H, \vee_e)$ and $(B', H', \vee'_e)$ are cancellative hoop triples, a *good morphism* between them is a pair (h, k) such that

- $h : B \to B'$ is a boolean homomorphism,
- $k : H \to H$ is a hoop homomorphism,
- $k(b \vee_e c) = h(b) \vee'_e k(c)$ for every $b \in B$ and $c \in H$.

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Cancellative hoop triples and good morphisms form a category (denoted $\mathcal{T}_{\mathbb{CH}}$) which is equivalent to a class of MTL-algebras called *strongly perfect with cancellative radical*. This class includes $\mathbb{DLMV}$ and the category $\mathbb{P}$ of product algebras.

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[S. Aguzzoli, T. Flaminio, S. Ugolini, Equivalences between subcategories of MTL-algebras via Boolean algebras and prelinear semihoops. Manuscript 2016.]

Given any strongly perfect MTL-algebra A with cancellative radical, corresponding to a triple $(B, H, \vee_e)$, a state of A should be made of

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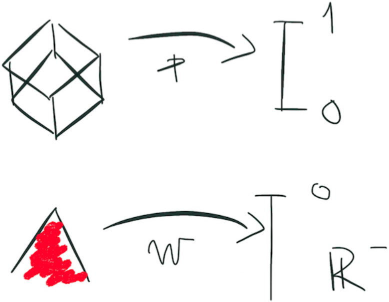

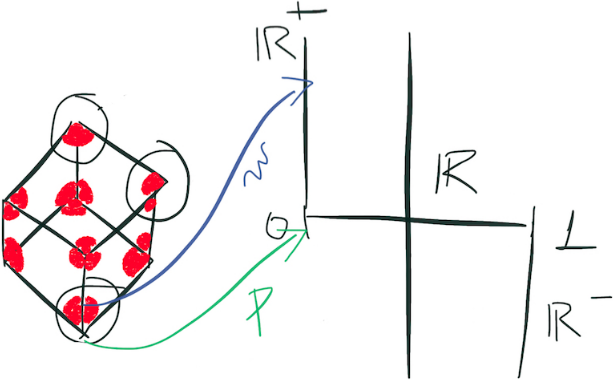
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Let $A = (A, \odot, \to, \wedge, \vee, 0, 1)$ be a strongly perfect MTL-algebra with cancellative radical. A *hyperstate* of A is a map

$$s : A \to \mathcal{L}(\mathbb{R})$$

where $\mathcal{L}(\mathbb{R}) = \Gamma(\mathbb{R} \times_{lex} \mathbb{R}, (1, 0))$ and such that

- $s(1) = 1$,
- $s(a \oplus b) + s(a \odot b) = s(a) + s(b)$ (where $a \oplus b = \neg(\neg a \odot \neg b)$),
- if $a \vee \neg a = 1$, then either $s(a) = 0$, or there is $n \in \mathbb{N}$ such that $n.s(a) = 1$ (where $n.x = x \oplus \ldots \oplus x$ n-times).

Any hyperstate of a strongly perfect MTL-algebra with cancellative radical satisfies:

(I) $s(\neg x) = 1 - s(x)$, and hence $s(0) = 0$,

(II) if $a \leq b$, then $s(a) \leq s(b)$,

(III) if $a \odot b = 0$, $s(a \oplus b) = s(a) + s(b)$,

(IV) if $a \oplus b = 1$, $s(a \odot b) = s(a) \odot s(b)$,

(V) $s(a \wedge b) + s(a \vee b) = s(a) + s(b)$,

(VI) The restriction p of s to $\mathscr{B}(A)$ is a $[0, 1]$-valued and finitely additive probability measure.

More interestingly we can prove the following:

Theorem

For every strongly perfect MTL-algebra A with cancellative radical and for every hyperstate s of A, there is a probability measure p on $\mathscr{B}(A)$ and a state w of $\mathscr{H}(A)$ such that, for every $a \in A$,

$$s(a) = p(b_a) + \varepsilon w(c).$$

where $c = [b_a \vee_e \delta(c_a), \neg b_a \vee_e c_a]$ is an element of the abelian ℓ-group generated by the radical $\mathscr{H}(A)$.

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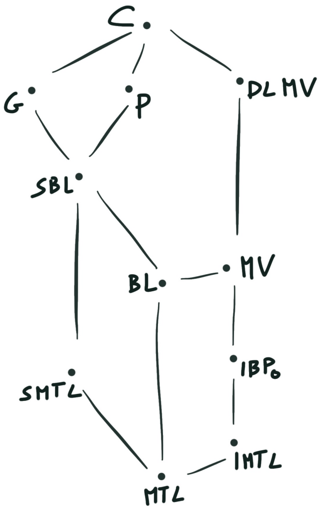

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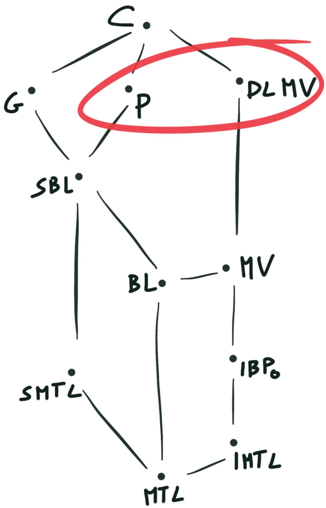

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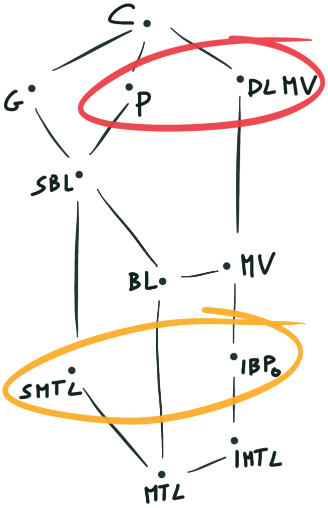

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Let $M = (M, +, 0)$ *be a monoid. Then, there is an abelian group* $\mathbf{K}(M)$ *and a monoid homomorphism* $h : M \to \mathbf{K}(M)$ *which is injective iff* M *is cancellative.*

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Theorem

Let $M = (M, +, \wedge, \vee, 0)$ *be a lattice-ordered monoid. Then, there is an abelian* ℓ*-group* $\mathbf{K}(M)$ *and a* ℓ*-monoid homomorphism* $h : M \to \mathbf{K}(M)$ *which is injective iff* M *is cancellative.*

Let $H = (H, \odot, \to, \wedge, 1)$ be a prelinear semihoop and consider the ℓ-monoid

$$M_H = (H, \odot, \wedge, \vee, 1)$$

Then, there is an ℓ-monoid homomorphism $h : H \to \mathbf{K}(M_H)$. Now, if $\sigma : \mathbf{K}(M_H) \to \mathbb{R}$, states of prelinear semihoop should correspond to the composition maps

$$\sigma \circ h : H \to \mathbb{R}^-.$$

We can hence define a *state of a prelinear semihoop H* as a map $w : H \to \mathbb{R}^-$ such that

- $w(1) = 0$,
- $w(a \odot b) = w(a) + w(b)$,
- if $a \leq b$, then $w(a) \leq w(b)$ (this condition is redundant if H is cancellative!)

Thank you.