On Paraconsistent Weak Kleene Logic and Involutive Bisemilattices

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(Joint work with J. Gil-Férez, L. Peruzzi, and F. Paoli)

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Outline

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2. Involutive bisemilattices

3. AAL approach to Paraconsistent Weak Kleene
Paraconsistent Week Kleene:
Introduction

- The language: $\land, \lor, \neg, 0, 1$
Paraconsistent Week Kleene: Introduction

- The language: $\land, \lor, \neg, 0, 1$
- The algebra $WK$
Paraconsistent Week Kleene: Introduction

• The **language**: $\land, \lor, \neg, 0, 1$

• The **algebra** \textbf{WK}

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A closer look to $\textbf{WK}$

$$\textbf{WK} = \langle \{0, 1, \frac{1}{2}\}, \lor, \land, \neg, 0, 1 \rangle$$
A closer look to \textbf{WK}

\[ WK = \langle \{0, 1, \frac{1}{2}\}, \lor, \land, \neg, 0, 1 \rangle \]

\[ a \leq b \iff a \lor b = b \]
A closer look to **WK**

\[
WK = \langle \{0, 1, \frac{1}{2}\}, \lor, \land, \neg, 0, 1 \rangle
\]

\[
a \leq b \iff a \lor b = b \quad \text{and} \quad a \leq b \iff a \land b = a
\]

\[
\begin{array}{c|c}
\frac{1}{2} & 1 \\
\hline
1 & 0 \\
\hline
0 & \frac{1}{2}
\end{array}
\]
A closer look to $\mathbf{WK}$

$\mathbf{WK} = \langle \{0, 1, \frac{1}{2}\}, \lor, \land, \neg, 0, 1 \rangle$

\[
a \leq b \iff a \lor b = b \quad \text{and} \quad a \leq b \iff a \land b = a
\]

$\frac{1}{2}$

\[\begin{array}{c}
0 \\
\downarrow \\
1 \\
\downarrow \\
0
\end{array}
\quad
\begin{array}{c}
1 \\
\downarrow \\
\frac{1}{2}
\end{array}
\quad
\begin{array}{c}
\frac{1}{2} \\
\downarrow \\
0
\end{array}
\quad
\begin{array}{c}
0 \\
\downarrow \\
\frac{1}{2}
\end{array}
\]

\[
a \leq b \iff \neg b \leq \neg a
\]
A closer look to **WK**

**WK** = $\langle\{0, 1, \frac{1}{2}\}, \lor, \land, \neg, 0, 1\rangle$

\[a \leq b \iff a \lor b = b\]

and

\[a \leq b \iff a \land b = a\]

\[a \leq b \iff \neg b \leq \neg a\]

Counterexample to absorption:

\[1 \land (1 \lor \frac{1}{2}) = \frac{1}{2} \neq 1\]
Paraconsistent Weak Kleene: the logic

• The matrix: $\text{PWK} = \langle \text{WK}, \{1, 1/2\} \rangle$
Paraconsistent Weak Kleene: the logic

- The matrix: \( \text{PWX} = \langle \text{WK}, \{1, \frac{1}{2}\} \rangle \)

\[ \Gamma \models_{\text{PWX}} \alpha \iff \text{for every } v, \ v[\Gamma] \subseteq \{1, \frac{1}{2}\} \Rightarrow v(\alpha) \in \{1, \frac{1}{2}\} \]
Paraconsistent Weak Kleene: the logic

- **The matrix:** $\text{PWK} = \langle \text{WK}, \{1, \frac{1}{2}\} \rangle$

  $\Gamma \models_{\text{PWK}} \alpha \iff$ for every $v$, $v[\Gamma] \subseteq \{1, \frac{1}{2}\} \Rightarrow v(\alpha) \in \{1, \frac{1}{2}\}$

- **Hilbert system:** any set of axioms for Classical Logic and

  $[\text{RMP}] \quad \frac{\alpha}{\beta} \frac{\alpha \rightarrow \beta}{\beta} \quad \text{provided that } \text{var}(\alpha) \subseteq \text{var}(\beta)$
Involutive bisemilattices

Definition

An involutive bisemilattice is an algebra $\mathbf{B} = \langle B, \lor, \land, \neg, 0, 1 \rangle$ of type $(2, 2, 1, 0, 0)$, satisfying:

\begin{align*}
I1 & : x \lor x \approx x; \\
I2 & : x \lor y \approx y \lor x; \\
I3 & : x \lor (y \lor z) \approx (x \lor y) \lor z; \\
I4 & : \neg\neg x \approx x; \\
I5 & : x \land y \approx \neg(\neg x \lor \neg y); \\
I6 & : x \land (\neg x \lor y) \approx x \land y; \\
I7 & : 0 \lor x \approx x; \\
I8 & : 1 \approx \neg 0.
\end{align*}
Involutive bisemilattices

Definition

An involutive bisemilattice is an algebra $B = \langle B, \vee, \land, \neg, 0, 1 \rangle$ of type $(2,2,1,0,0)$, satisfying:

1. $x \lor x \approx x$;
2. $x \lor y \approx y \lor x$;
3. $x \lor (y \lor z) \approx (x \lor y) \lor z$;
4. $\neg \neg x \approx x$;
5. $x \land y \approx \neg (\neg x \lor \neg y)$;
6. $x \land (\neg x \lor y) \approx x \land y$;
7. $0 \lor x \approx x$;
Involution bisemilattices

Definition
An involutive bisemilattice is an algebra $B = \langle B, \lor, \land, \neg, 0, 1 \rangle$ of type $(2,2,1,0,0)$, satisfying:

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Involutive bisemilattices

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An involutive bisemilattice is an algebra $\mathbb{B} = \langle B, \lor, \land, \neg, 0, 1 \rangle$ of type $(2,2,1,0,0)$, satisfying:

\begin{align*}
\text{l1} & \quad x \lor x \cong x; \\
\text{l2} & \quad x \lor y \cong y \lor x; \\
\text{l3} & \quad x \lor (y \lor z) \cong (x \lor y) \lor z; \\
\text{l4} & \quad \neg \neg x \cong x; \\
\text{l5} & \quad x \land y \cong \neg (\neg x \lor \neg y); \\
\text{l7} & \quad 0 \lor x \cong x; \\
\text{l8} & \quad 1 \cong \neg 0.
\end{align*}

We denote by $\text{IBSL}$ the variety of involutive bisemilattices.
**Involutional bisemilattices**

**Definition**

An *involutional bisemilattice* is an algebra \( B = \langle B, \lor, \land, \neg, 0, 1 \rangle \) of type \((2,2,1,0,0)\), satisfying:

1. \( x \lor x \approx x \);
2. \( x \lor y \approx y \lor x \);
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5. \( x \land y \approx \neg (\neg x \lor \neg y) \);
6. \( x \land (\neg x \lor y) \approx x \land y \);
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Involutive bisemilattices

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An involutive bisemilattice is an algebra $\mathbf{B} = \langle B, \lor, \land, \neg, 0, 1 \rangle$ of type (2,2,1,0,0), satisfying:

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We denote by $\mathcal{IBSL}$ the variety of involutive bisemilattices.
Examples

Every Boolean algebra, in particular the 2-element Boolean algebra $B_2$, is an involutive bisemilattice.

$B_2 =$

\[
\begin{array}{c|c}
1 & \\
\hline
0 & 0
\end{array}
\]
Examples

Every Boolean algebra, in particular the 2-element Boolean algebra $B_2$, is an involutive bisemilattice.

Also, the 2-element semilattice with zero, endowed with identity as its unary fundamental operation, is an involutive bisemilattice.
Examples

Every Boolean algebra, in particular the 2-element Boolean algebra $B_2$, is an involutive bisemilattice.

Also, the 2-element semilattice with zero, endowed with identity as its unary fundamental operation, is an involutive bisemilattice.

$B_2 = \begin{array}{c}
1 \\
0
\end{array}$

$S_2 = \begin{array}{c}
a \\
1 = 0
\end{array}$

$WK = \begin{array}{c}
1/2 \\
1 \\
0
\end{array}$
Theorem

The only nontrivial subdirectly irreducible bisemilattices are $\text{WK}$, $S_2$, and $B_2$, up to isomorphism.
The only nontrivial subdirectly irreducible bisemilattices are $\text{WK}$, $S_2$, and $B_2$, up to isomorphism.

$\forall (\text{WK}) = IBSL$. 
Płonka sums: definition

A direct system of algebras: $T = \langle A_i, (\varphi_{ij} : i \leq j), I \rangle$ such that:

• $I = \langle I, \leq \rangle$ is a join semilattice with least element $i_0$;
• $\varphi_{ij} : A_i \rightarrow A_j$ is a homomorphism, for each $i \leq j$,
  $\varphi_{ii}$ is the identity in $A_i$ and $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$;
• If $i \neq j \in I$, then $A_i$ and $A_j$ are disjoint.

Płonka sum over $T$ is the algebra $T = \langle \bigcup I A_i, \{ g_T : g \in \nu \} \rangle$,

• for every $n$-ary $g \in \nu$, and $a_1, \ldots, a_n \in T$, where $n \geq 1$ and $a_r \in A_{i_r}$, we set $j = i_1 \vee \cdots \vee i_n$ and define $g_T(a_1, \ldots, a_n) = g_{A_j}(\varphi_{i_1 j}(a_1), \ldots, \varphi_{i_n j}(a_n))$;
• if $g \in \nu$ is a constant, then $g_T = g_{A_{i_0}}$. 
Płonka sums: definition

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**Płonka sums: definition**

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**Płonka sums: definition**

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  \[ \varphi_{ii} \text{ is the identity in } A_i \quad \text{and} \quad \varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}; \]

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Płonka sum over \( T \) is the algebra \( T = \langle \bigcup_I A_i, \{ g^T : g \in \nu \} \rangle \),
Płonka sums: definition

A direct system of algebras: \( \mathbf{T} = \langle \mathbf{A}_i, (\varphi_{ij} : i \leq j), \mathbf{I} \rangle \) such that:

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- \( \varphi_{ij} : \mathbf{A}_i \to \mathbf{A}_j \) is a homomorphism, for each \( i \leq j \),
  \[ \varphi_{ii} \text{ is the identity in } \mathbf{A}_i \quad \text{and} \quad \varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}; \]
- If \( i \neq j \in I \), then \( \mathbf{A}_i \) and \( \mathbf{A}_j \) are disjoint.

Płonka sum over \( \mathbf{T} \) is the algebra \( \mathbf{T} = \langle \bigcup_I \mathbf{A}_i, \{g^\mathbf{T} : g \in \nu\} \rangle \),

- for every \( n \)-ary \( g \in \nu \), and \( a_1, \ldots, a_n \in \mathbf{T} \), where \( n \geq 1 \) and \( a_r \in \mathbf{A}_{i_r} \), we set \( j = i_1 \vee \cdots \vee i_n \) and define
  \[ g^\mathbf{T}(a_1, \ldots, a_n) = g^\mathbf{A}_j(\varphi_{i_1 j}(a_1), \ldots, \varphi_{i_n j}(a_n)); \]
Płonka sums: definition

A direct system of algebras: $T = \langle \mathcal{A}_i, (\varphi_{ij} : i \leq j), I \rangle$ such that:

- $I = \langle I, \leq \rangle$ is a join semilattice with least element $i_0$;
- $\varphi_{ij} : \mathcal{A}_i \to \mathcal{A}_j$ is a homomorphism, for each $i \leq j$,
  
  $$
  \varphi_{ii} \text{ is the identity in } \mathcal{A}_i \text{ and } \varphi_{jk} \circ \varphi_{ij} = \varphi_{ik};
  $$

- If $i \neq j \in I$, then $\mathcal{A}_i$ and $\mathcal{A}_j$ are disjoint.

Płonka sum over $T$ is the algebra $\mathcal{T} = \langle \bigcup_I A_i, \{g^\mathcal{T} : g \in \nu\} \rangle$,

- for every $n$-ary $g \in \nu$, and $a_1, \ldots, a_n \in T$, where $n \geq 1$ and $a_r \in A_{i_r}$, we set $j = i_1 \lor \cdots \lor i_n$ and define
  
  $$
  g^\mathcal{T}(a_1, \ldots, a_n) = g^\mathcal{A}_j(\varphi_{i_1 j}(a_1), \ldots, \varphi_{i_n j}(a_n));
  $$

- if $g \in \nu$ is a constant, then $g^\mathcal{T} = g^{\mathcal{A}_{i_0}}$. 
Płonka sums: example

\[ \varphi_{01} \quad \varphi_{02} \quad \varphi_{12} \]

\[ \mathbf{A}_0 \rightarrow \mathbf{A}_1 \rightarrow \mathbf{A}_2 \]
Płonka sums representation

Theorem

1. If $T$ is a direct system of Boolean algebras, then the Płonka sum $T$ over $T$ is an involutive bisemilattice.
Płonka sums representation

Theorem

1. If $T$ is a direct system of Boolean algebras, then the Płonka sum $T$ over $T$ is an involutive bisemilattice.

2. If $B$ is an involutive bisemilattice, then $B$ is isomorphic to the Płonka sum over a direct system of Boolean algebras.
Płonka sums representation

Theorem

1. If $T$ is a direct system of Boolean algebras, then the Płonka sum $T$ over $T$ is an involutive bisemilattice.

2. If $B$ is an involutive bisemilattice, then $B$ is isomorphic to the Płonka sum over a direct system of Boolean algebras.

Corollary

$IBSL$ is the variety satisfying exactly the regular identities satisfied by $BA$. 
Leibniz Hierarchy

- regularly finitely algebraizable
  - finitely algebraizable
  - finitely equivalential
  - equivalential
  - protoalgebraic
- regularly algebraizable
  - algebraizable
  - weakly algebraizable
  - weakly equivalential
  - truth-equational
- regularly weakly algebraizable
  - assertional
AAL

- The Leibniz congruence of a matrix $\mathbf{M} = \langle \mathbf{A}, F \rangle$ is the largest congruence of $\mathbf{A}$ that is compatible with $F$. 

\[ \Omega^A F \]
AAL

- The Leibniz congruence of a matrix $M = \langle A, F \rangle$ is the largest congruence of $A$ that is compatible with $F$.
- Given a matrix $M = \langle A, F \rangle$, we define $\Gamma \models_M \alpha$ as follows:
  
  $$\Gamma \models_M \alpha \iff \text{for every valuation } v \text{ on } A,$$

  $$v[\Gamma] \subseteq F \text{ implies } v(\alpha) \in F.$$
AAL

- The Leibniz congruence of a matrix \( \mathbf{M} = \langle \mathbf{A}, F \rangle \) is the largest congruence of \( \mathbf{A} \) that is compatible with \( F \).
- Given a matrix \( \mathbf{M} = \langle \mathbf{A}, F \rangle \), we define \( \models_{\mathbf{M}} \) as follows:
  \[
  \Gamma \models_{\mathbf{M}} \alpha \iff \text{for every valuation } v \text{ on } \mathbf{A},
  v[\Gamma] \subseteq F \text{ implies } v(\alpha) \in F.
  \]
- A matrix \( \mathbf{M} = \langle \mathbf{A}, F \rangle \) is a model of a logic \( L \) if
  \[
  \Gamma \models_{L} \alpha \text{ implies } \Gamma \models_{\mathbf{M}} \alpha.
  \]
• The Leibniz congruence of a matrix $M = \langle A, F \rangle$ is the largest congruence of $A$ that is compatible with $F$.

• Given a matrix $M = \langle A, F \rangle$, we define $\models_M$ as follows:

$$\Gamma \models_M \alpha \iff \text{for every valuation } v \text{ on } A, v[\Gamma] \subseteq F \text{ implies } v(\alpha) \in F.$$ 

• A matrix $M = \langle A, F \rangle$ is a model of a logic $L$ if

$$\Gamma \vdash_L \alpha \implies \Gamma \models_M \alpha.$$ 

• $\mathcal{F}_{iL} A = \{F \subseteq A : \langle A, F \rangle \text{ is a model of } L\}$. 
• The Leibniz congruence of a matrix $\mathbf{M} = \langle \mathbf{A}, F \rangle$ is the largest congruence of $\mathbf{A}$ that is compatible with $F$.

• Given a matrix $\mathbf{M} = \langle \mathbf{A}, F \rangle$, we define $\models_{\mathbf{M}}$ as follows:
  \[ \Gamma \models_{\mathbf{M}} \alpha \iff \text{for every valuation } v \text{ on } \mathbf{A}, \]
  \[ v[\Gamma] \subseteq F \text{ implies } v(\alpha) \in F. \]

• A matrix $\mathbf{M} = \langle \mathbf{A}, F \rangle$ is a model of a logic $\mathbf{L}$ if
  \[ \Gamma \vdash_{\mathbf{L}} \alpha \text{ implies } \Gamma \models_{\mathbf{M}} \alpha. \]

• $\mathcal{F}_{\mathbf{L}} \mathbf{A} = \{ F \subseteq A : \langle \mathbf{A}, F \rangle \text{ is a model of } \mathbf{L} \}$. 
• The Leibniz congruence of a matrix $M = \langle A, F \rangle$ is the largest congruence of $A$ that is compatible with $F$.

• Given a matrix $M = \langle A, F \rangle$, we define $\models_M$ as follows:

$$\Gamma \models_M \alpha \iff \text{for every valuation } v \text{ on } A,$$

$$v[\Gamma] \subseteq F \implies v(\alpha) \in F.$$

• A matrix $M = \langle A, F \rangle$ is a model of a logic $L$ if

$$\Gamma \vdash_L \alpha \implies \Gamma \models_M \alpha.$$

• $\mathcal{F}_L A = \{ F \subseteq A : \langle A, F \rangle \text{ is a model of } L \}$.

Theorem (Iso Thm)

*If $L$ is an algebraizable logic with equivalent algebraic semantics $\mathcal{K}$, then for every $A \in \mathcal{K}$,

$$\Omega^A : \mathcal{F}_L A \to \text{Co}_\mathcal{K} A.$$*
Theorem

*IbSL* is not the equivalent algebraic semantics of any logic \( L \).
Theorem

\textit{IBSL is not the equivalent algebraic semantics of any logic }\mathbf{L}.

- Suppose \textit{IBSL} is the \textbf{equivalent algebraic semantics} of an algebraizable logic \(\mathbf{L}\).
Theorem

\[ \mathcal{IBSL} \text{ is not the equivalent algebraic semantics of any logic } \mathcal{L}. \]

- Suppose \( \mathcal{IBSL} \) is the equivalent algebraic semantics of an algebraizable logic \( \mathcal{L} \).
- Consider the algebra \( C \in \mathcal{IBSL} \) and its congruence lattice:

\[
C = \begin{array}{c}
3 \\
| \\
2 \\
| \\
1 \\
| \\
0
\end{array}
\]

\[
\begin{array}{c}
\cdots \\
\theta_3: \{0,1,2,3\} \\
\theta_1: \{0,1,2,3\} \\
\theta_2: \{0,1,2,3\} \\
\theta_4: \{0,1,2,3\}
\end{array}
\]

\[
\begin{array}{c}
\Delta \\
\nabla
\end{array}
\]

- There is a lattice isomorphism \( \Omega_{\mathcal{C}}: F_{\mathcal{L}}C \to \text{Co}_{\mathcal{IBSL}}C \).
- \( \{2\} \) is the only subset of \( C \) such that \( \Omega_{\mathcal{C}}\{2\} = \theta_2 \), and hence it is an \( \mathcal{L} \)-filter.
- It follows that \( \emptyset \) is also an \( \mathcal{L} \)-filter, \( \mathcal{L} \) is purely inferential, and this leads to a contradiction.
Theorem

\( \mathcal{IBSL} \) is not the equivalent algebraic semantics of any logic \( L \).

- Suppose \( \mathcal{IBSL} \) is the equivalent algebraic semantics of an algebrable logic \( L \).
- Consider the algebra \( C \in \mathcal{IBSL} \) and its congruence lattice:

\[
C = \begin{array}{c}
3 \\
2 \\
1 \\
0
\end{array}
\]

\[
\begin{array}{c}
\theta_3: |0, 1|2, 3| \\
\theta_1: |0|1|2, 3|
\end{array}
\]

\[
\begin{array}{c}
\theta_4: |0, 1, 2|3| \\
\theta_2: |0, 1|2, 3|
\end{array}
\]

- There is a lattice isomorphism \( \Omega^C: \mathcal{F}_{iL} C \rightarrow \mathcal{C}_{\mathcal{IBSL}} C \).
Theorem

$\mathcal{IBSL}$ is not the equivalent algebraic semantics of any logic $L$.

- Suppose $\mathcal{IBSL}$ is the equivalent algebraic semantics of an algebraizable logic $L$.
- Consider the algebra $C \in \mathcal{IBSL}$ and its congruence lattice:

\[
C = \begin{array}{c}
3 \\
2 \\
1 \\
0
\end{array}
\]

\[
\begin{array}{c}
\theta_1 : |0|1|2,3| \\
\theta_3 : |0,1|2,3| \\
\theta_4 : |0,1,2|3|
\end{array}
\]

\[
\begin{array}{c}
\nabla \\
\Delta
\end{array}
\]

- There is a lattice isomorphism $\Omega^C : \mathcal{Fi}_L C \rightarrow \mathcal{Co}_{\mathcal{IBSL}} C$.
- $\{2\}$ is the only subset of $C$ such that $\Omega^C \{2\} = \theta_2$, and hence it is an $L$-filter.
Theorem

\[ \text{IBSL is not the equivalent algebraic semantics of any logic } \mathcal{L}. \]

• Suppose \( \text{IBSL} \) is the equivalent algebraic semantics of an algebraizable logic \( \mathcal{L} \).
• Consider the algebra \( \mathbf{C} \in \text{IBSL} \) and its congruence lattice:

\[
\begin{array}{cccc}
3 & \succ & 0 \\
2 & \succ & 1 \\
1 & & 2, 3 \\
0 & & & 3
\end{array}
\]

\[
\begin{array}{cccccc}
\theta_3 : [0, 1, 2, 3] & \begin{array}{c}
\Downarrow
\end{array} & \theta_4 : [0, 1, 2, 3] \\
& & \\
\theta_1 : [0, 1, 2, 3] & \begin{array}{c}
\Delta
\end{array} & \theta_2 : [0, 1, 2, 3]
\end{array}
\]

• There is a lattice isomorphism \( \Omega^\mathbf{C} : \mathcal{F}_L \mathbf{C} \rightarrow \mathcal{C}_\text{IBSL} \mathbf{C} \).
• \( \{2\} \) is the only subset of \( \mathbf{C} \) such that \( \Omega^\mathbf{C} \{2\} = \theta_2 \), and hence it is an \( \mathcal{L} \)-filter.
• It follows that \( \emptyset \) is also an \( \mathcal{L} \)-filter, \( \mathcal{L} \) is purely inferential, and this leads to a contradiction.
Theorem

PWK is not protoalgebraic.
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  1 $\vdash_{PWK} p \Rightarrow p$,
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  2. $p, p \Rightarrow q \vdash_{\text{PWK}} q$.
- Consider the valuation $v$ on $\textbf{WK}$: $v(p) = 1/2$, $v(q) = 0$.
- Thus, $v[\{p\} \cup p \Rightarrow q] = \{1/2\}$, while $v(q) = 0$, which is a contradiction.
Theorem

PWK is not selfextensional, and therefore it is non-Fregean.
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- Notice that \( \neg p \lor p \not\models_{\text{PWK}} \neg q \lor q \).
**Theorem**

PWK *is not selfextensional, and therefore it is non-Fregean.*

- Notice that $\neg p \lor p \models_{PWK} \neg q \lor q$.
- Consider the valuation $v$ on WK: $v(p) = \frac{1}{2}$, $v(q) = 0$. 
Theorem

PWK is not selfextensional, and therefore it is non-Fregean.

• Notice that \( \neg p \lor p \not\models_{\text{PWK}} \neg q \lor q \).
• Consider the valuation \( v \) on WK: \( v(p) = 1/2, \ v(q) = 0 \).
  • \( v(\neg(\neg p \lor p)) = 1/2 \).
PWK in the Frege hierarchy

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**PWK in the Frege hierarchy**

**Theorem**

PWK *is not selfextensional, and therefore it is non-Fregean.*

- Notice that \( \neg p \lor p \not\models_{PWK} \neg q \lor q \).
- Consider the valuation \( v \) on WK: \( v(p) = 1/2, v(q) = 0 \).
  - \( v(\neg(\neg p \lor p)) = 1/2 \).
  - \( v(\neg(\neg q \lor q)) = 0 \).
- Therefore \( \neg(\neg p \lor p) \not\models_{PWK} \neg(\neg q \lor q) \), does not hold. That is, \( \not\models_{PWK} \) is not a congruence.
A matrix $M = \langle A, F \rangle$ is reduced if $\Omega^A F = Id_A$. 

\[ AAL \]
• A matrix $M = \langle A, F \rangle$ is reduced if $\Omega^A F = Id_A$.

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$\text{Mod}^*(L) = \text{class of reduced models of } L$.

$\text{Alg}^*(L) = \{ A : \text{there is a reduced model } \langle A, F \rangle \text{ of } L \}$. 
The Leibniz congruence

Lemma

If $A$ is an algebra of type of $\mathcal{IBSL}$ and $F \in \mathcal{F}_{\text{PWK}} A$, then for every $a, b \in A$, $\langle a, b \rangle \in \Omega^A F$ if and only if for every $c \in A$,

\[ a \lor c \in F \iff b \lor c \in F \quad \text{and} \quad \neg a \lor c \in F \iff \neg b \lor c \in F. \]
The Leibniz congruence

Lemma

If \( A \) is an algebra of type of \( \mathcal{IBSL} \) and \( F \in \mathcal{F}_{\text{PWK}} \), then for every \( a, b \in A \), \( \langle a, b \rangle \in \Omega^A F \) if and only if for every \( c \in A \),

\[
\begin{align*}
  a \lor c & \in F \iff b \lor c \in F \\
  \neg a \lor c & \in F \iff \neg b \lor c \in F.
\end{align*}
\]

Theorem

\( \text{Alg}^*(\text{PWK}) \subseteq \mathcal{IBSL} \).
Let $B \in \mathcal{IBSL}$, $b \in B$ is positive iff $1 \leq b$. 

Theorem

$B \in \text{Alg}^*(\text{PWK})$ if and only if $B \in \mathcal{IBSL}$ and for every $a < b$ positive elements, there is $c \in B$ such that $1 \leq \neg b \lor c$ but $1 \not\leq \neg a \lor c$.

Moreover, $\langle B, F \rangle \in \text{Mod}^*(\text{PWK})$ if and only if $B$ is an involutive bisemilattice satisfying the above condition and $F = \mathcal{P}(B)$, the set of positive elements, which is given by:

$$\mathcal{P}(B) = \{c \in B : 1 \lor c = c\}$$

Corollary

PWK is truth-equational.
Let $B \in \mathcal{IBSL}$, $b \in B$ is positive iff $1 \leq b$.

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**Theorem**

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$$P(B) = \{ c \in B : 1 \lor c = c \}$$
Let $B \in IBSL$, $b \in B$ is positive iff $1 \leq b$.

**Theorem**

$B \in \text{Alg}^*(PWK)$ if and only if $B \in IBSL$ and for every $a < b$ positive elements, there is $c \in B$ such that

$$1 \leq\neg b \lor c \quad \text{but} \quad 1 \not\leq \neg a \lor c.$$ 

Moreover, $\langle B, F \rangle \in \text{Mod}^*(PWK)$ if and only if $B$ is an involutive bisemilattice satisfying the above condition and $F = P(B)$, the set of positive elements, which is given by:

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**Corollary**

$PWK$ is truth-equational.
PWK in the Leibniz Hierarchy

- Regularly finitely algebraizable
- Finitely algebraizable
- Finitely equivalential
- Equivalential
  - Protoalgebraic
- Algebraizable
  - Weakly algebraizable
    - Assertional
- Regularly weakly algebraizable
- Regularly algebraizable
- Equivalential
  - Protoalgebraic
  - Truth-equational
Work in progress

- Natural duality for involutive bisemilattices (joint with A. Loi and L. Peruzzi)
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- Natural duality for involutive bisemilattices (joint with A. Loi and L. Peruzzi)

- Sequent calculi for PWK and Gentzen algebraizability (joint with M. Pra Baldi)
Thank you!