Modal operators for meet-complemented lattices

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Skolem's expansion

In 1919 Skolem[1] considers an expansion of lattices with both the meet and the join relative complements. The latter is the binary operation

$$a - b = min\{x : a \leq b \vee x\}.$$

As a particular case, we have the join complement, that is,

$$1 - b = Db = min\{x : b \vee x = 1\}.$$

Note that $-$ is the dual of the relative meet complement (intuitionistic conditional, from a logical point of view) and that D is the dual of intuitionistic negation.

[1] T. Skolem. Untersuchungen über die Axiome des Klassenkalküls und über Produktations- und Summationsprobleme, welche gewisse Klassen von Aussagen betreffen, *Skrifter utgit av Videnskabsselskapet i Kristiania*, **3**, pp. 1-37, 1919.

Moisil's modal operators

With no mention of Skolem, in 1942 Moisil [1] considers a bi-intuitionistic logic, where, apart from the usual connectives for conjunction, disjunction and the conditional, he has a connective for the dual of the conditional. In that context, he defines both intuitionistic negation $\neg$ and its dual D.

He considers DD and $\neg\neg$ as operators for necessity and possibility, respectively.

However, for instance, $DD(\alpha \to \beta) \nvdash DD\alpha \to DD\beta$.

He observs that $\neg\neg\alpha \vdash D\neg\alpha$ and that $\neg D\alpha \vdash DD\alpha$,
but does not study $D\neg$ and $\neg D$ as modal connectives.

[1] G. Moisil. Logique modale, *Disquisitiones math. et phys.*, **II**:1, 3-98, 1942.

Rauszer's approach

In 1974 Rauszer [1] considers lattices expanded with both the meet and the join relative complements, where, as we have seen, both $\neg$ and D are easily definable.

She neither mentions Skolem nor Moisil. Also, she does not seem to be interested in necessity or possibility.

Her logic has two rules, *modus ponens* and $\varphi/\neg D\varphi$.

She proves soundness, completeness and a variant of the Deduction Theorem:

if $\Gamma, \varphi \vdash \psi$, then $\Gamma \vdash (\neg D)^n\varphi \to \psi$, for some natural number n.

[1] C. Rauszer. Semi-Boolean algebras and their applications to intuitionistic logic with dual operations, *Fundamenta Mathematicae*, **83**, 219-235, 1974.

López Escobar's modal operators

In 1985 López-Escobar[1] studies $\neg D$ and $D\neg$ as modal connectives of necessity and possibility, respectively.

He works in the context of Beth structures.

He neither mentions Skolem nor Moisil.
However, many papers by Rauszer appear in the list of references.

[1] K. López-Escobar. On intuitionistic sentential connectives I, *Revista colombiana de matemáticas*, **XIX**, 117-130, 1985.

Some more references

There many other papers on $\neg D$ and $D\neg$, some treating them as necessity and possibility, respectively. [1,2,3,4]

[1] J. Varlet. A regular variety of type $< 2, 2, 1, 1, 0, 0 >$, *Algebra universalis*, **2**, 1, 218-223, 1972.

[2] T. Katriňák. Subdirectly irreducible distributive double p-algebras, *Algebra universalis*, **10**, 195-219, 1980.

[3] H. P. Sankappanavar. Heyting algebras with dual pseudocomplementation, *Pacific Journal of Mathematics*, **117**, 2, 405-415, 1985.

[4] G. E. Reyes and H. Zolfaghari. Bi-Heyting algebras, toposes and modalities, *Journal of Philosophical Logic*, **25**, 25-43, 1996.

Another operation

Another operation we will have ocassion to mention in the context of a meet-complemented lattice $\mathbf{A}$, is the greatest boolean below a given element $a \in A$[1,2]:

$$Ba = max\{b \in A : b \leq a \text{ and } b \vee \neg b = 1\}.$$

It was suggested to me by Franco Montagna.

[1] G. E. Reyes and H. Zolfaghari. Bi-Heyting algebras, toposes and modalities, *Journal of Philosophical Logic*, **25**, 25-43, 1996.

[2] R. C. Ertola-Biraben, F. Esteva, and L. Godo. Expanding FL_{ew} with a boolean connective, *Soft Computing*, 2016.

Our work

In this talk we introduce modal operators of necessity and possibility that are similar to the mentioned $\neg D$ and $D\neg$, respectively.

Our operators are defined in the context of a (not necessarily distributive) meet-complemented lattice, that is, the usual algebraic counterpart of the connectives of conjunction, disjunction, and negation in intuitionistic logic. We also consider the distributive extension and the expansion with the relative meet complement, that is, Heyting algebras.

Our operators of necessity and possibility are defined as maximum and minimum, respectively. So, when they exist, there cannot be two different operations satisfying their definition.

Meet-complemented lattices

As well known, a meet-complemented lattice $\mathbf{A}$ is a lattice such that there exists

$$\neg a = max\{b \in A : a \wedge b \leq c, \text{ for all } c \in A\}, \text{ for any } a \in A.$$

It is equivalent to state both

($\neg$E) $a \wedge \neg a \leq c$, for all $a, c \in A$ and

($\neg$I) for any $a, b \in A$, if $a \wedge b \leq c$, for all $c \in A$, then $b \leq \neg a$.

We use $\mathbb{ML}$ for the class of meet complemented lattices.

As very well known, the class $\mathbb{ML}$ is an equational class.

As in the context of a lattice the existence of $\neg$ implies the existence of both bottom $\perp$ and top $\top$, in what follows we are allowed to use them.

Adding necessity

A meet complemented lattice with necessity is a meet complemented lattice $\mathbf{A}$ such that there exists

$$\Box a = max\{b \in A : a \vee \neg b = \top\}, \text{ for any } a \in A.$$

It is equivalent to state both

($\Box$E) $a \vee \neg\Box a = \top$ and

($\Box$I) if $a \vee \neg b = \top$, then $b \leq \Box a$.

We have Monotonicity: if $a \leq b$, then $\Box a \leq \Box b$.

It follows that $\Box(a \wedge b) \leq \Box a \wedge \Box b$. However, we are ashamed we have not been able to decide the reciprocal!

We use $\mathbb{ML}^{\Box}$ for the class of meet complemented lattices with necessity.

An equational class

$\mathbb{ML}^{\Box}$ is an equational class adding to any set of identities for $\mathbb{ML}$ the following (independent) ones:

(□E) $x \vee \neg\Box x \approx 1$,

(□I1) $\Box 1 \approx 1$, and

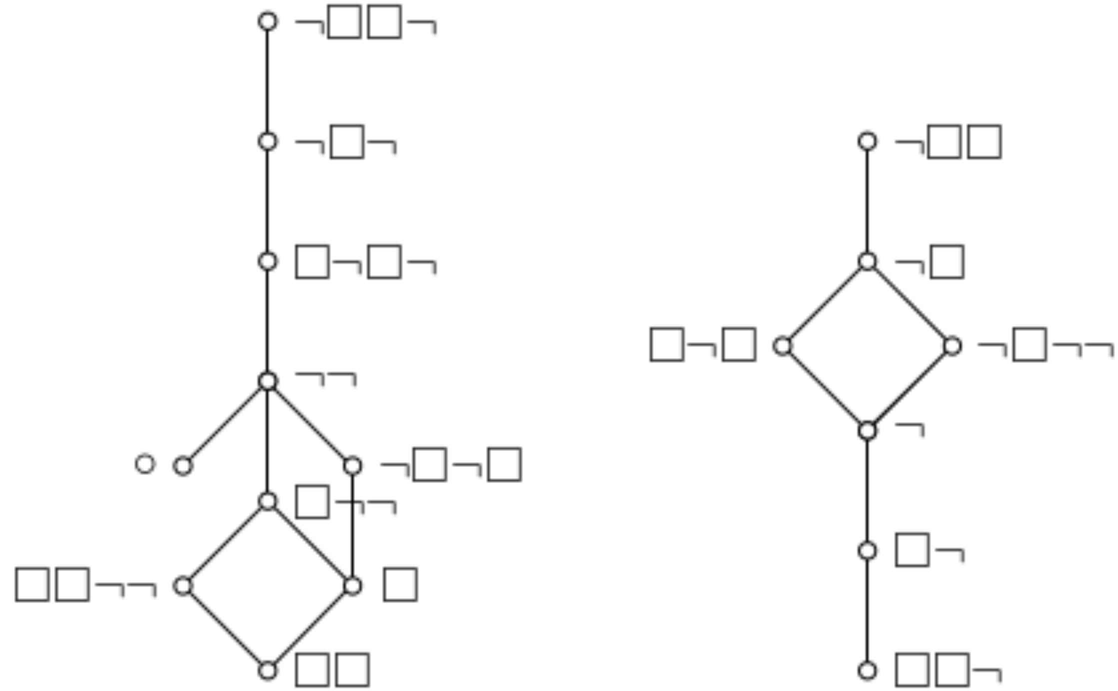
(□I2) $\Box(x \vee \neg y) \wedge y \approx \Box x \wedge y$.

Modalities in $\mathbb{ML}^{\Box}$

We will be interested in modalities, that is, finite combinations of unary operators, at the present stage, $\neg$ and $\Box$.

We will use $\circ$ for the identity modality.

We distinguish between *positive* and *negative* modalities.



Positive and negative modalities of ¬ and □ for up to two boxes

Adding possibility

A meet-complemented lattice with possibility is a meet-complemented lattice $\mathbf{A}$ such that there exists

$\Diamond a = min\{b \in A : \neg a \vee b = \top\}$, for any $a \in A$.

It is equivalent to state both

($\Diamond$I) $\neg a \vee \Diamond a = \top$ and

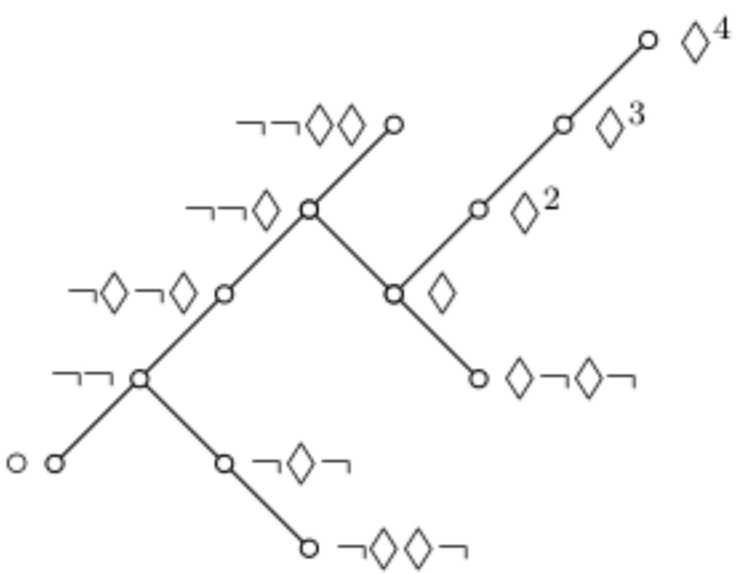
($\Diamond$E) if $\neg a \vee b = \top$, then $\Diamond a \leq b$.

We have Monotonicity: if $a \leq b$, then $\Diamond a \leq \Diamond b$.

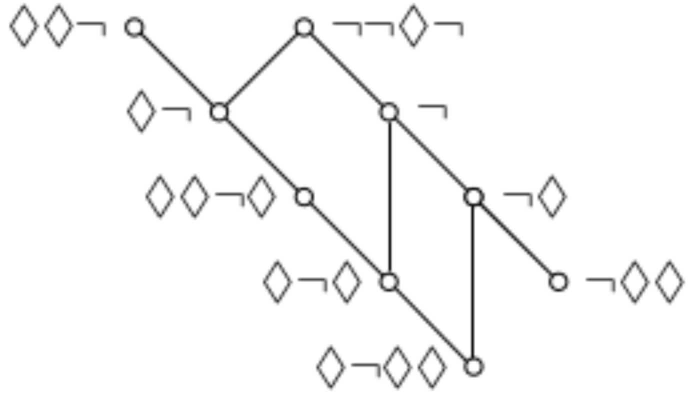
It follows that $\Diamond a \vee \Diamond b \leq \Diamond(a \vee b)$. However, the reciprocal does not hold.

We use the notation $\mathbb{ML}^{\Diamond}$ for the class of meet complemented lattices with possibility.

$\mathbb{ML}^{\Diamond}$ is *not* an equational class.



Positive modalities for $\neg$ and $\Diamond$ with maximum length 4



Negative modalities for ¬ and ◊ with maximum length 4

Comparing $\Box$ and $\Diamond$ with other operators

Let $\mathbf{A} \in \mathbb{ML}$. If D exists in A, then $\Box$ also exists in A with $\Box = \neg D$.

So, If both D and $\Box$ exist in a meet-complemented lattice, then $\Box = \neg D$.

Let $A \in \mathbb{ML}^{\Box}$. If B exists in A, then $B \leq \Box$.

The reciprocal is not the case.

Let $\mathbf{A} \in \mathbb{ML}^{\Diamond}$. If D exists in A, then $\Diamond$ also exists in A with $\Diamond = D\neg$.

So, If both D and $\Diamond$ exist in a meet-complemented lattice, then $\Diamond = D\neg$.

Necessity and possibility together

Let us now consider meet complemented lattices with necessity and possibility.

We use the notation $\mathbb{ML}^{\Box\Diamond}$ for the corresponding class.

Some properties of $\mathbb{ML}^{\Box\Diamond}$ are the following:

(B1) $\circ \leq \Box\Diamond$,

(B2) $\Diamond\Box \leq \circ$.

(A) $\Diamond a \leq b$ iff $a \leq \Box b$,

$\Diamond\Box\Diamond = \Diamond$ and $\Box\Diamond\Box = \Box$.

Notation: Above we use “B” for the schemas corresponding to the modal logic B and “A” for adjunction.

Some other facts about $\mathbb{ML}^{\Box\Diamond}$

$\mathbb{ML}^{\Box\Diamond}$ is an equational class adding to a set of identities for $\mathbb{ML}^{\Box}$ the following ones:

($\Diamond$I) $x \vee \neg\Diamond x \approx 1$,

($\Diamond$E1) $\Diamond x \preccurlyeq \Diamond(x \vee y)$, and

($\Diamond$E2) $\Diamond\Box x \preccurlyeq x$.

We have $\Box(a \wedge b) = \Box a \wedge \Box b$.

We also have that $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$, which does not hold for $\mathbb{ML}^{\Diamond}$. So, $\mathbb{ML}^{\Box\Diamond}$ is not a conservative expansion of $\mathbb{ML}^{\Diamond}$.

The distributive extension

Let us now consider meet-complemented *distributive* lattices with necessity and possibility.

We use the notation $\mathbb{ML}_d^{\Box\Diamond}$ for the corresponding class.

Operations $\Box$ and $\Diamond$ exist in every finite meet-complemented distributive lattice.

There is an (infinite) meet-complemented distributive lattice where $\Box$ does not exist (Franco Montagna).

There is also an (infinite) meet-complemented distributive lattice where $\Diamond$ does not exist.

We have both $\Box \leq \circ$ and $\circ \leq \Diamond$.

Using representation theory, it may be seen that there are infinite modalities: $\circ$, $\Box$, $\Box\Box$, etc.

The S-extension

We define the S-extension by adding to $\mathbb{ML}^{\Box\Diamond}$ the algebraic version of the S4-schema:

(S) $\Box \preccurlyeq \Box\Box$.

We use the notation $\mathbb{ML}_S^{\Box\Diamond}$ for the class of meet-complemented lattices expanded with both $\Box$ and $\Diamond$ that satisfy (S).

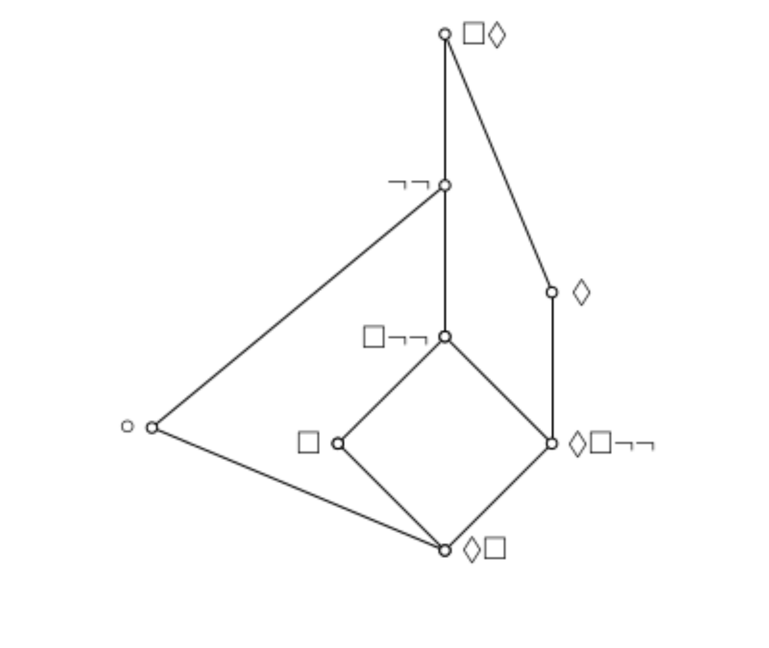
It is equivalent to extend with any of the following

$\Diamond\Diamond \leq \Diamond$,

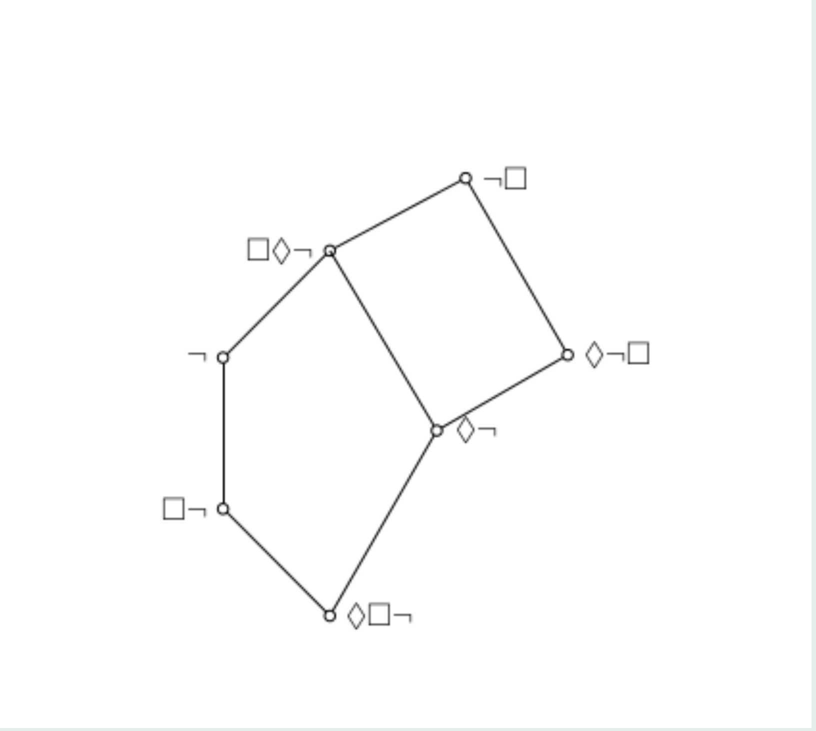
$\Box a \vee \neg\Box a = 1, \qquad \Diamond a \vee \neg\Diamond a = 1,$

$\Diamond \leq \Box\Diamond, \qquad \Diamond\Box \leq \Box.$

Somehow surprisingly not having distributivity, we have finite modalities.



Positive modalities for the S-extension



Negative modalities for the S-extension

The distributive S-extension

Let us now extend with both distributivity and the S-schema.

We use the notation $\mathbb{ML}_{dS}^{\Box\Diamond}$ for the class of meet-complemented distributive lattices expanded with both $\Box$ and $\Diamond$ satisfying S.

In $\mathbb{ML}_{dS}^{\Box\Diamond}$ possibility turns out to be definable: $\Diamond = \neg\Box\neg$.

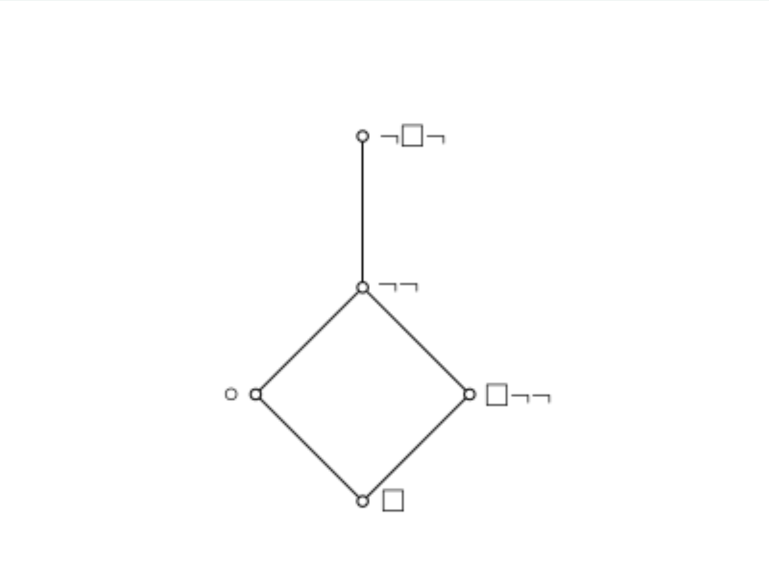
In $\mathbb{ML}_{dS}^{\Box\Diamond}$ the following equations hold [1]:

$$\Box(a \wedge \Diamond b) = \Box a \wedge \Diamond b, \qquad \Diamond(a \vee \Box b) = \Diamond a \vee \Box b,$$

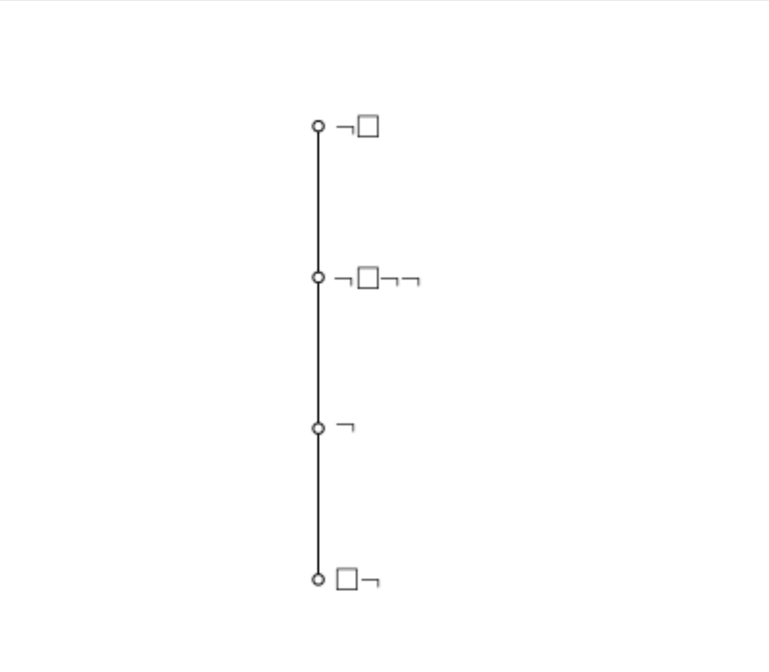
$$\Box(a \vee \Diamond b) = \Box a \vee \Box b, \qquad \Diamond(a \wedge \Diamond b) = \Diamond a \wedge \Diamond b.$$

In $\mathbb{ML}_{dS}^{\Box\Diamond}$ we have that B exists, with $B = \Box$.

[1] Dunn, J. M. and Hardegree, G. *Algebraic Methods in Philosophical Logic*. Fisrt Edition. Oxford University Press, 2001.



Positive modalities for the distributive S-extension



Negative modalities for the distributive S-extension

Adding the relative meet-complement

As well known, we get distributivity for free.

The following hold:

$$\Box(x \to y) \preccurlyeq \Box x \to \Box y,$$

$$\Box(x \to y) \preccurlyeq \Diamond x \to \Diamond y,$$

$$\Diamond a \to \Box b \preccurlyeq \Box(x \to y).$$

The given properties maybe obtained without using the (S)-schema.

The logical versions of the given inequalities appear in a work by Simpson[1].

[1] Simpson, Alex K. *The proof theory and semantics of intuitionistic modal logic*. PhD Thesis, University of Edinburgh, 1994.

Intuitionistic logic expanded with both $\Box$ and $\Diamond$

Take an axiomatization of intuitionistic logic and add the following axiom schemas:

($\Box$A1) $\alpha \vee \neg\Box\alpha$,

($\Box$A2) $(\Box(\alpha \vee \neg\beta) \wedge \beta) \to \Box\alpha$,

($\Diamond$A1) $\neg\alpha \vee \Diamond\alpha$,

($\Diamond$A2) $\Diamond\Box\alpha \to \alpha$,

and the rules:

($\Box$R) $\alpha/\Box\alpha$,

($\Diamond$R) $\alpha \to \beta/\Diamond\alpha \to \Diamond\beta$.

Properties of intuitionistic logic with $\Box$ and $\Diamond$

We have the following form of the Deduction Theorem:

If $\Gamma, \alpha \vdash \beta$, then $\Gamma \vdash \Box\alpha \to \beta$.

We have the Conservative Expansion result at the propositional level. However, the Disjunction Property does not hold.

We have soundness and completeness with the following usual definition of algebraic consequence $\models$:

$\Gamma \models \alpha$ iff for all $\Box\Diamond$-algebras $\mathbf{A}$, for all $a \in A$,
if $v\gamma = 1$, for all $\gamma \in \Gamma$, then $v\alpha = 1$.

Intuitionistic logic with $\Box$ in the S-extension

Take an axiomatization of intuitionistic logic and add the following axiom schemas:

($\Box$A1) $\alpha \vee \neg\Box\alpha$,

($\Box$A2) $(\Box(\alpha \vee \neg\beta) \wedge \beta) \to \Box\alpha$,

($\Box$A3) $\Box\alpha \to \Box\Box\alpha$,

and the rule:

($\Box$R) $\alpha/\Box\alpha$.

Reference

Castiglioni, J. L. and Ertola-Biraben, R. C.
Modal operators in meet-complemented lattices.
Preprint available as arXiv:1603.02489 [math.LO]
(http://arxiv.org/abs/1603.02489)

Thanks for coming!

Adding a weak relative meet-complement

In the context of a lattice, we looked for an arrow such that

1) it is a restriction of the relative meet-complement,

2) if it exists, there cannot be two operations satisfying its definition, and

3) it does not imply distributivity.

We found the following operation, given a lattice L:

$$a \rightarrow_w b = max\{x \in L : b \leq x \text{ and } a \wedge x \leq b\}.$$

It turns out that it equals

$$a \rightarrow_S b = max\{x \in L : a \wedge x = a \wedge b\},$$

which appears in a paper by Jürgen Schmidt.[1]

[1] Schmidt, Jürgen. *Binomial pairs, semi-Brouwerian and Brouwerian semilattices*. Notre Dame Journal of Formal Logic, XIX, 3, July 1978.