Evaluation Driven Proof-Search in Natural Deduction Calculi for Intuitionistic Propositional Logic

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SYSMICS 2016
Barcelona, September 5th, 2016
Motivations

Natural deduction and sequent calculus are dual presentations of Intuitionistic Logic with different features:

- **Sequent calculus**
  More appropriate for meta-theoretical reasoning and proof-search (cut-elimination).

- **Natural Deduction calculus**
  Natural and immediate computational interpretation via the Curry-Howard isomorphism (normalization).

Lot of work has been done to extend the Curry-Howard isomorphism to sequent calculi, see the Herbelin’s permutation-free sequent calculus and its successive refinements

- *H. Herbelin. A lambda-calculus structure isomorphic to Gentzen-style sequent calculus structure. CSL, 1994*

Motivations

Instead, proof-search in natural deduction calculi has been scarcely investigated.

Main references:

- **Proof-search procedures based on the intercalation calculus**
  
  

  These procedures are highly inefficient and do not provide a clear characterization of the proof-search space.

- **Recent work:**
  
  *G. Mints and SH. Steinert-Threlkeld. ADC method of proof search for intuitionistic propositional natural deduction. JLC, 2016.*

  The authors introduce the ADC method (Analysis and Direct Chaining) to construct natural derivations where there are not I-rules above E-rules.

  Derivations of this kind can be built in polynomial time, but ADC method is incomplete for Intuitionistic Logic.
Our contribution

We reconsider the problem of proof-search in the Natural Deduction calculus for Intuitionistic Propositional Logic (IPL).

- We introduce **Nbu**, a variant of the standard Natural Deduction calculus for IPL.
- We define a goal-oriented proof-search procedure for **Nbu**, which considerably improves the one based on intercalation calculus.
  
  The key point is that **Nbu** internalizes some aspects of the proof-search procedure.

This show that goal-oriented proof-search is not a distinctive feature of sequent calculus but can be recovered also in the context of natural deduction.
The natural deduction calculus has been introduced to capture logical mathematical reasoning.

*The formalization of logical deduction, especially as it has been developed by Frege, Russel, and Hilbert, is rather far removed from the forms of deduction used in practice in mathematical proofs. Considerable formal advantages are achieved in return.*

*I intended, first of all, to set up a formal system which comes as close as possible to actual reasoning. The result was a calculus of natural deduction (NJ for intuitionist, NK for classical predicate logic).*

*[Gentzen, “Investigations into logical deduction”, 1934]*
Formulas $A, B, \ldots$ of IPL are built starting from a set $\mathcal{V}$ of propositional variables:

$$A, B ::= \bot \mid p \mid A \land B \mid A \lor B \mid A \rightarrow B \quad p \in \mathcal{V}$$

$$\neg A ::= A \rightarrow \bot$$

For each logical connective it is defined an introduction rule (I-rule) and an elimination rule (E-rule)

- **I-rule**
  How to introduce a compound formula.
  *Infer a complex formula from already established components*

- **E-rule**
  How to de-construct information about a compound formula.
  *Specify how components of assumed or established complex formulas can be used as arguments.*
A derivation of $B$ having open assumptions $A_1, \ldots, A_n$ is represented by a proof-tree $D$ of the form

$$
A_1, \ldots, A_n
\quad D
\quad B
$$

In our presentation, it is more convenient to localize hypothesis and present derivations in sequent style:

$$
D
\quad \Gamma \Rightarrow B
$$

The context $\Gamma$ (multiset) contains the assumptions $A_1, \ldots, A_n$ on which $B$ depends.

**NJ$_0$**: Natural Deduction calculus for **IPL** in sequent style
• Axiom rule

\[
\frac{}{A, \Gamma \Rightarrow A} \text{Id}
\]

It represents a single-node derivation, where \( A \) is both an assumption and the conclusion.
Rules of $\textbf{NJ}_0$

- **Axiom rule**

\[
\frac{}{A, \Gamma \Rightarrow A} \text{Id}
\]

It represents a single-node derivation, where $A$ is both an assumption and the conclusion.

- **I-rules for $\land$, $\lor$**

\[
\frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \land B} \land I
\]

\[
\frac{D_1 \quad D_2}{A \land B} \land I
\]

\[
\frac{\Gamma \Rightarrow A_k}{\Gamma \Rightarrow A_0 \lor A_1} \lor I
\]

\[
\frac{D}{A_k \lor A_0 \lor A_1} \lor I
\]

$k \in \{0, 1\}$
I-rules for $\rightarrow$

\[
\begin{array}{c}
[A] \\
\mathcal{D} \\
\frac{B}{A \rightarrow B} \rightarrow I
\end{array}
\]

The assumption $A$ of $\mathcal{D}$ can be discharged by the rule application. We split the rule into $\rightarrow l_1$ and $\rightarrow l_2$:

\[
\begin{array}{c}
\Gamma \Rightarrow A \rightarrow B \\
\Gamma \Rightarrow B
\end{array} \rightarrow l_1
\]

$A$ is not discharged

\[
\begin{array}{c}
\Gamma \Rightarrow A \rightarrow B \\
A, \Gamma \Rightarrow B
\end{array} \rightarrow l_2
\]

$A$ is discharged
E-rules for $\land$, $\to$

$$\frac{\Gamma \Rightarrow A_0 \land A_1}{\Gamma \Rightarrow A_k} \quad \land E$$

$$\frac{\Gamma \Rightarrow A \rightarrow B \quad \Gamma \Rightarrow A}{\Gamma \Rightarrow B} \quad \rightarrow E$$

modus ponens

$$\frac{A_0 \land A_1}{A_k} \quad \land E$$

$$\frac{\mathcal{D}_1 \quad \mathcal{D}_2}{A \rightarrow B \quad B \quad A} \quad \rightarrow E$$
Rules of $\mathbf{NJ}_0$

- E-rule for $\lor$

\[
\Gamma \Rightarrow A \lor B \quad A, \Gamma \Rightarrow C \quad B, \Gamma \Rightarrow C
\]

\[
\Delta_1 \quad \Delta_2 \quad \Delta_3
\]

\[
\Gamma \Rightarrow C \quad \lor E
\]

\[
A \lor B \quad C \quad C
\]

\[
\lor E
\]

Falsehood corresponds to a disjunction with no alternatives:
- no I-rule
- E-rule has no cases

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Rules of $\textbf{NJ}_0$

- **E-rule for $\lor$**

  \[
  \begin{array}{c}
  \Gamma \Rightarrow A \lor B \\
  A, \Gamma \Rightarrow C \\
  B, \Gamma \Rightarrow C
  \end{array}
  \quad \Rightarrow
  \begin{array}{c}
  \Gamma \Rightarrow C
  \end{array}
  \quad \lor E
  \]

- **Rules for $\bot$**

  Falsehood corresponds to a disjunction with no alternatives:
  - no I-rule
  - E-rule has no cases

  \[
  \begin{array}{c}
  \Gamma \Rightarrow \bot
  \end{array}
  \quad \Rightarrow
  \begin{array}{c}
  \Gamma \Rightarrow C
  \end{array}
  \quad \bot E
  \]

  \[
  \begin{array}{c}
  \bot
  \end{array}
  \quad \Rightarrow
  \begin{array}{c}
  C
  \end{array}
  \quad \bot E
  \]
The calculus $\textbf{NJ}_0$

\[
\begin{align*}
\frac{A, \Gamma \Rightarrow A}{\Gamma \Rightarrow A} & \quad \text{Id} \\
\frac{\Gamma \Rightarrow \bot}{\Gamma \Rightarrow A} & \quad \bot E \\
\frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \land B} & \quad \land I \\
\frac{\Gamma \Rightarrow A_k}{\Gamma \Rightarrow A_0 \lor A_1} & \quad \lor I \\
\frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B} & \quad \rightarrow I_1 \\
\frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B} & \quad \rightarrow I_2 \\
\frac{\Gamma \Rightarrow A \land A_1}{\Gamma \Rightarrow A_k} & \quad \land E \\
\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow B} & \quad \rightarrow E
\end{align*}
\]

**Theorem (Completeness of $\textbf{NJ}_0$)**

$A \in \textbf{IPL}$ *iff* there exists a derivation of $\cdot \Rightarrow A$ in $\textbf{NJ}_0$
A naïve proof-search strategy for $\textbf{NJ}_0$

We perform two expansion steps:

- $\uparrow$-expansion: apply I-rules bottom-up
- $\downarrow$-expansion: apply the E-rules top-down
- Goal: meet in the middle!

To get a derivation $\mathcal{D}$, one has to alternate $\uparrow$-expansions and $\downarrow$-expansion phases (*intercalation calculus*).

By definition, $\mathcal{D}$ is in normal form.
A naïve proof-search strategy for $\textbf{NJ}_0$

To formalize the strategy, we orient the sequents:

- $\Gamma \Rightarrow A \uparrow$
  - $A$ has a *normal derivation (nd)* from assumptions $\Gamma$
- $\Gamma \Rightarrow A \downarrow$
  - $A$ can be *extracted* from the assumptions $\Gamma$

$$\textbf{NJ}_0 + \text{arrows} \downarrow, \uparrow = \textbf{NJ}$$

- **Proof-search:**
  
  

- **Arrow notation:**
  
  
We rewrite the rules of the calculus by orienting the sequents. Arrows suggest in which direction a rule must be applied in proof-search.

- \( \land I \)

\[
\frac{\Gamma \Rightarrow A \uparrow \quad \Gamma \Rightarrow B \uparrow}{\Gamma \Rightarrow A \land B \uparrow} \land I
\]

\( A \land B \) has a nd from \( \Gamma \) if both \( A \) and \( B \) have a nd from \( \Gamma \) (hence \( \land I \) must be applied bottom-up).
Rules for Normal Derivations

We rewrite the rules of the calculus by orienting the sequents. Arrows suggest in which direction a rule must be applied in proof-search.

- **∧I**

\[
\frac{\Gamma \Rightarrow A \uparrow \quad \Gamma \Rightarrow B \uparrow}{\Gamma \Rightarrow A \land B \uparrow} \quad \text{∧I}
\]

- A \land B has a nd from \( \Gamma \) if both \( A \) and \( B \) have a nd from \( \Gamma \) (hence \( \text{∧I} \) must be applied bottom-up).

- **∧E**

\[
\frac{\Gamma \Rightarrow A \land B \downarrow}{\Gamma \Rightarrow A \downarrow} \quad \text{∧E} \quad \frac{\Gamma \Rightarrow A \land B \downarrow}{\Gamma \Rightarrow B \downarrow} \quad \text{∧E}
\]

- If \( A \land B \) can be extracted from \( \Gamma \), then both \( A \) and \( B \) can be extracted from \( \Gamma \) (hence \( \text{∧E} \) must be applied top-down).
Rules for Normal Derivations

- **∨I**

\[
\frac{\Gamma \Rightarrow A \uparrow}{\Gamma \Rightarrow A \lor B \uparrow} \quad \frac{\Gamma \Rightarrow B \uparrow}{\Gamma \Rightarrow A \lor B \uparrow}
\]

\( A \lor B \) has a nd from \( \Gamma \) if \( A \) or \( B \) has a nd from \( \Gamma \).
Rules for Normal Derivations

• $\lor I$

$$
\frac{\Gamma \Rightarrow A \uparrow}{\Gamma \Rightarrow A \lor B \uparrow} \lor I
\quad
\frac{\Gamma \Rightarrow B \uparrow}{\Gamma \Rightarrow A \lor B \uparrow} \lor I
$$

$A \lor B$ has a nd from $\Gamma$ if $A$ or $B$ has a nd from $\Gamma$.

• $\lor E$

$$
\frac{\Gamma \Rightarrow A \lor B \downarrow}{\Gamma \Rightarrow C \uparrow}
\quad
\frac{A, \Gamma \Rightarrow C \uparrow}{\Gamma \Rightarrow C \uparrow}
\quad
\frac{B, \Gamma \Rightarrow C \uparrow}{\Gamma \Rightarrow C \uparrow} \lor E
$$

$C$ has a nd from $\Gamma$ if $A \lor B$ can be extracted from $\Gamma$ and $C$ has both a nd from $A, \Gamma$ and a nd from $B, \Gamma$ ($\uparrow$-expansion and $\downarrow$-expansion meet)
Rules for Normal Derivations

- **→ I₁**

\[
\frac{\Gamma \Rightarrow B \uparrow}{\Gamma \Rightarrow A \rightarrow B \uparrow} \rightarrow I₁
\]

A → B has a nd from Γ if B has a nd from Γ.

- **→ I₂**

\[
\frac{A, \Gamma \Rightarrow B \uparrow}{\Gamma \Rightarrow A \rightarrow B \uparrow} \rightarrow I₂
\]

A → B has a nd from Γ if B has a nd from A, Γ.
Rules for Normal Derivations

- $\rightarrow I_1$

\[
\frac{\Gamma \Rightarrow B \uparrow}{\Gamma \Rightarrow A \rightarrow B \uparrow} \rightarrow I_1
\]

$A \rightarrow B$ has a nd from $\Gamma$ if $B$ has a nd from $\Gamma$.

- $\rightarrow I_2$

\[
\frac{A, \Gamma \Rightarrow B \uparrow}{\Gamma \Rightarrow A \rightarrow B \uparrow} \rightarrow I_2
\]

$A \rightarrow B$ has a nd from $\Gamma$ if $B$ has a nd from $A, \Gamma$.

- $\rightarrow E$

\[
\frac{\Gamma \Rightarrow A \rightarrow B \downarrow}{\Gamma \Rightarrow B \downarrow} \frac{\Gamma \Rightarrow A \uparrow}{\rightarrow E}
\]

If $A \rightarrow B$ can be extracted from $\Gamma$, then $B$ can be extracted from $\Gamma$, provided that $A$ has a nd from $\Gamma$ (start a new $\uparrow$-expansion phase).
Rules for Normal Derivations

- **Axiom Id**

\[
\frac{A, \Gamma \Rightarrow A \downarrow}{\text{Id}}
\]

*A can be extracted from* \(A, \Gamma\) (*\(\downarrow\)-expansion starts by selecting an assumption)*.
Rules for Normal Derivations

- **Axiom Id**

\[
\frac{}{A, \Gamma \Rightarrow A \downarrow} \text{Id}
\]

*A can be extracted from \(A, \Gamma\)* (*↓*-expansion starts by selecting an assumption).

- **⊥ E**

\[
\frac{\Gamma \Rightarrow \bot \downarrow}{\Gamma \Rightarrow A \uparrow} \text{⊥E}
\]

*If \(\bot\) can be extracted from \(\Gamma\), then \(A\) has a nd from \(\Gamma\) (meet point).*

- **Coercion ↓↑**

\[
\frac{\Gamma \Rightarrow A \downarrow}{\Gamma \Rightarrow A \uparrow} \text{↓↑}
\]

*If \(A\) can be extracted from \(\Gamma\), then \(A\) has nd from \(\Gamma\) (meet point).*
The calculus **NJ**

\[
\begin{align*}
\text{Id} & \quad \frac{A, \Gamma \Rightarrow A \downarrow}{\Gamma \Rightarrow A \uparrow} \\
\wedge I & \quad \frac{\Gamma \Rightarrow A \uparrow}{\Gamma \Rightarrow A \wedge B \uparrow} \\
\vee I & \quad \frac{\Gamma \Rightarrow A \downarrow}{\Gamma \Rightarrow A \vee B \uparrow} \\
\rightarrow I & \quad \frac{\Gamma \Rightarrow B \uparrow}{\Gamma \Rightarrow A \rightarrow B \uparrow} \\
\wedge E & \quad \frac{\Gamma \Rightarrow A \uparrow}{\Gamma \Rightarrow A \downarrow \lor_A B \uparrow} \\
\vee E & \quad \frac{A, \Gamma \Rightarrow C \uparrow}{\Gamma \Rightarrow B \uparrow} \\
\rightarrow E & \quad \frac{\Gamma \Rightarrow A \uparrow}{\Gamma \Rightarrow B \downarrow}
\end{align*}
\]

**Theorem (Completeness of NJ)**

\(A \in \text{IPL} \text{ iff there exists a derivation of } \cdot \Rightarrow A \uparrow \text{ in NJ} \text{ (i.e., a nd of } A)\).
On normal forms

Derivations in **NJ** are *by definition* in normal form.

For instance, let us consider the following detour which introduces a maximal formula $p \rightarrow q$:

\[
\begin{align*}
\vdash & p, \Gamma \Rightarrow q \\
\Gamma & \Rightarrow p \rightarrow q \\
\Gamma & \Rightarrow p \\
\Gamma & \Rightarrow q \\
\end{align*}
\]

\[\vdash \Gamma \Rightarrow p \rightarrow q \Rightarrow I \quad \Gamma \Rightarrow p \Rightarrow E \]

This derivation cannot be replicated in **NJ**:

\[
\begin{align*}
\vdash & p, \Gamma \Rightarrow q \\
\Gamma & \Rightarrow p \rightarrow q \uparrow \\
\Gamma & \Rightarrow p \downarrow \\
\Gamma & \Rightarrow q \downarrow \\
\end{align*}
\]

\[\vdash \Gamma \Rightarrow p \rightarrow q \Rightarrow I \\
\Gamma \Rightarrow p \Rightarrow E \]

Application of $\rightarrow I_2$ is not allowed ($p \rightarrow q$ must be *extracted* from $\Gamma$).
To build non-normal derivations, we should add to $\textbf{NJ}$ the rule

$$
\frac{\Gamma \Rightarrow A^{\uparrow}}{\Gamma \Rightarrow A^{\downarrow}} \text{ converse of coercion}
$$

Example of use of $^{\uparrow\downarrow}$ to introduce a maximal $p \rightarrow q$

\[
\begin{array}{c}
\vdots \\
p, \Gamma \Rightarrow q^{\uparrow} \\
\Gamma \Rightarrow p \rightarrow q^{\uparrow} \quad \leadsto l_2 \\
\Gamma \Rightarrow p \rightarrow q^{\downarrow} \\
\Gamma \Rightarrow q^{\downarrow} \\
\Gamma \Rightarrow p^{\uparrow} \quad \leadsto E
\end{array}
\]

- The presence of coercion rule $^{\uparrow\downarrow}$, and the lack of $^{\uparrow\downarrow}$, is crucial to “coerce” derivations in normal form.
A proof-search example

Let us search for a proof of

\[ A \rightarrow q_1 \quad A = p_1 \land (p_1 \lor p_2 \rightarrow q_1 \land q_2) \]

Proof-search starts from the $\uparrow$-sequent

\[ \cdot \Rightarrow A \rightarrow q_1 \uparrow \]
A proof-search example

Let us search for a proof of

\[ A \rightarrow q_1 \quad A = p_1 \land (p_1 \lor p_2 \rightarrow q_1 \land q_2) \]

Proof-search starts from the \( \uparrow \)-sequent

\[ \cdot \Rightarrow A \rightarrow q_1 \uparrow \]

\( \uparrow \)-expansion

\[ \begin{array}{c}
A \Rightarrow q_1 \uparrow \\
\cdot \Rightarrow A \rightarrow q_1 \uparrow \end{array} \rightarrow I_2 \]
A proof-search example

Let us search for a nd of

\[ A \rightarrow q_1 \quad A = p_1 \land (p_1 \lor p_2 \rightarrow q_1 \land q_2) \]

Proof-search starts from the ↑-sequent

\[ \rightarrow A \rightarrow q_1 \uparrow \]

\[ \uparrow \text{-expansion} \]

\[ \begin{array}{c}
A \Rightarrow q_1 \uparrow \\
\rightarrow A \rightarrow q_1 \uparrow \rightarrow l_2
\end{array} \]

We need a ↓-expansion step to extract \( q_1 \) from \( A \):

\[ \begin{array}{c}
\vdots \\
A \Rightarrow q_1 \downarrow \\
A \Rightarrow q_1 \uparrow \\
\rightarrow A \rightarrow q_1 \uparrow \rightarrow l_2
\end{array} \]
To extract $q_1$ from $A$, we can build the following proof-tree with root-sequent $A \Rightarrow q_1 \downarrow$:

$$A = p_1 \land (p_1 \lor p_2 \rightarrow q_1 \land q_2)$$

$$\begin{array}{c}
A \Rightarrow A \downarrow \\
\text{Id} \\
A \Rightarrow p_1 \lor p_2 \rightarrow q_1 \land q_2 \downarrow \\
\wedge E \\
A \Rightarrow q_1 \land q_2 \downarrow \\
\wedge E \\
A \Rightarrow q_1 \downarrow
\end{array}$$

The proof-tree has the open-leaf $A \Rightarrow p_1 \lor p_2 \uparrow$, which must be $\uparrow$-expanded.
Gluing the above derivations by rule $\downarrow\uparrow$ we get:

$$A = p_1 \land (p_1 \lor p_2 \rightarrow q_1 \land q_2)$$

$$\frac{\frac{A \Rightarrow A\downarrow \quad \text{Id}}{A \Rightarrow p_1 \lor p_2 \rightarrow q_1 \land q_2\downarrow \quad \land E}}{A \Rightarrow p_1 \lor p_2 \Uparrow \quad \rightarrow E} \land E$$

$$\frac{A \Rightarrow q_1 \land q_2\downarrow}{A \Rightarrow q_1 \downarrow \quad \uparrow E}$$

$$\frac{A \Rightarrow q_1 \downarrow \quad \uparrow E}{A \Rightarrow q_1 \Uparrow \quad \rightarrow l_2}$$

$$\therefore A \Rightarrow q_1 \Uparrow \quad \rightarrow l_2$$
Gluing the above derivations by rule $\downarrow\uparrow$ we get:

$$A = p_1 \land (p_1 \lor p_2 \rightarrow q_1 \land q_2)$$

\[
\frac{A \Rightarrow A\downarrow}{A \Rightarrow p_1 \lor p_2 \rightarrow q_1 \land q_2 \downarrow} \quad \frac{\land E}{A \Rightarrow q_1 \land q_2 \downarrow} \quad \frac{\land E}{A \Rightarrow q_1 \downarrow} \quad \frac{\uparrow \uparrow}{A \Rightarrow q_1 \uparrow} \quad \frac{\Rightarrow \rightarrow l_2}{
\Rightarrow A \rightarrow q_1 \uparrow}
\]

We have to $\uparrow$-expand $A \Rightarrow p_1 \lor p_2 \uparrow$
A proof-search example

\[ A \Rightarrow p_1 \uparrow \]

\[
\frac{A \Rightarrow p_1 \uparrow}{A \Rightarrow p_1 \lor p_2 \uparrow} \lor I
\]
A proof-search example

- ↑-expansion

\[ A \Rightarrow p_1 \uparrow \]
\[ \frac{}{A \Rightarrow p_1 \lor p_2 \uparrow} \lor I \]

We get

\[ A = p_1 \land (p_1 \lor p_2 \rightarrow q_1 \land q_2) \]

\[ \frac{A \Rightarrow A \downarrow}{A \Rightarrow p_1 \lor p_2 \rightarrow q_1 \land q_2 \downarrow} \land E \]
\[ \frac{A \Rightarrow p_1 \uparrow}{A \Rightarrow p_1 \lor p_2 \uparrow} \lor I \]
\[ \frac{A \Rightarrow q_1 \land q_2 \downarrow}{A \Rightarrow q_1 \downarrow} \land E \]
\[ \frac{A \Rightarrow q_1 \downarrow}{A \Rightarrow q_1 \uparrow} \uparrow \]
\[ \frac{}{\Rightarrow A \rightarrow q_1 \uparrow} \rightarrow I_2 \]
A proof-search example

- ↑-expansion

\[
\frac{A \Rightarrow p_1 \uparrow}{A \Rightarrow p_1 \lor p_2 \uparrow} \lor I
\]

We get

\[
A = p_1 \land (p_1 \lor p_2 \rightarrow q_1 \land q_2)
\]

\[
\frac{A \Rightarrow A\downarrow}{A \Rightarrow p_1 \lor p_2 \rightarrow q_1 \land q_2 \downarrow} \land E
\]

\[
\frac{A \Rightarrow q_1 \land q_2 \downarrow}{A \Rightarrow q_1 \downarrow} \land E
\]

\[
\frac{A \Rightarrow q_1 \downarrow}{A \Rightarrow q_1 \uparrow} \uparrow
\]

\[
\therefore \Rightarrow A ightarrow q_1 \uparrow \rightarrow I_2
\]

We close the leaf $A \Rightarrow p_1 \uparrow$ by extracting $p_1$ from $A$. 
A proof-search example

\[\text{\(-expansion\)}\]

\[
\frac{A \Rightarrow A\downarrow}{A \Rightarrow p_1\downarrow} \quad \text{Id} \quad \land E
\]

\[A = p_1 \land (p_1 \lor p_2 \rightarrow q_1 \land q_2)\]
A proof-search example

- \( \downarrow \)-expansion

\[
\frac{A \Rightarrow A \downarrow}{A \Rightarrow p_1 \downarrow} \quad \text{Id} \quad \wedge E \\
A = p_1 \land (p_1 \lor p_2 \rightarrow q_1 \land q_2)
\]

By applying \( \downarrow \uparrow \), we get the closed derivation:

\[
\frac{A \Rightarrow A \downarrow}{A \Rightarrow p_1 \downarrow} \quad \text{Id} \quad \wedge E \\
\frac{A \Rightarrow p_1 \downarrow}{A \Rightarrow p_1 \uparrow} \\
\frac{A \Rightarrow p_1 \lor p_2 \rightarrow q_1 \land q_2 \downarrow}{A \Rightarrow p_1 \lor p_2 \uparrow} \quad \wedge E \\
\frac{A \Rightarrow q_1 \land q_2 \downarrow}{A \Rightarrow q_1 \downarrow} \\
\frac{A \Rightarrow q_1 \downarrow}{A \Rightarrow q_1 \uparrow} \quad \uparrow \\
\frac{\cdot \Rightarrow A \rightarrow q_1 \uparrow}{\Rightarrow I_2} \quad \rightarrow E
\]
Some problems

- The proof-search space is very huge.

Recall that a ↓-search phase starts selecting a formula from the context (don’t know non-determinism) and decomposing it by applying elimination rules.

To keep the contexts small and minimize the possible choices, we have to avoid as much as possible the application of the context-extending rules:

\[
\begin{align*}
A, \Gamma \Rightarrow B & \uparrow \\
\Gamma \Rightarrow A \rightarrow B & \uparrow \\
\end{align*}
\rightarrow \text{I}_2
\]

\[
\begin{align*}
\Gamma \Rightarrow A \lor B & \downarrow \\
A, \Gamma \Rightarrow C & \uparrow \\
B, \Gamma \Rightarrow C & \uparrow \\
\Gamma \Rightarrow C & \uparrow \\
\end{align*}
\lor \text{E}
\]
Some problems

- Too many non-deterministic choices.

For instance, in $\uparrow$-expansion of

$$\Gamma \Rightarrow C \land D \uparrow$$

we have three choices:

\[
\begin{align*}
\vdots & \quad \vdots \\
\Gamma \Rightarrow C \uparrow & \quad \Gamma \Rightarrow D \uparrow \\
\hline
\Gamma \Rightarrow C \land D \uparrow & \quad \land I \quad (1) \\
\end{align*}
\]

\[
\begin{align*}
\vdots & \quad \vdots \\
\Gamma \Rightarrow C \land D \downarrow & \quad \Gamma \Rightarrow C \land D \uparrow \\
\hline
\Gamma \Rightarrow C \land D \uparrow & \quad \uparrow \quad (2) \\
\end{align*}
\]

\[
\begin{align*}
\vdots & \quad \vdots \\
\Gamma \Rightarrow A \lor B \downarrow & \quad A, \Gamma \Rightarrow C \land D \uparrow & \quad B, \Gamma \Rightarrow C \land D \uparrow \\
\hline
\Gamma \Rightarrow C \land D \uparrow & \quad \lor E \quad (3) \\
\end{align*}
\]
Our proposal

- We decorate the $\uparrow$-arrow with one of the labels $b$, $u$.

  We have now three kinds of sequents:

  $\Gamma \Rightarrow A\downarrow$ \hspace{1cm} $\Gamma \Rightarrow A\uparrow^b$ \hspace{1cm} $\Gamma \Rightarrow A\uparrow^u$

  Label $b$ blocks some rule applications during $\uparrow$-expansion, and this decreases the degree of non-determinism.

- We impose additional constraints on rule applications.

\[ \text{NJ} + \text{labels } b, u + \text{constraints} = \text{Nbu} \]

Labels $b$, $u$ have been introduced to get a terminating proof-search procedure for the intuitionistic sequent calculus:

- M. Ferrari, C. Fiorentini, and G. Fiorino. A terminating evaluation-driven variant of G3i. TABLEAUX 2013
In the conclusion, we constraint the form of the right formula and we add a label $l \in \{b, u\}$ to the $\uparrow$-arrow.
From NJ to Nbu

Rules of Nbu

- \( \land I \)
  - \( \ast \) NJ
    \[
    \frac{\Gamma \Rightarrow A^\uparrow \quad \Gamma \Rightarrow B^\uparrow}{\Gamma \Rightarrow A \land B^\uparrow} \land I
    \]
  - \( \ast \) Nbu
    \[
    \frac{\Gamma \Rightarrow A^{\uparrow/l} \quad \Gamma \Rightarrow B^{\uparrow/l}}{\Gamma \Rightarrow A \land B^{\uparrow/l}} \land I \quad l \in \{b, u\}
    \]

Note that the label of the \( \land \)-arrow in the premise is set to \( b \).
From NJ to Nbu

Rules of Nbu

- $\land I$
  - * NJ
    \[
    \frac{\Gamma \Rightarrow A^\uparrow \quad \Gamma \Rightarrow B^\uparrow}{\Gamma \Rightarrow A \land B^\uparrow} \quad \land I
    \]
  - * Nbu
    \[
    \frac{\Gamma \Rightarrow A^l \quad \Gamma \Rightarrow B^l}{\Gamma \Rightarrow A \land B^l} \quad \land I \quad l \in \{b, u\}
    \]

- $\lor I$
  - * NJ
    \[
    \frac{\Gamma \Rightarrow A_k^\uparrow}{\Gamma \Rightarrow A_0 \lor A_1^\uparrow} \quad \lor I
    \]
  - * Nbu
    \[
    \frac{\Gamma \Rightarrow A_k^{\uparrow^b}}{\Gamma \Rightarrow A_0 \lor A_1^l} \quad \lor I \quad l \in \{b, u\}
    \]

Note that the label of the $\uparrow$-arrow in the premise is set to $b$. 
From **NJ** to **Nbu**

- $\lor E$
  - **NJ**
    
    $$
    \Gamma \Rightarrow A \lor B \\
    A, \Gamma \Rightarrow C \\
    B, \Gamma \Rightarrow C \\
    \Gamma \Rightarrow C
    $$
    
    This is the most problematic rule in proof-search.
  - **Nbu**
    
    $$
    \Gamma \Rightarrow A \lor B \\
    A, \Gamma \Rightarrow D^u \\
    B, \Gamma \Rightarrow D^u \\
    \Gamma \Rightarrow D^u
    $$

  - $\checkmark$ Condition on $D$:
    
    $$
    D \in \mathcal{V} \cup \{\bot\} \quad \text{or} \quad D = D_0 \lor D_1
    $$
  - $\checkmark$ Conditions on $A \lor B$:
    
    $$
    A \notin \Gamma \\
    B \notin \Gamma
    $$
  - $\checkmark$ The label of the $\uparrow$-arrow in the conclusion must be $u$

These constraints strongly reduce the non-determinism.
From $NJ$ to $Nbu$

- $\rightarrow I_1$ and $\rightarrow I_2$
  - $NJ$
    
    \[
    \begin{align*}
    \Gamma \Rightarrow B^\uparrow & \quad \rightarrow I_1 \\
    \Gamma \Rightarrow A \rightarrow B^\uparrow & \quad \rightarrow I_1 \\
    \end{align*}
    \]

    Non-deterministic choice between the two rules
  - $Nbu$
    
    \[
    \begin{align*}
    \Gamma \Rightarrow B^\uparrow & \quad \rightarrow I_1 \\
    \Gamma \Rightarrow A \rightarrow B^\uparrow & \quad \rightarrow I_1 \\
    \end{align*}
    \]

    \[
    \begin{align*}
    \Gamma \Rightarrow A \Rightarrow B^\uparrow & \quad \rightarrow I_1 \\
    \Gamma \Rightarrow A \Rightarrow B^\uparrow & \quad \rightarrow I_1 \\
    \end{align*}
    \]

    \[
    \begin{align*}
    \Gamma \Rightarrow B^\uparrow & \quad \rightarrow I_2 \\
    \Gamma \Rightarrow A \rightarrow B^\uparrow & \quad \rightarrow I_2 \\
    \end{align*}
    \]

    \[
    \begin{align*}
    \Gamma \Rightarrow A \Rightarrow B^\uparrow & \quad \rightarrow I_2 \\
    \Gamma \Rightarrow A \Rightarrow B^\uparrow & \quad \rightarrow I_2 \\
    \end{align*}
    \]

    $A \in \Gamma$

    $A \notin \Gamma$

- $\sqrt{\text{The choice between the two version is deterministic.}}$
- $\sqrt{\text{In bottom-up application, } \rightarrow I_1 \text{ preserves the label, whereas } \rightarrow I_2 \text{ always sets the label to } u.}$
The calculus **Nbu**

\[
\begin{align*}
A, \Gamma \Rightarrow A \downarrow & \quad \text{Id} \\
\Gamma \Rightarrow A \uparrow' & \quad \Gamma \Rightarrow p \downarrow \quad \Uparrow \\
\Gamma \Rightarrow \bot \downarrow & \quad \down E \\
\Gamma \Rightarrow A \uparrow' & \quad \Gamma \Rightarrow F \uparrow' \quad \Box E \\
\Gamma \Rightarrow A \uparrow' & \quad \Gamma \Rightarrow B \uparrow' \quad \land I \\
\Gamma \Rightarrow \land E & \quad \Gamma \Rightarrow A_0 \land A_1 \downarrow \\
\Gamma \Rightarrow A \uparrow' & \quad \Gamma \Rightarrow \land E \\
\Gamma \Rightarrow A \lor B \downarrow & \quad \land E \quad \Gamma \Rightarrow A_0 \lor A_1 \uparrow' \\
\Lambda, \Gamma \Rightarrow D \uparrow' & \quad \lor I \\
\Gamma \Rightarrow D \uparrow' & \quad \lor E \\
\Gamma \Rightarrow B \uparrow' & \quad \rightarrow I_1 \\
\Lambda, \Gamma \Rightarrow B \uparrow' & \quad \rightarrow I_2 \\
\Gamma \Rightarrow A \rightarrow B \downarrow & \quad \rightarrow E \\
\Gamma \Rightarrow A \rightarrow B \uparrow' & \quad \Gamma \Rightarrow A \uparrow' \\
\end{align*}
\]

\[\begin{align*}
p & \in \mathcal{V}, \quad F \in \mathcal{V} \cup \{\bot\}, \quad D \in \mathcal{V} \cup \{\bot\} \text{ or } D = D_0 \lor D_1
\end{align*}\]

**Theorem (Completeness of Nbu)**

\[A \in \text{IPL} \iff \text{there exists a derivation of } \cdot \Rightarrow A \uparrow' \text{ in Nbu}\]
The calculus \textbf{Nbu}

- There is a trivial translation from \textbf{Nbu} into \textbf{NJ}:

  \[
  \begin{array}{ccc}
  \text{Nbu} & \rightarrow & \text{NJ} \\
  \Gamma \Rightarrow A^\uparrow & \ldots \text{ erase the labels } \ldots & \Gamma \Rightarrow A^\uparrow
  \end{array}
  \]

- The definition of the converse map is more demanding. Indeed, before labeling an \textbf{NJ}-derivation, some preliminary non-trivial reduction steps could be required.

- Derivations in \textbf{Nbu} are in \textit{long normal form}, since the switch from \(\uparrow\)-expansion and \(\downarrow\)-expansion (rules \(\uparrow\), \(\downarrow\) \textit{E}) is marked by an atomic formula.
Example

The **Nbu**-derivation

$$
\begin{align*}
p \lor q & \Rightarrow p \lor q \\
p \lor q & \Rightarrow p \lor q \\
\downarrow & \\
\downarrow & \\
\cdot & \Rightarrow p \lor q \rightarrow p \lor q \\
\downarrow & \\
\rightarrow l_2
\end{align*}
$$

cannot be labelled (rule $\downarrow \uparrow$ is applied to non atomic formula).

To get a **Nbu**-derivation, we need some reduction steps, so that $\downarrow \uparrow$ is applied to an atomic formula (long normal form):

$$
\begin{align*}
p \lor q & \Rightarrow p \lor q \\
p \lor q & \Rightarrow p \lor q \\
\downarrow & \\
\downarrow & \\
\cdot & \Rightarrow p \lor q \rightarrow p \lor q \\
\downarrow & \\
\rightarrow l_2
\end{align*}
$$
Note that label $b$ introduce a sort of *focus* on the right. Indeed, in $\uparrow$-expansion of $\Gamma \Rightarrow A^{\uparrow b}$, with $A$ non-atomic, we are forced to apply an $I$-rule to decompose $A$.

\[
\frac{A \rightarrow p, \Gamma \Rightarrow A \rightarrow p \downarrow }{ \text{Id} } \frac{A \rightarrow p, \Gamma \Rightarrow A^{\uparrow b} }{ \rightarrow E } \]

\[
\frac{A \rightarrow p, \Gamma \Rightarrow p \downarrow }{ \uparrow } \frac{A \rightarrow p, \Gamma \Rightarrow p^{\uparrow u} }{ \uparrow }
\]

In (⋆), we have to decompose $A$ as much as possible; application of $\lor E$ is not allowed.
Proof-search

- We can define a goal-oriented proof-search strategy for \textbf{Nbu}, where all the rules are applied bottom-up.

- \textbf{Nbu}-trees can contain loops:

\[
\Gamma = p_1, \ p_1 \rightarrow p_2, \ p_2 \rightarrow p_1
\]

\[
\Gamma \Rightarrow p_1 \downarrow \quad \Gamma \Rightarrow p_1 \uparrow^b \quad \Gamma \Rightarrow p_2 \downarrow \quad \Gamma \Rightarrow p_2 \uparrow^b
\]

\[
\Gamma \Rightarrow p_1 \downarrow \quad \Gamma \Rightarrow p_1 \uparrow^b \quad \Gamma \Rightarrow p_2 \downarrow \quad \Gamma \Rightarrow p_2 \uparrow^b
\]

\[
\Gamma \Rightarrow p_1 \downarrow \quad \Gamma \Rightarrow p_1 \uparrow^u
\]

To get termination, we make use of \textit{history sets} (sets of atoms).
Evaluations

We have observed that in proof-search it is desirable to extend a context as little as possible, so to narrow the search space.

In \textbf{Nbu}, the context-extending rules $\rightarrow I_2$ and $\lor E$ can only be applied if the formulas to be added are not in the context.

\[
\frac{A, \Gamma \Rightarrow B \uparrow^u}{\Gamma \Rightarrow A \rightarrow B \uparrow^l} \quad \rightarrow I_2 \quad \text{and} \quad A \not\in \Gamma
\]

\[
\frac{\Gamma \Rightarrow A \lor B \downarrow}{A, \Gamma \Rightarrow D \uparrow^u} \quad \frac{\Gamma \Rightarrow D \uparrow^u}{A, \Gamma \Rightarrow B \uparrow^u} \quad \frac{B, \Gamma \Rightarrow D \uparrow^u}{\lor E} \quad \text{and} \quad A, B \not\in \Gamma
\]
Evaluations

We have observed that in proof-search it is desirable to extend a context as little as possible, so to narrow the search space.

In **Nbu**, the context-extending rules \( \rightarrow I_2 \) and \( \lor E \) can only be applied if the formulas to be added are not in the context.

\[
\begin{align*}
\frac{A, \Gamma \Rightarrow B \uparrow^u}{\Gamma \Rightarrow A \rightarrow B \uparrow^l} & \rightarrow I_2 \\
\frac{\Gamma \Rightarrow A \lor B \downarrow}{A, \Gamma \Rightarrow D \uparrow^u} & \lor E \\
\frac{B, \Gamma \Rightarrow D \uparrow^u}{\Gamma \Rightarrow D \uparrow^u}
\end{align*}
\]

\( A \notin \Gamma \) \hspace{1cm} \( A, B \notin \Gamma \)

We introduce a more general condition which prevents the application of \( \rightarrow I_2 \) and \( \lor E \) whenever the formulas \( A, B \) do not add “significant information” to the context \( \Gamma \).

We formalize this by the notion of evaluation relation.
Evaluations

Definition (Evaluation)

An evaluation is any decidable relation $\models_\theta$ between finite multisets of formulas $\Gamma$ and formulas $A$ such that:

1. $A \in \Gamma$ implies $\Gamma \models_\theta A$
2. $\Gamma \models_\theta A$ and $\text{Nbu} \vdash_d A, \Gamma \Rightarrow B \uparrow^l$ imply $\text{Nbu} \vdash_d \Gamma \Rightarrow B \uparrow^l$

(a sort of cut rule).

$\text{Nbu} \vdash_d \Gamma \Rightarrow A \uparrow^l \sim \text{there is an Nbu-derivation of } \Gamma \Rightarrow A \uparrow^l$
with depth at most $d$

Intuitively, $\Gamma \models_\theta A$ means:

the information conveyed by $A$ is already available in $\Gamma$

We avoid to extend a context $\Gamma$ with $A$ whenever $\Gamma \models_\theta A$. 
The calculus $\text{Nbu}_\theta$

Let $\models_\theta$ be an evaluation

$$\text{Nbu}_\theta = \text{Nbu}$$

with context-extending rules modified as follows:

$$\Gamma \Rightarrow \mathcal{B} \uparrow^l \quad \rightarrow \quad I_1 \quad \Gamma \models_\theta A \quad \text{no need to add } A$$

$$\frac{\Gamma \Rightarrow \mathcal{A} \Rightarrow \mathcal{B} \uparrow^l}{\Gamma \models_\theta \mathcal{A}} \quad \rightarrow \quad I_2 \quad \Gamma \not\models_\theta A$$

$$\Gamma \models_\theta \mathcal{A} \lor \mathcal{B} \downarrow \quad \frac{\mathcal{A}, \Gamma \Rightarrow \mathcal{D} \uparrow^u \quad \mathcal{B}, \Gamma \Rightarrow \mathcal{D} \uparrow^u}{\mathcal{D} \in \mathcal{V} \cup \{\bot\} \text{ or } \mathcal{D} = \mathcal{D}_0 \lor \mathcal{D}_1} \quad \rightarrow \quad \vee E \quad \Gamma \not\models_\theta \mathcal{A} \quad \Gamma \not\models_\theta \mathcal{B}$$
Examples of evaluation relations:

- **Minimum evaluation relation** \( \Vdash_{\theta_{\text{min}}} \)

  \[ \Gamma \Vdash_{\theta_{\text{min}}} A \iff A \in \Gamma \quad \text{membership} \]

Note that \( \text{Nbu}_{\theta_{\text{min}}} \equiv \text{Nbu} \)
Examples of evaluation relations:

- **Minimum evaluation relation** $\vdash_{\theta_{\text{min}}}$
  
  $$\Gamma \vdash_{\theta_{\text{min}}} A \iff A \in \Gamma \quad \text{membership}$$

  Note that $\text{Nbu}_{\theta_{\text{min}}} \equiv \text{Nbu}$

- **Cover relation** $\vdash_{\theta_{\text{cov}}}$
  
  $$\Gamma \vdash_{\theta_{\text{cov}}} E \iff E \text{ has the form}$$

  $$E ::= G \mid E \land E \mid E \lor A \mid A \lor E \mid A \rightarrow E \quad G \in \Gamma \quad A \text{ any formula}$$

  The cover relation has been introduced in


  to study the depth of derivations in sequent calculi.
The calculus $\textbf{Nbu}_\theta$

- The stronger the evaluation relation is, the better is the gain in proof-search.

For instance, let $\Gamma = p, q$

\[ \Gamma \not\models_{\theta_{\text{min}}} p \land q \quad (p \land q \not\in \Gamma) \]
\[ \Gamma \models_{\theta_{\text{cov}}} p \land q \]

Thus, $\models_{\theta_{\text{cov}}}$ is better than $\models_{\theta_{\text{min}}}$.

- Can we define evaluation relations stronger than cover?

Can *semantics* help?

We remark that to prove our results (completeness of $\textbf{Nbu}$ and of the related proof-search strategy) we have only used *syntax* (basically, rule permutations).
Concluding remarks

- We have presented Nbu, a variant of the Natural Deduction calculus for IPL which allows goal-oriented proof-search.
- Nbu internalizes some aspects of such proof-search procedure using three mechanisms:
  1. orientation of sequents by labelled arrows $\uparrow^u$, $\uparrow^b$ and $\downarrow$;
  2. side conditions involving rules $\lor E$, $\rightarrow\_1$ and $\rightarrow\_2$;
  3. restrictions on the conclusion of the rules $\downarrow\uparrow$, $\bot E$ and $\lor E$
- The sequent image of Nbu is a labelled variant of Herbelin’s calculus [CSL,94] (the sequent calculus isomorphic to Natural Deduction).
- We have implemented the proof-search procedure in JTabWb (a Java framework for developing provers):
  
  http://www.dista.uninsubria.it/~ferram/

We have performed some experiments and the results are competitive with those of the state-of-the-art provers for IPL.