A geometrical representation of the basic laws of Categorial Grammar

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Abstract. We present a geometrical analysis of the principles that lay at the basis of Categorial Grammar and of the Lambek Calculus. In [3] it is shown that the basic properties known as *Residuation* laws can be characterized in the framework of Cyclic Multiplicative Linear Logic, a purely non-commutative fragment of Linear Logic. We present a summary of this result and, pursuing this line of investigation, we analyze a well-known set of categorial grammar laws: *Monotonicity*, *Application*, *Expansion*, *Type-raising*, *Composition*, *Geach* laws and *Switching* laws.

Keywords Categorial grammar, cyclic linear logic, proof-net

1 Introduction

We propose a geometrical representation of the set of laws that are at the basis of Categorial Grammar and of the Lambek Calculus, developing our analysis in the framework of *Cyclic Multiplicative Linear Logic*, a purely non-commutative fragment of Linear Logic [1, 5, 4]. The rules we intend to investigate are known as *Residuation* laws, *Monotonicity* laws, *Application* laws, *Expansion* laws, *Type-raising* laws, *Composition* laws, *Geach* laws, *Switching* laws [13, 14, 8, 9, 18, 21].

1.1 Formulation of basic laws in an algebraic style

In an algebraic style, the basic laws of Categorial Grammar involve:

- a binary operation on a set M, the *product* or the *residuated* operation, denoted by $\cdot$;
- two binary *residual* operations on the same set M: $\backslash$ (the *left residual* operation of the product) and $/$ (the *right residual* operation of the product);
- a partial ordering on the same set M, denoted by $\leq$.

The following is the algebraic formulation of these laws (cf. [8], pp. 17-19):

(a) *Residuation laws*

- (RES) $a \cdot b \leq c$ iff $b \leq a\backslash c$ iff $a \leq c/b$

(b) *Monotonicity laws*[1]

- (MON1.1) if $a \leq b$ then $a \cdot c \leq b \cdot c$ (MON1.2) if $a \leq b$ then $c \cdot a \leq c \cdot b$
- (MON2.1) if $a \leq b$ then $c \backslash a \leq c \backslash b$ (MON2.2) if $a \leq b$ then $b \backslash c \leq a \backslash c$
- (MON3.1) if $a \leq b$ then $a/c \leq b/c$ (MON3.2) if $a \leq b$ then $c/b \leq c/a$

(c) *Application laws*

- (APP1) $a \cdot a \backslash b \leq b$
- (APP2) $b/a \cdot a \leq b$

(d) *Expansion laws*

- (EXP1) $a \leq b \backslash (b \cdot a)$
- (EXP2) $a \leq (a \cdot b)/b$

(e) *Type-raising laws*

- (TYR1) $a \leq (b/a) \backslash b$
- (TYR2) $a \leq b/(a \backslash b)$

(f) *Composition laws*

- (COM1) $(a \backslash b) \cdot (b \backslash c) \leq (a \backslash c)$
- (COM2) $(a/b) \cdot (b/c) \leq (a/c)$

(g) *Geach laws*

- (GEA1) $b \backslash c \leq (a \backslash b) \backslash (a \backslash c)$
- (GEA2) $a/b \leq (a/c)/(b/c)$

(h) *Switching laws*

- (SWI1) $(a \backslash b) \cdot c \leq a \backslash (b \cdot c)$
- (SWI2) $a \cdot (b/c) \leq (a \cdot b)/c$

1.2 Formulation of the basic laws in a sequent calculus style

The basic laws of Categorial Grammar can also be expressed in a sequent calculus style. The sequent calculus for Lambek Calculus (L) [13] is the multiplicative fragment of intuitionistic non-commutative Linear Logic [5, 11, 12], where one deals with:

- a binary connective, the multiplicative conjunction, denoted by $\otimes$;
- two binary connectives:
  - the linear *retro*-implication denoted by $\circ\!-$
  - the linear *post*-implication denoted by $-\!\circ$
- a derivability relation $\vdash$ between formulas A, B: $A \vdash B$

Formulas of L are constructed from non-negated atoms, by means of the connectives $\otimes$, $-\!\circ$, $\circ\!-$. Sequents of L are expressions $A_1, \ldots, A_n \vdash B$ where $A_1, \ldots, A_n$ and B are formulas of L. In the semantics:

- the multiplicative conjunction $\otimes$ corresponds to the operation $\cdot$
- the linear retro-implication $\circ\!-$ corresponds to the operation $/$
- the linear post-implication $-\!\circ$ corresponds to the operation $\backslash$

[1]Monotonicity can also be introduced by rules with two premises, as pointed out by one of the referees, however following [8], we prefer the present formulation.

– the derivability relation $\vdash$ corresponds to the partial order $\leq$

The following is the formulation of the basic laws of categorial grammar in a sequent calculus style:

(a) *Residuation laws*

- (RES) $A \otimes B \vdash C$ iff $B \vdash A \multimap C$ iff $A \vdash C \circ\!\!- B$

(b) *Monotonicity laws*

- (MON1.1) $\dfrac{A \vdash B}{A \otimes C \vdash B \otimes C}$ (MON1.2) $\dfrac{A \vdash B}{C \otimes A \vdash C \otimes B}$
- (MON2.1) $\dfrac{A \vdash B}{C \multimap A \vdash C \multimap B}$ (MON2.2) $\dfrac{A \vdash B}{B \multimap C \vdash A \multimap C}$
- (MON3.1) $\dfrac{A \vdash B}{A \circ\!\!- C \vdash B \circ\!\!- C}$ (MON3.2) $\dfrac{A \vdash B}{C \circ\!\!- B \vdash C \circ\!\!- A}$

(c) *Application laws*

- (APP1) $A \otimes (A \multimap B) \vdash B$
- (APP2) $(B \circ\!\!- A) \otimes A \vdash B$

(d) *Expansion laws*

- (EXP1) $A \vdash B \multimap (B \otimes A)$
- (EXP2) $A \vdash (A \otimes B) \circ\!\!- B$

(e) *Type-raising laws*

- (TYR1) $A \vdash (B \circ\!\!- A) \multimap B$
- (TYR2) $A \vdash B \circ\!\!- (A \multimap B)$

(f) *Composition laws*

- (COM1) $(A \multimap B) \otimes (B \multimap C) \vdash A \multimap C$
- (COM2) $(A \circ\!\!- B) \otimes (B \circ\!\!- C) \vdash A \circ\!\!- C$

(g) *Geach laws*

- (GEA1) $B \multimap C \vdash (A \multimap B) \multimap (A \multimap C)$
- (GEA2) $A \circ\!\!- B \vdash (A \circ\!\!- C) \circ\!\!- (B \circ\!\!- C)$

(h) *Switching laws*

- (SWI1) $(A \multimap B) \otimes C \vdash A \multimap (B \otimes C)$
- (SWI2) $A \otimes (B \circ\!\!- C) \vdash (A \otimes B) \circ\!\!- C$

The sequents occurring in the formulation of these rules are of the form $C \vdash D$ where C, D are formulas of L, i.e. sequents with exactly one formula on the left side and exactly one formula on the right side.

Residuation laws state an equivalence between sequents in L: a sequent $C \vdash D$ is equivalent to a sequent $E \vdash F$ iff every proof in L of $C \vdash D$ can be transformed into a proof in L of $E \vdash F$ and every proof in L of $E \vdash F$ can be transformed into a proof in L of $C \vdash D$.

Each Monotonicity law states a derived unary rule of L, a rule where the premise is the sequent above the line and the conclusion is the sequent below the line, i.e. each Monotonicity law states that every proof in L of the premise of the rule can be transformed into a proof in L of the conclusion of the rule.

All the other laws state the provability of a sequent in L, i.e. the existence of a proof of a sequent in L .

1.3 Overview of the paper

The paper introduces a geometrical representation of the basic laws of Categorial Grammar and of the Lambek Calculus by means of geometric objects called *cyclic multiplicative proof-nets* (CyM-PN's).

In section 2, we characterize the notion of a *cyclic multiplicative proof-net*. In Linear Logic proof-nets are geometrical representations of proofs [2, 19, 20]. Cyclic multiplicative proof-nets represent proofs in Cyclic Multiplicative Linear Logic (CyMLL), a purely non-commutative fragment of Linear Logic. The conclusions of a CyM-PN may be described in different ways corresponding to different sequents of CyMLL. A subset of the sequents of CyMLL represent the sequents of the Lambek Calculus L, and a subset of the CyM-PN's represent proofs in L.

In section 3, we recall the result presented in [3], where is given the geometrical representation of *Residuation* laws and it is explained that Residuation laws correspond to different ways to read the conclusions of a *single* CyM-PN.

In section 4, we show (theorem 1) that the geometrical representation of *Monotonicity* laws is given by the CyM-PN's obtained from an arbitrary CyM-PN (corresponding to the premise of the law) and a single axiom link.

In section 5, we show (theorem 2) that the geometrical representations of *Application* laws, *Expansion* laws and *Type-raising* laws, are given by the CyM-PN's obtained from two axiom links, one $\otimes$-link and one $⅋$-link.

Finally, in section 6, we show that the geometrical representations of *Composition* laws (theorem 3), *Geach* laws (theorem 4) and *Switching* laws (theorem 5), are given by the CyM-PN's obtained from three axiom links, two $\otimes$-links and two $⅋$-links.

2 Cyclic multiplicative proof-nets

Cyclic multiplicative proof-nets are a subclass of multiplicative proof nets. Multiplicative proof-nets are defined by means of the language of Multiplicative Linear Logic (MLL), a fragment of Linear Logic. Formulas of MLL are defined by using atoms and the binary connectives: $\otimes$ (*multiplicative conjunction*), $⅋$ (*multiplicative disjunction*). The language of MLL has the following features:

- for each atom X there is another atom which is the dual of X and is denoted by $X^\perp$, in such a way that for every atom X, $X^{\perp\perp} = X$;
- for each formula A the linear negation $A^\perp$ is defined as follows, in order to satisfy the principle $A^{\perp\perp} = A$:
  - if A is an atom, $A^\perp$ is the atom which is the dual of A,
  - $(B \otimes C)^\perp = C^\perp ⅋ B^\perp$
  - $(B ⅋ C)^\perp = C^\perp \otimes B^\perp$.

Left and *right* residual connectives, i.e. the *left implication* $\multimap$ and the *right implication* $\multimapinv$, can be defined by means of the linear negation $()^\perp$ and $⅋$:

$$A \multimap B = A^\perp ⅋ B \quad ; \quad B \multimapinv A = B ⅋ A^\perp$$

Thus, under this definition of $\multimap$ and $\multimapinv$, the class of formulas of L is the subclass of formulas of NMLL inductively defined as follows:

– if X is an atom, then X is a formula of L;
– if A and B are formulas of MLL which are formulas of L, then $A \otimes B$, $A^\perp \parr B$ i.e. $A \multimap B$ and $B \parr A^\perp$ i.e. $B \multimapinv A$, are formulas of L;
– no other formula of MLL is a formula of L.

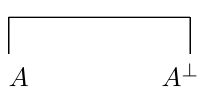
Moreover the class of formulas which are *linear negations* of formulas of L is the subclass of formulas of MLL inductively defined as follows:

– if X is an atom, then $X^\perp$ is the linear negation of a formula of L;
– if A and B are formulas of MLL which are formulas of L, then $A^\perp \parr B^\perp$ i.e. $(B \otimes A)^\perp$, $B^\perp \otimes A$ i.e. $(A \multimap B)^\perp$ and $B \otimes A^\perp$ i.e. $(A \multimapinv B)^\perp$ are linear negations of formulas of L;
– no other formula of MLL is the linear negation of a formula of L.

2.1 Multiplicative proof-nets

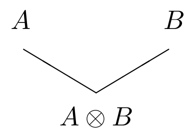
A *multiplicative proof-net* is a graph such that:

– the nodes are decorated by formulas of *Multiplicative Linear Logic*
– the edges are grouped by *links* and the links are:
  - the *axiom-link*, a binary link (i.e. a link with two nodes and one edge) with no premise, in which both nodes are conclusions of the link and each node is decorated by the linear negation of the formula decorating the other one; i.e. the conclusions of an axiom link are decorated by two formulas $A, A^\perp$

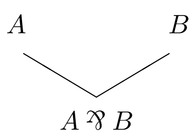


  - the cut-link, a binary link with no conclusion and both nodes are premises of the link: each node is decorated by the linear negation of the formula decorating the other one; i.e. the premises of a cut link are decorated by two formulas $A, A^\perp$

  - the $\otimes$-*link*, a ternary link (i.e. a link with three nodes and two edges), where two nodes are premises (the first premise and the second premise) and the other node is the conclusion, there is one edge between the first premise and the conclusion and another edge between the second premise and the conclusion; the conclusion is decorated by the formula $A \otimes B$, where A is the formula decorating the first premise and B is the formula decorating the second premise



- the *$\parr$-link*, a ternary link in which two nodes are premises (the first premise and the second premise) and the other node is the conclusion, there is one edge between the first premise and the conclusion and another edge between the second premise and the conclusion; the conclusion is decorated by the formula $A \parr B$, where A is the formula decorating the first premise and B is the formula decorating the second premise



– each node is the premise of at most one link, and is the conclusion of exactly one link; the nodes which are not premises of links are called the *conclusions of the proof-net*;

– for each "switching", the graph is acyclic and connected, where a "switching" of the graph is the removal of one edge in each $\parr$-link of the graph.

A multiplicative proof-net is *cut-free* iff it contains no cut-link.

An important theorem (*cut-elimination theorem* or *normalization theorem* for proof-nets) states that every multiplicative proof-net can be transformed in a cut-free multiplicative proof-net with the same conclusions. We may therefore restrict our attention to cut-free multiplicative proof-nets.

A *sequent* of MLL is a set of occurrences of formulas of MLL represented by $\vdash \Gamma$, where Γ is a finite sequence of all the elements of the sequent.

A proof of a sequent $\vdash \Gamma$ is a proof of one of the formulas of Γ from the linear negation of the other formulas of Γ. Therefore, a proof of the sequent $\vdash A, B$ may be considered as a proof of A from $B^\perp$ (i.e. a proof of $B^\perp \vdash A$) or a proof of B from $A^\perp$ (i.e. a proof of $A^\perp \vdash B$).

The rules of the sequent calculus of MLL allow one to generate proofs of sequents in MLL.

Multiplicative proof-nets are *geometrical representations* of proofs in MLL. This result is a consequence of the following theorems about the relationship between multiplicative proof-nets and the sequent calculus for MLL:

– each proof of $\vdash \Gamma$ in the sequent calculus for MLL can be transformed in a multiplicative proof-net in which the conclusions are the formulas of Γ;

– every multiplicative proof-net can be considered as the multiplicative proof net coming from a proof of the sequent $\vdash \Gamma$ in the sequent calculus for MLL, where Γ is a finite sequence of all the occurrences of formulas which are the conclusions of the given multiplicative proof-net.

The multiplicative disjunction $\parr$ is a *reversible* connective, i.e.:

- the sequent $\vdash \Gamma, A \parr B, \Delta$ is provable in MLL iff the sequent $\vdash \Gamma, A, B, \Delta$ is provable in MLL;
- π is a multiplicative proof-net with conclusions $\Gamma, A \parr B, \Delta$ iff the graph obtained from π by deleting the terminal $\parr$-link with conclusion $A \parr B$ is a multiplicative proof-net with conclusions $\vdash \Gamma, A, B, \Delta$.

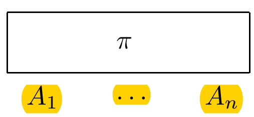
Therefore, given a multiplicative proof net π with conclusions Γ, A, B, Δ, the sequence $\Gamma, A \parr B, \Delta$ may be considered as a way to read the conclusions of π.

2.2 Cyclic Multiplicative Proof-Nets

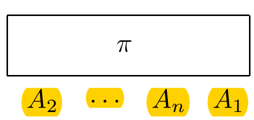
A *cyclic multiplicative proof-net* (CyM-PN) is a multiplicative proof-net s. t.

- the graph is *planar*, i. e. the graph may be drawn on the plane with no crossing of edges;
- the conclusions of the graph are in a *cyclic order*, induced by the "trips" inside the proof-net (as defined in [5]); this cyclic order corresponds to the order of the conclusions going from left to right, when the graph is written on the plane and one considers the "rightmost" conclusion before the "leftmost" one.

We may represent a CyM-PN π as follows:



where $A_1, \ldots, A_n$ are the conclusions of π in their cyclic order (A_2 is the immediate successor of A_1, A_n is the immediate successor of A_{n-1}, A_1 is the immediate successor of A_n). For each conclusion A of a CyM-PN, we may draw the graph on the plane - with no crossing of edges - in such a way that A is the first conclusion going from left to right. For example, we may draw the CyM-PN π considered above in such a way that the first conclusion (from left to right) is A_2 and the last conclusion is A_1, i.e. :



Cut-elimination theorem or *normalization theorem* holds also for CyM-PN's[2]: every CyM-PN can be transformed in a cut-free multiplicative proof-net with the same conclusions and the same cyclic order of the conclusions. We may therefore restrict our attention to cut-free CyM-PN's.

Other important theorems state the relationship between CyM-PN's and the sequent calculus for a refinement of MLL, called *Cyclic Multiplicative Linear Logic* (CyMLL): CyM-PN's are geometrical representations of proofs in CyMLL.

[2]This result is proved in [1, 5]. Cut elimination for non-commutative proof nets is rather tricky, as shown in [17]. We thank one of the referees for this observation.

A sequent of CyMLL is a finite *cyclic order* of occurrences of formulas of MLL, and is represented by $\vdash \Gamma$, where Γ is a finite sequence of occurrences of formulas of MLL and the cyclic finite order of the sequent is the cyclic order induced by Γ.

Given a sequent $\vdash A_1, \cdots, A_n, B, C_1, \cdots, C_m$, the linear order of the predecessors of B inside the sequent is $A_n, \cdots, A_1, C_m, \cdots, C_1$.

A proof of a sequent is, for each formula A, a proof of A from the linear negations of the predecessors of A inside the sequent; so a proof of a sequent $\vdash A_1, \cdots, A_n, B, C_1, \cdots, C_m$ corresponds to the proof of B from the linear order of the premisses $(A_n)^\perp, \cdots, (A_1)^\perp, (C_m)^\perp, \cdots, (C_1)^\perp$ i.e. a proof of $(A_n)^\perp, \cdots, (A_1)^\perp, (C_m)^\perp, \cdots, (C_1)^\perp \vdash B$.

Rules of the sequent calculus of CyMLL are refinements of rules of the sequent calculus of MLL and allow one to generate proofs of sequents of CyMLL.

CyM-PN's are *geometrical representations* of proofs in CyMLL, as a consequence of the following results:

– every proof of a sequent $\vdash \Gamma$ in the sequent calculus for CyMLL can be transformed in a CyM-PN with conclusions Γ;
– every CyM-PN with conclusions Γ can be considered as the CyM-PM coming from a proof of the sequent $\vdash \Gamma$ in the sequent calculus for Cy-MLL.

Multiplicative disjunction $⅋$ is a *reversible* connective in CyMLL, i.e.:

– the sequent $\vdash \Gamma,\ A ⅋ B,\ \Delta$ is provable in CyMLL iff $\vdash \Gamma,\ A,\ B,\ \Delta$ is provable in CyMLL;
– if π is a CyM-PN with conclusions $\Gamma,\ A ⅋ B,\ \Delta$, then the graph obtained from π by deleting the terminal $⅋$-link with conclusion $A ⅋ B$ is a CyM-PN with conclusions $\vdash \Gamma,\ A,\ B,\ \Delta$.

Therefore, given a proof net π with conclusions $\Gamma,\ A,\ B,\ \Delta$, the sequence Γ, $A ⅋ B,\ \Delta$ may be considered another way to read the conclusions of π.

We know that, under the definition of $\multimap$ and $\circ\!-$ by means of multiplicative disjunction and linear negation, the set of the formulas of the Lambek Calculus L and the set of the linear negations of formulas of L are subsets of the set of the formulas of MLL. It is easy to see that, under the same definition of $\multimap$ and $\circ\!-$, a subset of the sequents of CyMLL represent the sequents of L. Indeed:

– each sequent $A_1, \ldots, A_n \vdash B$ of L corresponds to the sequent $\vdash A_n^\perp, \ldots, A_1^\perp, B$ of CyMLL;
– each sequent, where there is exactly one formula of L and all the other formulas are linear negations of formulas of L, represents a sequent of L; in particular, the sequent $\vdash A_n^\perp, \ldots, A_1^\perp, B$, when $A_1, \ldots, A_n, B$ are formulas of L, represents the sequent $A_1, \ldots, A_n \vdash B$ of L.

Moreover, as proven in [2, 3] , when $A_1, \ldots, A_n, B$ are formulas of L:

– each proof in L of a sequent $A_1, \ldots, A_n \vdash B$ may be transformed into a proof in CyMLL of the corresponding sequent $\vdash A_n^\perp, \ldots, A_1^\perp, B$, and therefore there is a CyM-PN which is the geometrical representation of such a proof;

– each proof in CyM-LL of the sequent $\vdash A_n^\perp, \ldots, A_1^\perp, B$ may be transformed into a proof in L of the corresponding sequent $A_1, \ldots, A_n \vdash B$, and the CyM-PN which is the geometrical representation of the proof in CyM-LL of $\vdash A_n^\perp, \ldots, A_1^\perp, B$ is also the geometrical representation of a proof in L of $A_1, \ldots, A_n \vdash B$.

From these results, it is easy to see that a CyM-PN π is a geometrical representation of a proof in L when:

– there is exactly one conclusion A of π which is a formula of L and all the other conclusions of π are negations of formulas of L;
– no point of the graph π is labeled by formulas $C \parr D$ or $C^\perp \otimes D^\perp$, where C and D are formulas of L.

Indeed, Lambek Calculus is an intuitionistic fragment of CyMLL. The conditions required for being a geometrical representation of a proof in L reflects the features of intuitionistic systems: in each proof there is only one conclusion whereas the number of hypotheses is an arbitrary natural number and there is no duality (i.e. no way to consider a conclusion as an hypothesis and an hypothesis as a conclusion, no way to change the role - hypothesis vs. conclusion - of a formula). Each CyM-PN representing a proof π in L must have just one conclusion which is a formula of L (the conclusion of π), whereas all the other conclusions are linear negations of formulas of L (the hypotheses of π).

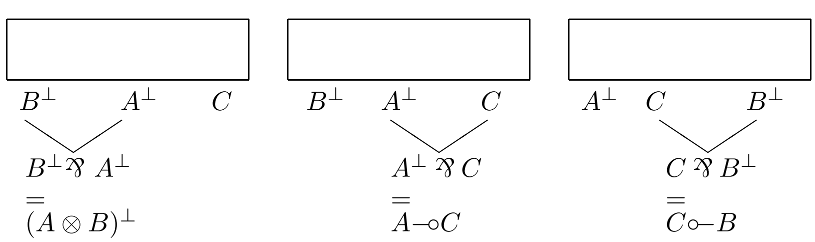
3 Geometrical Representation of Residuation laws

In [3] it is studied the question of offering a geometrical representation of the *Residuation* laws of L, i.e. the laws which refer to the sequents:

$$A \otimes B \vdash C \quad , \quad B \vdash A \multimap C \quad , \quad A \vdash C \circ\!\!- B$$

stating their equivalence in L (i.e. the proof in L of one of these sequents can be transformed into the proof in L of each of the other sequents). As shown in the previous section, CyM-PN's are geometrical representations of proofs in L. So, every possible proof in L of one of these sequents is a CyM-PN with 2 conclusions: the formula which is on the right side of the sequent, and the linear negation of the formula which is on the left side of the sequent. Therefore:

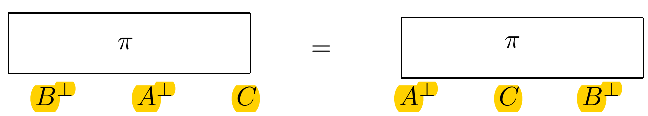
– possible proofs of $A \otimes B \vdash C$ are the CyM-PN's with conclusions $B^\perp \parr A^\perp, C$ i.e. $(A \otimes B)^\perp, C$,
– possible proofs of $B \vdash A \multimap C$ are the CyM-PN's with conclusions $B^\perp, A^\perp \parr C$ i.e. $B^\perp, A \multimap C$,
– possible proofs of $A \vdash C \circ\!\!- B$ are the CyM-PN's with conclusions $A^\perp, C \parr B^\perp$ i.e. $A^\perp, C \circ\!\!- B$.



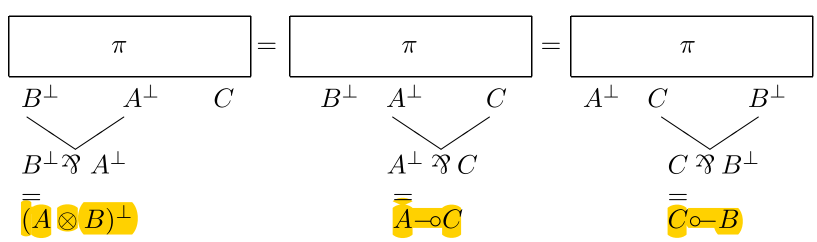
In particular, it is interesting to show how the geometrical representation of a proof of one of these sequents can be transformed - by means of purely geometrical methods - into a geometrical representation of a proof of the other two sequents, i.e. how to transform by means of purely geometrical methods

- each CyM-PN with conclusions $B^\perp \parr A^\perp, C$ into a CyM-PN with conclusions $B^\perp, A \multimap C$ and a CyM-PN with conclusions $A^\perp, C \multimapinv B$;
- each CyM-PN with conclusions $B^\perp, A \multimap C$ into a CyM-PN with conclusions $B^\perp \parr A^\perp, C$ and a CyM-PN with conclusions $A^\perp, C \multimapinv B$;
- each CyM-PN with conclusions $A^\perp, C \multimapinv B$ into a CyM-PN with conclusions $B^\perp \parr A^\perp, C$ and a CyM-PN with conclusions $A^\perp, C \multimapinv B$.

In [3] it is shown that Residuation laws are all the possible descriptions - by means of formulas of L - of the conclusions of *just one* CyM-PN with *three* conclusions as a CyM-PN with *two* conclusions. Let us consider a CyM-PN π with three conclusions and let us suppose that π is the geometrical representation of a proof in L of the sequent $A, B \vdash C$:



The three possible descriptions of this CyM-PN π as a CyM-PN with two conclusions are exactly the following ones:



which are the geometrical representations of proofs in L of the sequents:

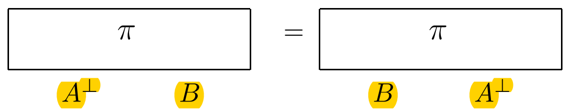
$$A \otimes B \vdash C \qquad B \vdash A \multimap C \qquad A \vdash C \multimapinv B$$

4 Geometrical representation of Monotonicity laws

Let's now consider the geometrical representation of *Monotonicity* laws, which are unary *rules* of L.

Each Monotonicity law is a rule with a premise $A \vdash B$, stating the existence in L of a proof of a sequent $E \vdash F$ from the existence in L of a proof of the premise $A \vdash B$. Thus, to get a representation of this rule, we have to consider how one gets a CyM-PN representing a proof in L of the conclusion $E \vdash F$ from a CyM-PN representing a proof in L of the premise $A \vdash B$: both the CyM-PN's have two conclusions, one of the conclusion is a formula of L and the other conclusion is the negation of a formula of L.

A CyM-PN π representing a proof in L of the premise $A \vdash B$ must have two conclusions, $A^\perp$ and B:



and each Monotonicity law with conclusion $E \vdash F$ and premise $A \vdash B$ states that such a CyM-PN π may be transformed in another CyM-PN with conclusions $E^\perp, F$.

We will show that all the CyM-PN's corresponding to the conclusions of some Monotonicity law with premise $A \vdash B$ belong to the class MON$(\pi, A^\perp, B, C)$ of all the graphs obtained as follows:

- take one Axiom link (Ax) with conclusions $C, C^\perp$ and the CyM-PN π with conclusions $A^\perp, B$,
- then connect one conclusion of Ax with one conclusion of π by means of a $\otimes$-link,
- finally, connect the other conclusion of π and the other conclusion of Ax by means of a $\parr$-link.

It is easy to check that MON$(\pi, A^\perp, B, C)$ contains 8 elements and all these elements are CyM-PN's.

Now, let us consider the subclass MON-L$(\pi, A^\perp, B, C)$ of MON$(\pi, A^\perp, B, C)$, obtained by taking the elements of MON$(\pi, A^\perp, B, C)$ which satisfy the following requirement: no $\otimes$-link occurs between $C^\perp$ and $A^\perp$.

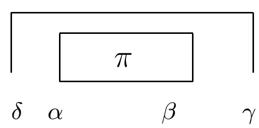
Lemma 1. *All the elements of MON-L$(\pi, A^\perp, B, C)$ are CyM-PN's representing a proof in L.*

Proof. We show that each CyM-PN belonging to MON-L$(\pi, A^\perp, B, C)$ satisfies the conditions stated above about the CyM-PN's representing a proof in L. Indeed, each graph belonging to MON-L$(\pi, A^\perp, B, C)$ is constructed by starting from a CyM-PN π which represents a proof in L, and moreover - since no $\otimes$-link occurs between $C^\perp$ and $A^\perp$ - the terminal $\otimes$-link is between $C^\perp$ and B (first case), or between C and $A^\perp$ (second case), or between C and B (third case):

1. in the first case, the terminal $\parr$-link is between $A^\perp$ and C, so that the conclusions of the graph are a formula of L: $A^\perp \parr C = A \multimap C$ or $C \parr A^\perp = C \multimapinv A$, and the linear negation of a formula of L: $C^\perp \otimes B = (B^\perp \parr C)^\perp = (B \multimap C)^\perp$ or $B \otimes C^\perp = (C \parr B^\perp)^\perp = (C \multimapinv B)^\perp$;

2. in the second case, the terminal $\parr$-link is between $C^\perp$ and B, so that the conclusions of the graph are a formula of L: $C^\perp \parr B = C \multimap B$ or $B \parr C^\perp = B \circ\!- C$ and the linear negation of a formula of L: $A^\perp \otimes C = (C^\perp \parr A)^\perp = (C \multimap A)^\perp$ or $C \otimes A^\perp = (A \parr C^\perp)^\perp = (A \circ\!- C)^\perp$;
3. in the third case, the terminal $\parr$-link is between B and C, so that the conclusions of the graph are a formula of L: $B \otimes C$ or $C \otimes B$ and the linear negation of a formula of L: $C^\perp \parr A^\perp = (A \otimes C)^\perp$ or $A^\perp \parr C^\perp = (C \otimes A)^\perp$.

In order to explore the elements of MON$(\pi, A^\perp, B, C)$ and MON-L$(\pi, A^\perp, B, C)$ it is useful to use the following configuration:



where δ, γ are the conclusions of the axiom $\vdash C, C^\perp$ (i.e. $\delta = C$ and $\gamma = C^\perp$, or $\delta = C^\perp$ and $\gamma = C$) and α, β are the conclusions of π (i.e. $\alpha = B$ and $\beta = A^\perp$, or $\alpha = A^\perp$ and $\beta = B$) . Remark that in this configuration the axiom $C \vdash C$ *wraps itself* around the CyM-PN which is the geometrical representation of the proof in L of the sequent $A \vdash B$.

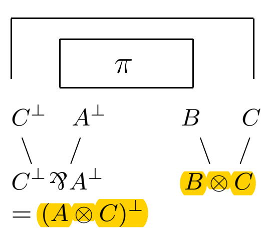
The following theorem shows that MON-L$(\pi, A^\perp, B, C)$ are the geometrical representations of the six Monotonicity rules with premise $A \vdash B$.

Theorem 1. *The eight elements of MON$(\pi, A^\perp, B, C)$ are:*

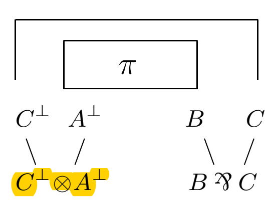
1. *six CyM-PN's which belong to MON-L$(\pi, A^\perp, B, C)$ and are geometrical representations of proofs in L, corresponding to the six Monotonicity laws.*
2. *two CyM-PN's which do not belong to MON-L$(\pi, A^\perp, B, C)$ i.e. are not representations of proofs in L.*

Proof. We consider the eight elements of MON$(\pi, A^\perp, B, C)$, and we prove that MON-L$(\pi, A^\perp, B, C)$ contains exactly six of the eight CyM-PN's belonging to MON$(\pi, A^\perp, B, C)$.

Case (1): (MON1.1)



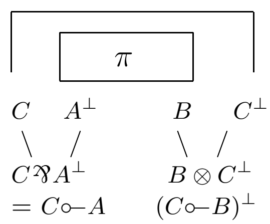
Case (2)*



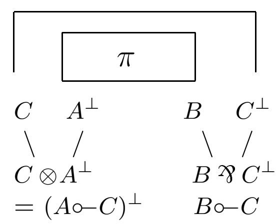
Case (1) is a CyM-PN belonging to MON-L$(\pi, A^\perp, B, C)$ and is the geometrical representation of the conclusion of the Monotonicity law MON1.1: one of the conclusions is the formula $B \otimes C$ of L and the other conclusion is the formula $C^{\perp} \parr A^{\perp}$ which is the negation of the formula $A \otimes C$ of L, therefore the CyM-PN is the geometrical representation of a proof of the sequent $A \otimes C \vdash B \otimes C$ of L.

Case (2) is a CyM-PN which does not belong to MON-L($\pi, A^{\perp}, B, C$), since there is a $\otimes$-link between $C^{\perp}$ and $A^{\perp}$: the conclusions $C^{\perp} \otimes A^{\perp}$, $B \parr C$ are not formulas of L and are not negations of formulas of L.

Case (3): (MON3.2)



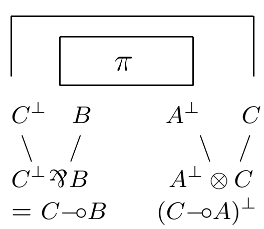
Case (4): (MON3.1)



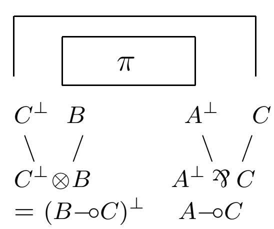
Case (3) is a CyM-PN belonging to MON-L($\pi, A^{\perp}, B, C$) and is the geometrical representation of the conclusion of the Monotonicity law MON3.2: one conclusion is the formula $C \multimapinv A$ of L and the other conclusion is the negation of the formula $C \multimapinv B$ of L, thus this CyM-PN is the geometrical representation of a proof of the sequent $C \multimapinv B \vdash C \multimapinv A$ of L.

Case (4), similarly, is a CyM-PN belonging to MON-L($\pi, A^{\perp}, B, C$): it is the geometrical representation of the conclusion of the Monotonicity law MON3.1.

Case (5): (MON2.1)



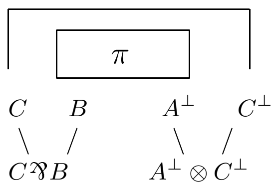
Case (6): (MON2.2)



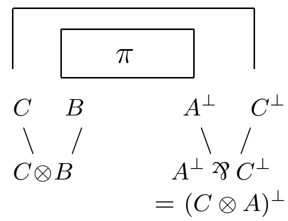
Case (5) is a CyM-PN belonging to MON-L($\pi, A^{\perp}, B, C$) and is the geometrical representation of the conclusion of the Monotonicity law MON2.1: one conclusion is the formula $C \multimap B$ of L and the other conclusion is the negation of the formula $C \multimap A$ of L, therefore this CyM-PN is the geometrical representation of a proof of the sequent $C \multimap A \vdash C \multimap B$ of L.

Case (6), by following a similar reasoning, is a CyM-PN belonging to MON-L($\pi, A^{\perp}, B, C$) and is the geometrical representation of the conclusion of the Monotonicity law MON2.1, i.e. a geometrical representation of a proof of the sequent $C \multimap A \vdash C \multimap B$ of L.

Case (7)*



Case (8): (MON1.2)



Case (7) is a CyM-PN not belonging to MON-L$(\pi, A^{\perp}, B, C)$, since it contains a $\otimes$-link between $C^{\perp}$ and $A^{\perp}$ (the conclusions are not formulas of L and are not negations of formulas of L).

Case (8) is a CyM-PN belonging to MON-L$(\pi, A^{\perp}, B, C)$ and is the geometrical representation of the conclusion of the Monotonicity law MON1.2: one conclusion is the formula $C \otimes B$ of L and the other conclusion is the negation of the formula $C \otimes A$ of L, so that the CyM-PN is the geometrical representation of a proof of the sequent $A \vdash C{\otimes}B$ of L.

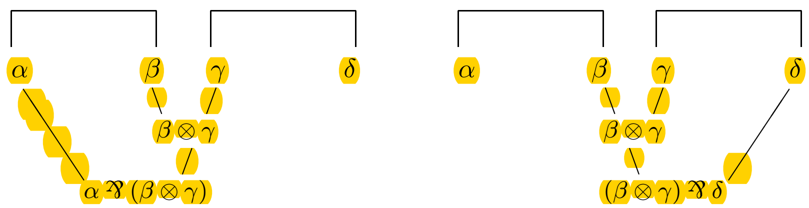
5 Geometrical Representation of Application Laws, Expansion Laws and Type-raising Laws

We will now consider *Application* laws, *Expansion* laws, and *Type-raising* laws and we will give a geometrical representation of these laws by showing the CyM-PN which corresponds to a proof of each law.

Let us consider a class of graphs ID(A, B)= $(\mathrm{ID}_A \times \mathrm{ID}_B) \cup (\mathrm{ID}_B \times \mathrm{ID}_A)$, where:

- A is a formula of L and ID_A is the axiom link with conclusions $A^{\perp}$ and A (corresponding to the axiom $A \vdash A$ of sequent calculus of L);
- B is a formula of L and ID_B is the axiom link with conclusions $B^{\perp}$ and B (corresponding to the axiom $B \vdash B$ of sequent calculus of L);
- each graph belonging to $\mathrm{ID}_A \times \mathrm{ID}_B$ is obtained from ID_A and ID_B as follows:
  - firstly, connect - by means of a $\otimes$-link - one conclusion of ID_A (first premise) with one conclusion of ID_B (second premise), so that one obtains a CyM-PN with 3 conclusions in a cyclic order;
  - secondly, connect - by means of a $⅋$-link - the conclusion of the $\otimes$-link with one of the other conclusions, by respecting the cyclic order of the conclusions, so that one obtains a CyM-PN with 2 conclusions.
- each graph belonging to $\mathrm{ID}_B \times \mathrm{ID}_A$ is obtained from ID_A and ID_B as follows:
  - firstly, connect - by means of a $\otimes$-link - one conclusion of ID_B (first premise) with one conclusion of ID_A (second premise), so that one obtains a CyM-PN with 3 conclusions in a cyclic order;
  - secondly, connect - by means of a $⅋$-link - the conclusion of the $\otimes$-link with one of the other conclusions, by respecting the cyclic order of the conclusions, so that one obtains a CyM-PN with 2 conclusions.

The graphs belonging to ID(A, B) have one of the following forms:



where one of the following cases occurs:

– α, β are A, $A^\perp$ or $A^\perp$,A and γ, δ are B, $B^\perp$ or $B^\perp$,B
– α, β are B, $B^\perp$ or $B^\perp$,B and γ, δ are A, $A^\perp$ or $A^\perp$,A

Lemma 2. *ID(A, B) has 16 elements and all the elements of ID(A, B) are CyM-PN's.*

Proof. Easy to check.

Let us consider a subclass ID-L(A, B) of ID(A, B), defined as the class of the CyM-PN's which belong to ID(A, B) and satisfy the following requirement: no conclusions $A^\perp \otimes B^\perp$ or $B^\perp \otimes A^\perp$ are admitted.

Lemma 3. *All the elements of ID-L(A, B) are CyM-PN's which are geometrical representations of proofs in L.*

Proof. In each CyM-PN belonging to ID-L(A, B) no $\otimes$-link occurs with conclusion $A^\perp \otimes B^\perp$ or $B^\perp \otimes A^\perp$, and moreover one of the conclusions of the graph is a formula of L and the other conclusion of the graph is the linear negation of a formula of L:

– if the conclusion of $\otimes$ is $A^\perp \otimes B$, then after this link the cyclic order of the 3 conclusions is $A, A^\perp \otimes B, B^\perp$, so that we get the following 2 conclusions of the graph: $A \parr (A^\perp \otimes B) = A \multimapinv (B \multimap A), B^\perp$ or $A, (A^\perp \otimes B) \parr B^\perp = (B \otimes (B \multimap A))^\perp$;
– if the conclusion of $\otimes$ is $A \otimes B^\perp$, then after this link the cyclic order of the 3 conclusions is $A^\perp, A \otimes B^\perp, B$, so that we get the following 2 conclusions of the graph: $A^\perp \parr (A \otimes B^\perp) = ((B \multimapinv A) \otimes A)^\perp, B$ or $A^\perp, (A \otimes B^\perp) \parr B = (B \multimapinv A) \multimap B$;
– if the conclusion of $\otimes$ is $A \otimes B$, then after this link the cyclic order of the 3 conclusions is $A^\perp, A \otimes B, B^\perp$, so that we get the following 2 conclusions of the graph: $A^\perp \parr (A \otimes B) = A \multimap A \otimes B, B^\perp$ or $A^\perp, (A \otimes B) \parr B^\perp = A \otimes B \multimapinv B$;
– if the conclusion of $\otimes$ is $B^\perp \otimes A$, then after this link the cyclic order of the 3 conclusions is $B, B^\perp \otimes A, A^\perp$, so that we get the following 2 conclusions of the graph: $B \parr (B^\perp \otimes A) = B \multimapinv (A \multimap B), A^\perp$ or $B, (B^\perp \otimes A) \parr A^\perp = (A \otimes (A \multimap B))^\perp$;

- if the conclusion of $\otimes$ is $B \otimes A^{\perp}$, then after this link the cyclic order of the 3 conclusions is $B^{\perp}, B \otimes A^{\perp}, A$, so that we get the following 2 conclusions of the graph: $B^{\perp} \parr (B \otimes A^{\perp}) = ((A \multimapinv B) \otimes B)^{\perp}, A$ or $B^{\perp}, (B \otimes A^{\perp}) \parr A = (A \multimapinv B) \multimap A$;
- if the conclusion of $\otimes$ is $B \otimes A$, then after this link the cyclic order of the 3 conclusions is $B^{\perp}, B \otimes A, A^{\perp}$, so that we get the following 2 conclusions of the graph: $B^{\perp} \parr (B \otimes A) = B \multimap B \otimes A, A^{\perp}$ or $B^{\perp}, (B \otimes A) \parr A^{\perp} = B \otimes A \multimapinv A$.

The following theorems says that the elements of ID-L(A, B) are the geometrical representations of Application laws, Expansion laws. Type-raising laws.

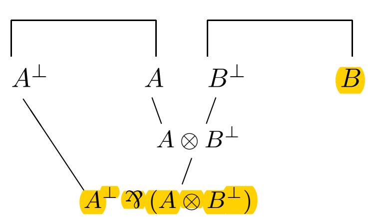
Theorem 2. *The 16 elements of ID(A, B) are:*

1. *the twelve elements of ID-L(A, B), i.e.*
   - *four CyM-PN's (two belonging to* $ID_A \times ID_B$*, two belonging to* $ID_B \times ID_A$*) which correspond to the proofs in L of the four Application laws (i.e. two Application laws with B at the left-side of the sequent, and two Application laws with A at the left side of the sequent),*
   - *four CyM-PN's (two belonging to* $ID_A \times ID_B$*, two belonging to* $ID_B \times ID_A$*)) which correspond to the proofs in L of the four Expansion laws,*
   - *four CyM-PN's (two belonging to* $ID_A \times ID_B$*, two belonging to* $ID_B \times ID_A$*)) which correspond to the four Type-raising laws (i.e. two Type-raising laws with A at the left-side of the sequent, and two Type-raising laws with B at the left side of the sequent).*
2. *other four elements which are CyM-PNs not corresponding to proofs in L.*

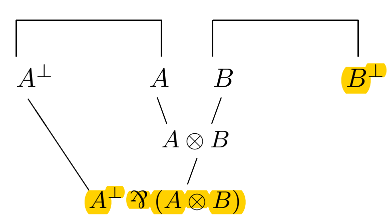
Proof. We list all the elements of ID(A, B), by showing for each element of ID-L(A, B) the corresponding law of categorial grammar:

- the cases 1-8 are the elements of $\mathrm{ID}_A \times \mathrm{ID}_B$,
- the cases 9-16 are the elements of $\mathrm{ID}_B \times \mathrm{ID}_A$.

Case (1): (APP2)

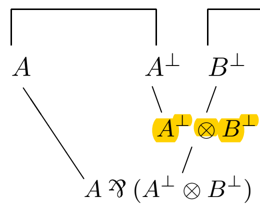


Case (2): (EXP1)

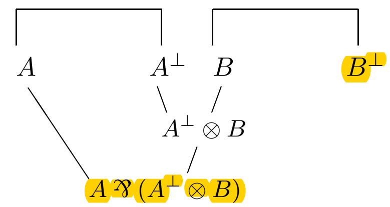


Case (1): this CyM-PN belongs to ID-L(A, B) and corresponds to a proof in L of the sequent $(A^{\perp} \parr (A \otimes B^{\perp}))^{\perp} \vdash B = (A \otimes B^{\perp})^{\perp} \otimes A \vdash B = (B \parr A^{\perp}) \otimes A \vdash B$ $= (B \multimapinv A) \otimes A \vdash B$ i.e. Application law (APP2). - Case (2): this CyM-PN belongs to ID-L(A, B) and corresponds to a proof in L of the sequent $B \vdash A^{\perp} \parr (A \otimes B)$ $= B \vdash A \multimap (A \otimes B)$ i.e. Expansion law (EXP1).

*Case (3)

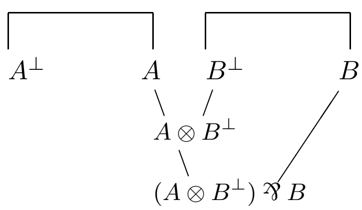


Case (4): (TYR2)

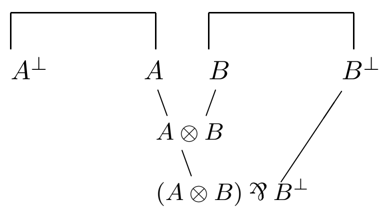


Case (3): this CyM-PN does not belong to ID-L(A, B), since it contains a $\otimes$-link with conclusion $A^\perp \otimes B^\perp$ (the conclusion B is a formula of L, whereas the other conclusion of the graph is not the linear negation of a formula of L). - Case (4): this CyM-PN belongs to ID-L(A, B) and corresponds to a proof in L of the sequent $B \vdash A \parr (A^\perp \otimes B) = B \vdash A \parr (B^\perp \parr A)^\perp = B \vdash A \multimapinv (B \multimap A)$ i.e. Type-raising law (TYR2) with B at the left-side of the sequent.

Case (5): (TYR1)

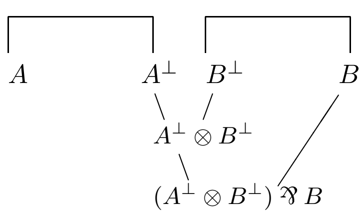


Case (6): (EXP2)

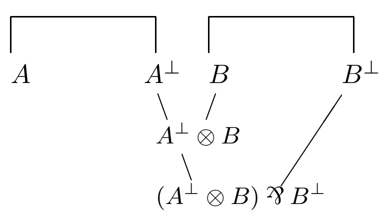


Case (5): this CyM-PN belongs to ID-L(A, B) and corresponds to a proof in L of the sequent $A \vdash (A \otimes B^\perp) \parr B = A \vdash (B \parr A^\perp)^\perp \parr B = A \vdash (B \multimapinv A) \multimap B$ i.e. (TYR1). - Case (6): this CyM-PN belongs to ID-L(A, B) and corresponds to a proof in L of the sequent $A \vdash (A \otimes B) \parr B^\perp = A \vdash (A \otimes B) \multimapinv B$ i.e. Expansion law (EXP2).

*Case (7)

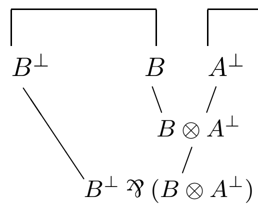


Case (8): (APP1)

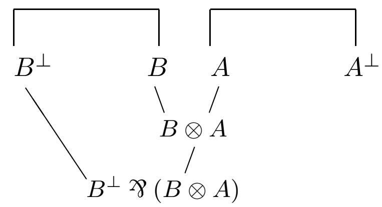


Case (7): this CyM-PN does not belong to ID-L(A, B), since it contains a $\otimes$-link with conclusion $A^\perp \otimes B^\perp$ (the conclusion A is a formula of L whereas the other conclusion of the graph is not the linear negation of a formula of L). - Case (8): this CyM-PN belongs to ID-L(A, B) and corresponds to a proof in L of the sequent $((A \otimes B) \parr B^\perp)^\perp \vdash A = B \otimes (B^\perp \parr A) \vdash A = B \otimes (B \multimap A) \vdash A$ i.e. Application law (APP1) with the formula A at the right side of the sequent.

Case (9): (APP2)

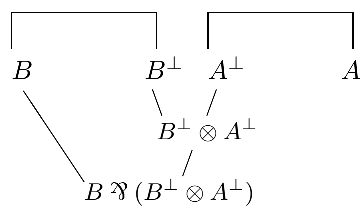


Case (10): (EXP1)

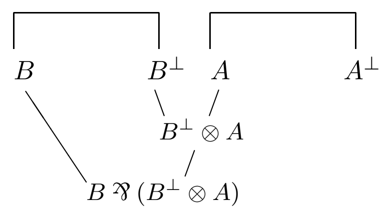


Case (9): this CyM-PN belongs to ID-L(A, B) and corresponds to a proof in L of the sequent$(B^\perp \parr (B \otimes A^\perp))^\perp \vdash A = (A \parr B^\perp) \otimes B \vdash A = (A \multimapinv B) \otimes B \vdash A$ i.e. Application law (APP2) with the formula A at the right side of the sequent. - Case (10): this CyM-PN belongs to ID-L(A, B) and corresponds to a proof in L of the sequent $A \vdash B^\perp \parr (B \otimes A) = A \vdash B \multimap (B \otimes A)$ i.e. Expansion law (EXP1).

*Case (11)

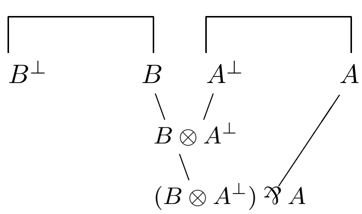


Case (12): (TYR2)

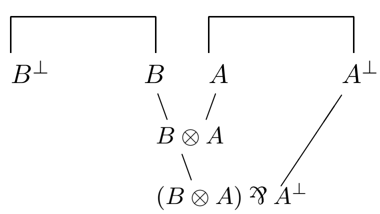


Case (11): this CyM-PN does not belong to ID-L(A, B), since it contains a $\otimes$-link with conclusion $B^\perp \otimes A^\perp$. Remark that the conclusion A is a formula of L whereas the other conclusion of the graph is not the linear negation of a formula of L. - Case (12): this CyM-PN belongs to ID-L(A, B) and corresponds to a proof in L of the sequent $A \vdash B \parr (B^\perp \otimes A) = A \vdash B \parr (A^\perp \parr B)^\perp = A \vdash B \multimapinv (A \multimap B)$ i.e. Type-raising law (TYR2).

Case (13): (TYR1)

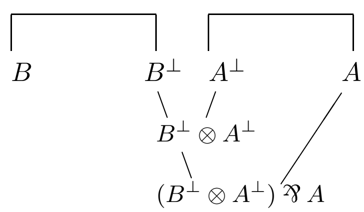


Case (14): (EXP2)

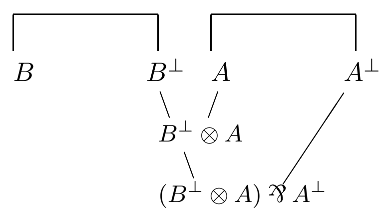


Case (13): this CyM-PN belongs to ID-L(A, B) and corresponds to a proof in L of the sequent$B \vdash (B \otimes A^\perp) \parr A = B \vdash (A \parr B^\perp)^\perp \parr A = B \vdash (A \multimapinv B) \multimap A$ i.e. Type-raising law (TYR1) with the formula B at the left side of the sequent. - Case (14): this CyM-PN belongs to ID-L(A, B) and corresponds to a proof in L of the sequent$B \vdash (B \otimes A) \parr A^\perp = B \vdash (B \otimes A) \multimapinv A$ i.e. Expansion law (EXP2).

*Case (15)



Case (16): (APP1)



Case (15): this CyM-PN does not belong to ID-L(A, B), since it contains a $\otimes$-link with conclusion $B^\perp \otimes A^\perp$ (the conclusion B is a formula of L whereas the other conclusion is not the linear negation of a formula of L). - Case (16): this CyM-PN belongs to ID-L(A, B) and corresponds to a proof in L of the sequent $((B^\perp \otimes A) \parr A^\perp)^\perp \vdash B = A \otimes (A^\perp \parr B) \vdash B = \ A \otimes (A \multimap B) \vdash B$ i.e. Application law (APP1).

6 Geometrical representation of Composition Laws, Geach Laws and Switching Laws

We will now consider another class of graphs, SYL(C, B, A). A graph belongs to the class SYL (C, B, A) iff:

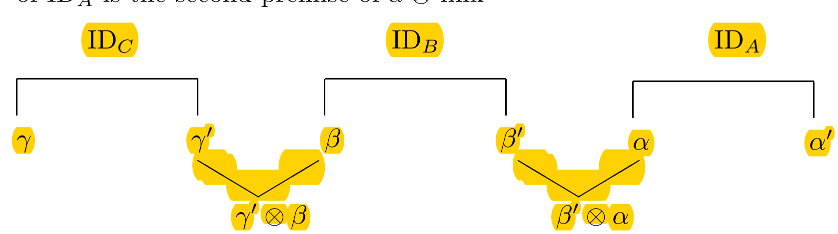
- A, B, C are formulas of MLL which are also formulas of L;
- the graph is constructed by starting from three Axiom links (ID_A, ID_B, ID_C) as follows: there is a $\otimes$-link whose premises are one of the conclusions of ID_C and one of the conclusions of ID_B, there is a $\otimes$-link whose premises are the other conclusion of ID_B and one of the conclusions of ID_A
- the conclusions of the graph are the conclusions of the two $\otimes$-links, one of the conclusions of ID_A and one of the conclusions of ID_C.

Lemma 4. *Each element of SYL(C, B, A) is a CyM-PN.*

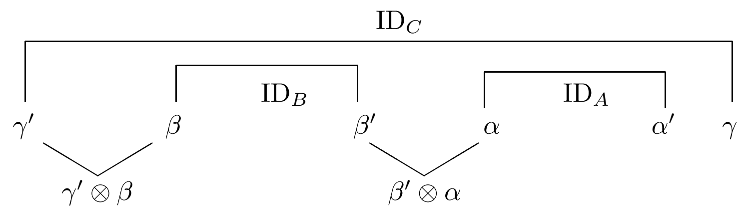
Proof. Easy to check.

When α, α' are the conclusions of ID_A, β, β' are the conclusions of ID_B and γ, γ' are the conclusions of ID_C, each element of SYL(C, B, A) belongs to one of the following cases:

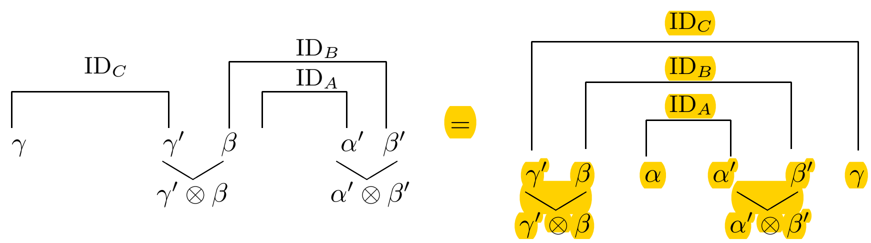
- Case 1: a conclusion of ID_C is the first premise of a $\otimes$-link, and a conclusion of ID_A is the second premise of a $\otimes$-link



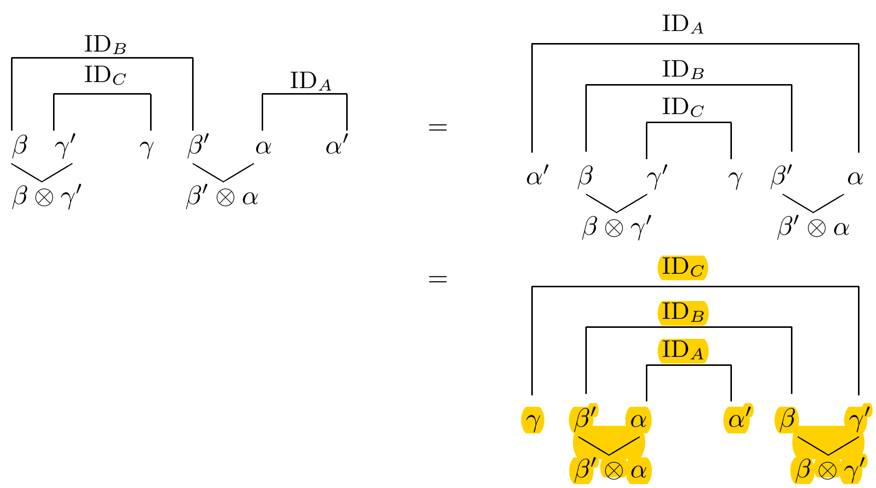
i.e.



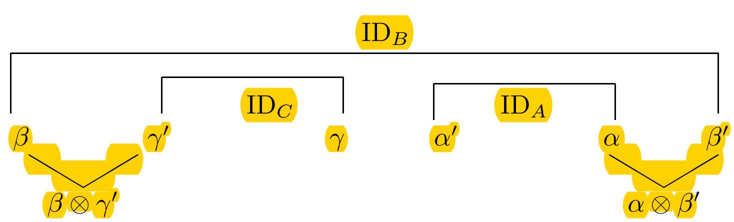
– Case 2: a conclusion of ID_C is the first premise of a $\otimes$-link, and a conclusion of ID_A is the first premise of a $\otimes$-link:



– Case 3: one of the conclusions of ID_C is the second premise of a $\otimes$-link, and one of the conclusions of ID_A is the second premise of a $\otimes$-link:



– Case 4: one of the conclusions of ID_C is the second premise of a $\otimes$-link, and one of the conclusions of ID_A is the first premise of a $\otimes$-link:



So, this case can be reduced to case (1).

As it is shown in [4], Aristotle's Syllogisms may be represented by means of graphs belonging to this class (we have only to connect through a ⅋-link the terminal points which are not conclusions of ⊗-links).

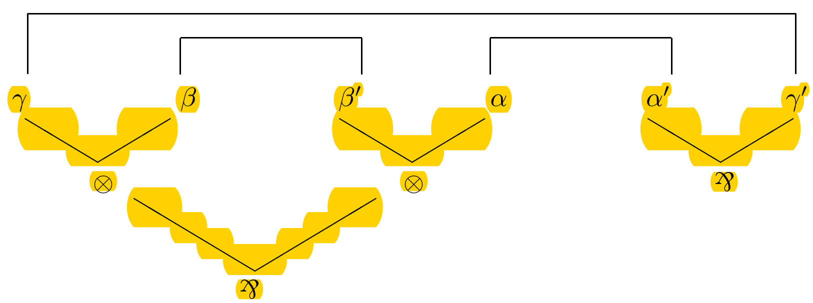
The geometrical representation of Composition laws and Geach laws is given by CyM-PN's which are obtained by adding two ⅋-links to CyM-PN's belonging to SYL(C, B, A) - case 1, whereas the geometrical representation of Switching laws is given by CyM-PN's which are obtained by adding two ⅋-links to CyM-PN's belonging to SYL(C, B, A) - case 2.

6.1 Composition Laws

We introduce a class of graphs COM(C, B, A) which are obtained from a graph belonging to the case 1 of SYL(C, B, A) by adding:

- a ⅋-link where the first premise is the conclusion of the ⊗-link connecting one of the conclusions of ID$(_C)$ and the second premise is the conclusion of the ⊗-link connecting one of the conclusions of ID$(_A)$
- a ⅋-link whose first premise is the other conclusion of ID$(_A)$ and the second premise is the other conclusion of ID$(_C)$.

Each graph belonging to COM(C, B, A) looks as follows:



where α, α' are the conclusions of ID$_A$, β, β' are the conclusions of ID$_B$ and γ, γ' are the conclusions of ID$_C$.

COM-L(C, B, A) is the subclass of COM(C, B, A), whose elements satisfy the following requirement: no ⊗-link has the formula $C^\perp \otimes B^\perp$ or the formula $B^\perp \otimes A^\perp$ as conclusion.

The following theorem states that the elements of COM-L(C, B, A) are geometrical representations of the proofs in L of *Composition* laws.

Theorem 3. *The eight elements of COM(C, B, A) are:*

1. *the four elements of COM-L(C, B, A) which are CyM-PN's and correspond to proofs in L of the two Composition laws ((COM1) and (COM2)) and two other laws ((COM*1) and (COM*2)),*
2. *four CyM-PN's which do not correspond to proofs in L.*

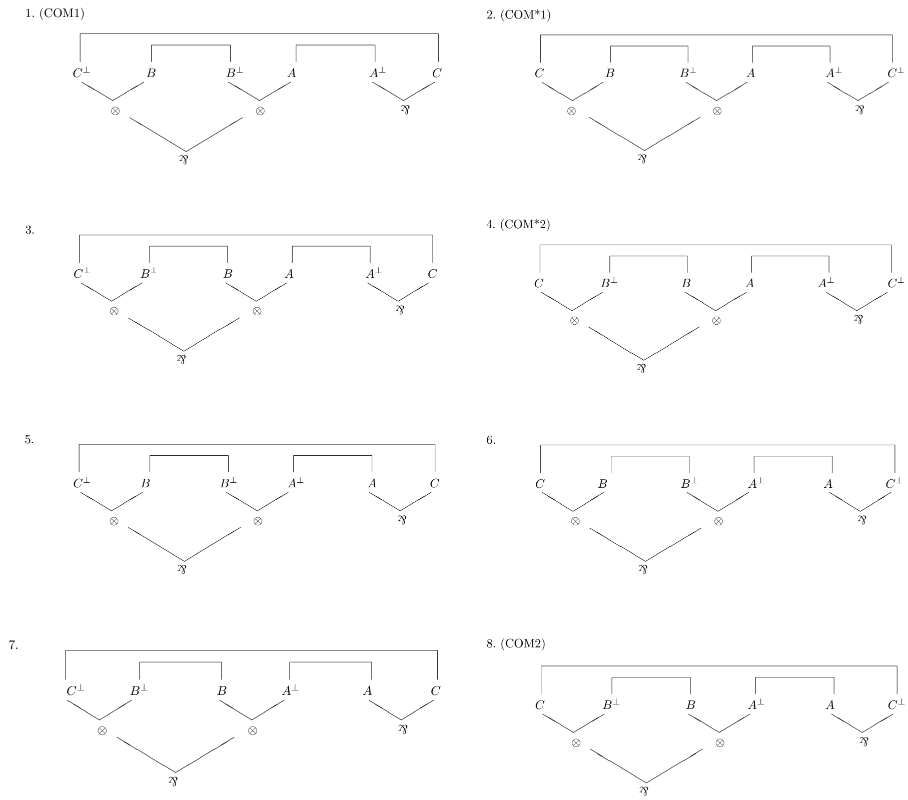


Fig. 1. Composition laws

Proof. It is easy to check that all the elements of COM(C, B, A) are CyM-PN's. We list all the elements of COM(C, B, A), presented in Figure 1, by giving, for each element of COM-L(C, B, A), the sequent which is proved in L.

PANEL 1: COM1. The conclusions of this CyM-PN belonging to COM-L(C, A, B) are: $(C^{\perp} \otimes B) \parr (B^{\perp} \otimes A)$, the linear negation of the formula of L $(A \multimap B) \otimes (B \multimap C)$, and $A^{\perp} \parr C$ i.e. $A \multimap C$, a formula of L. Thus this CyM-PM corresponds to a proof in L of the sequent $(A \multimap B) \otimes (B \multimap B) \vdash A \multimap C$, i.e. (COM1).

PANEL 2: COM*1. The conclusions of this CyM-PN belonging to COM-L(C, A, B) are: $(C \otimes B) \parr (B^{\perp} \otimes A)$ which is the formula of L $C \otimes B \multimapinv (A \multimap B)$, and $A^{\perp} \parr C^{\perp}$ which is the linear negation of the formula of L $C \otimes A$. Thus this CyM-PN corresponds to a proof in L of the sequent $C \otimes A \vdash C \otimes B \multimapinv (A \multimap B)$, and this sequent is very similar to the sequents introducing Composition laws.

PANEL 3. This CyM-PN does not belong to COM-L(C, A, B) since there is a $\otimes$-link with conclusion $C^{\perp} \otimes B^{\perp}$ (no conclusion of this CyM-PN is the linear negation of a formula of L).

PANEL 4: COM*2. The conclusions of this CyM-PN belonging to COM-L(C, A, B) are: $(C \otimes B^{\perp}) \parr (B \otimes A)$ which is the formula of L $(B \circ\!- C) -\!\circ B \otimes A$, and $A^{\perp} \parr C^{\perp}$ which is the linear negation of the formula of L $C \otimes A$. Thus this CyM-PM corresponds to a proof in L of the sequent $C \otimes A \vdash (B \circ\!- C) -\!\circ B \otimes A$, a sequent of L which is very close to Composition laws.

PANEL 5. This CyM-PN does not belong to COM-L(C, A, B) since there is a $\otimes$-link with conclusion $B^{\perp} \otimes A^{\perp}$ (no conclusion is a formula of L).

PANEL 6. This CyM-PN does not belong to COM-L(C, A, B) since there is a $\otimes$-link with conclusion $B^{\perp} \otimes A^{\perp}$ (no conclusion is the linear negation of a formula of L).

PANEL 7. This CyM-PN does not belong to COM-L(C, A, B) since there is a $\otimes$-link with conclusion $C^{\perp} \otimes B^{\perp}$ (no conclusion is the linear negation of a formula of L).

PANEL 8: (COM2). The conclusions of the CyM-PN belonging to COM-L(C, A, B) are: $(C \otimes B^{\perp}) \parr (B \otimes A^{\perp})$ which is the linear negation of the formula of L $(A \circ\!- B) \otimes (B \circ\!- C)$, and $A \parr C^{\perp}$ i.e. $A \circ\!- C$ which is a formula of L. Thus this CyM-PM corresponds to a proof in L of the sequent $(A \circ\!- B) \otimes (B \circ\!- C) \vdash A \circ\!- C$, i.e. it corresponds to a proof of (COM2).

Therefore, by means of CyM-PN's belonging to COM-L(C, B, A), we get geometrical representations of proofs in L of the following sequents which have to be considered as *Composition laws*:

- the law (COM1) $(A -\!\circ B) \otimes (B -\!\circ C) \vdash A -\!\circ C$,
- the law (COM2) $(A \circ\!- B) \otimes (B \circ\!- C) \vdash A \circ\!- C$,
- two other laws strictly related to the previous ones, at least from a geometrical point of view:
  - (COM*1) $C \otimes A \vdash C \otimes B \circ\!- (A -\!\circ B)$,
  - (COM*2) $C \otimes A \vdash (B \circ\!- C) -\!\circ B \otimes A$.

6.2 Geach Laws

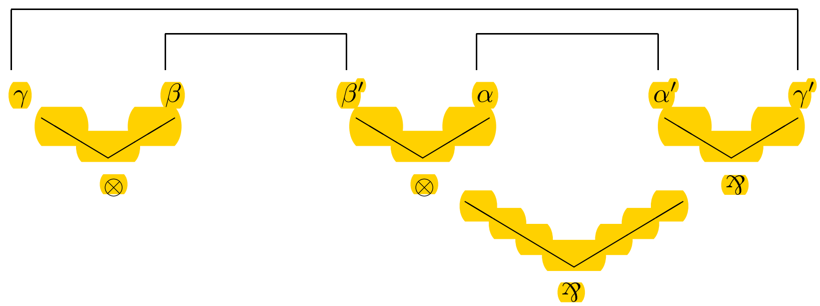
We introduce a class of graphs GEA(C, B, A), defined as follows:

- GEA(C, B, A) = GEA1$(C, B, A) \cup$ GEA2(C, B, A)
- A graph belongs to GEA1(C, B, A) iff it is constructed by starting from SYL(C, B, A) - case 1 and by adding:
  - a $\parr$-link whose first premise is the conclusion of ID$(_A)$ which is not the premise of a $\otimes$-link and the second premise is the conclusion of ID$(_C)$ which is not the premise of a $\otimes$-link;
  - a $\parr$-link where the first premise is the conclusion of the $\otimes$-link connecting one of the conclusions of ID$(_B)$ and the second premise is the conclusion of the $\parr$-link above described.
- A graph belongs to GEA2(C, B, A) iff it is constructed by starting from SYL(C, B, A) - case 1 and by adding:
  - a $\parr$-link whose first premise is the conclusion of ID$(_A)$ which is not the premise of a $\otimes$-link and the second premise is the conclusion of ID$(_C)$ which is not the premise of a $\otimes$-link;

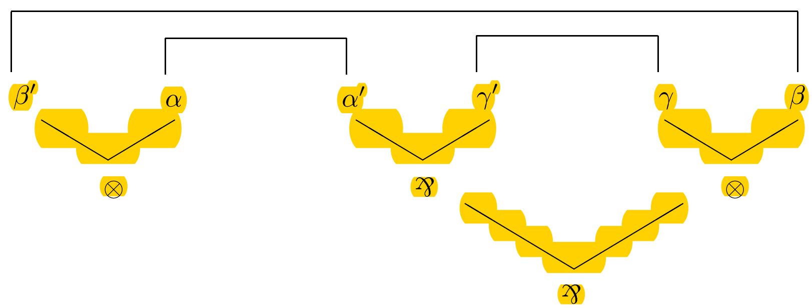
- a ⅋-link where the first premise is the conclusion of the ⅋-link above described and the second premise is the conclusion of the ⊗-link connecting one of the conclusions of $\mathrm{ID}(_B)$.

Remark that, when α, α' are the conclusions of ID_A, β, β' are the conclusions of ID_B and γ, γ' are the conclusions of ID_C,

– each graph belonging to GEA1(C, B, A) looks as follows:



– whereas each graph belonging to GEA2(C, B, A) looks as follows:



Lemma 5. *Every graph belonging to GEA(C, B, A) is a CyM-PN.*

Proof. Easy to check.

GEA-L(C, B, A) is the subclass of GEA(C, B, A) , whose elements satisfy the following requirement: no ⊗-link has as conclusion the formula $C^\perp \otimes B^\perp$ or the formula $B^\perp \otimes A^\perp$.

The following theorem says that the elements of GEA-L(C, B, A) are geometrical representations of Geach laws.

Theorem 4. *The elements of GEA(C, B, A) - eight elements of GEA1(C, B, A) (cf. Fig. 2) and eight elements of GEA2(C, B, A)) (cf. Fig. 3) - are:*

1. *the eight elements of GEA-L(C, B, A) (four inside GEA1(C, B, A), four inside GEA2(C, B, A)) which correspond to proofs in L of the two Geach laws ((GEA1) and (GEA2)) and six other laws strictly related to Geach laws,*
2. *eight CyM-PN's which do not correspond to proofs in L.*

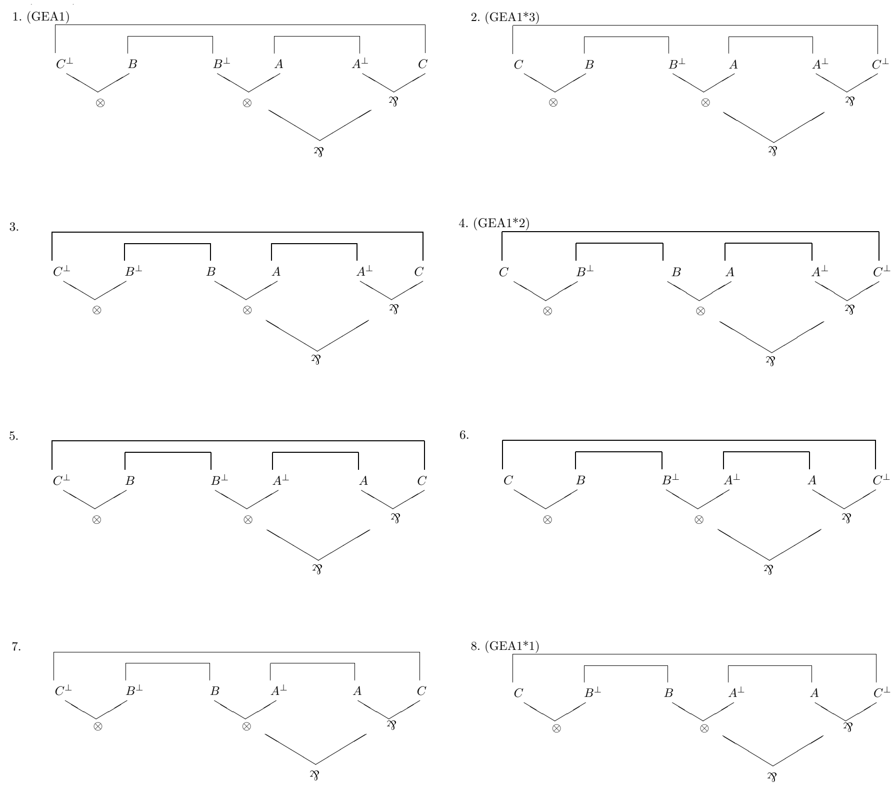


Fig. 2. Geach laws: GEA1(C, B, A)

Proof. We list all the elements of GEA(C, B, A), by showing for each element of GEA-L(C, B, A) the sequent of L which is proved. The first 8 CyM-PN's are the elements of GEA1(C, B, A), shown in Figure 2, and the other 8 CyM-PN's are the elements of GEA2(C, B, A), shown in Figure 3.

Panel 1: GEA1. The conclusions of this CyM-PN belonging to GEA-L(C, A, B) are: $C^\perp \otimes B$ which is the linear negation of the formula of L $B \multimap C$ and $(B^\perp \otimes A) \parr (A^\perp \parr C)$ which is the formula of L $(A \multimap B) \multimap (A \multimap C)$. Thus this CyM-PM corresponds to a proof in L of $B \multimap C \vdash (A \multimap B) \multimap (A \multimap C)$, i.e. Geach law (GEA1).

Panel 2: GEA1*3. The conclusions of this CyM-PN belonging to GEA-L(C, A, B) are: $C \otimes B$ which is a formula of L and $(B^\perp \otimes A) \parr A^\perp \parr C^\perp$ which is the linear negation of the formula of L $(C \otimes A) \otimes (A \multimap B)$. Thus this CyM-PM corresponds to a proof in L of $(C \otimes A) \otimes (A \multimap B) \vdash C \otimes B$, i.e. a poof in L of a sequent of L which is very close to Geach law (GEA1); we propose to call this sequent (GEA1*3).

Panel 3. This CyM-PN does not belong to COM-L(C, A, B) since there is a $\otimes$-link with conclusion $C^\perp \otimes B^\perp$ (no conclusion is a formula of L).

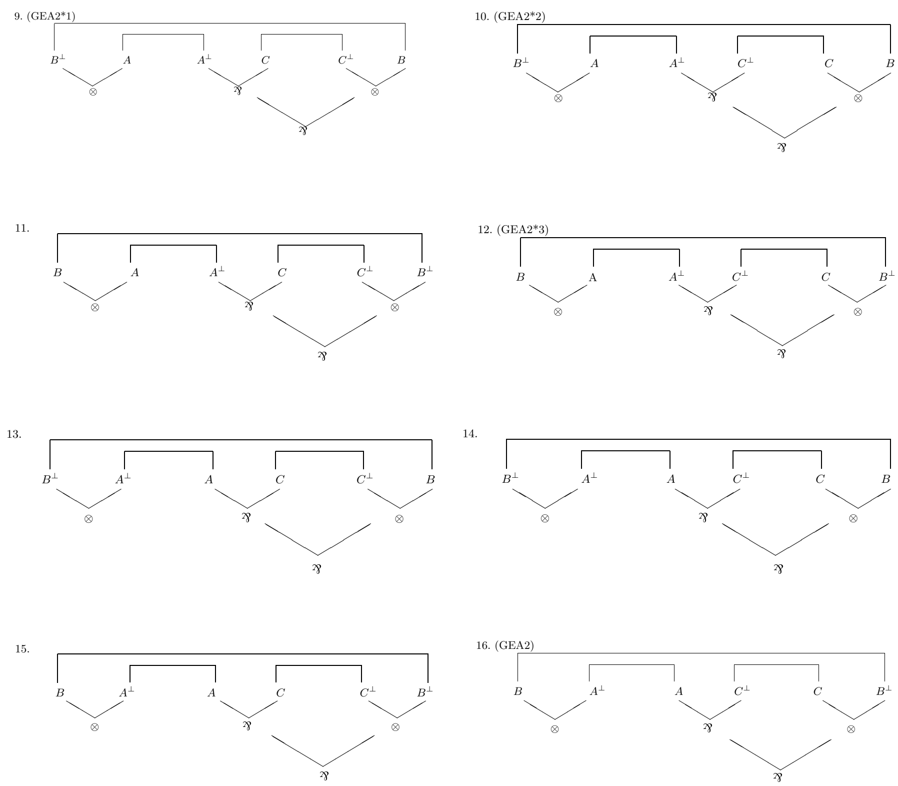


Fig. 3. Geach laws: GEA2(C, B, A)

Panel 4: GEA1*2. The conclusions of this CyM-PN belonging to GEA-L(C, A, B) are: $C \otimes B^{\perp}$, which is the linear negation of the formula of L $B \circ\!- C$, and $(B \otimes A) \parr (A^{\perp} \parr C^{\perp})$ i.e. the formula of L $(B \otimes A) \circ\!- (C \otimes A)$. Thus this CyM-PM corresponds to a proof in L of $B \circ\!- C \vdash B \otimes A) \circ\!- (C \otimes A)$, a sequent of L which is close to the Geach law (GEA1); we propose to call this sequent (GEA1*2).

Panel 5. This CyM-PN does not belong to GEA-L(C, A, B) since there is a $\otimes$-link with conclusion $B^{\perp} \otimes A^{\perp}$ (no conclusion is a formula of L).

Panel 6. This CyM-PN does not belong to GEA-L(C, A, B) since there is a $\otimes$-link with conclusion $B^{\perp} \otimes A^{\perp}$ (no conclusion is the linear negation of a formula of L).

Panel 7. This CyM-PN does not belong to GEA-L(C, A, B) since there is a $\otimes$-link with conclusion $C^{\perp} \otimes B^{\perp}$ (no conclusion is a formula of L).

Panel 8: GEA1*1. The conclusions of this CyM-PN belonging to GEA-L(C, A, B) are: $C \otimes B^{\perp}$ which is linear negation of the formula of L $B \circ\!- C$, and $(B \otimes A^{\perp}) \parr (A \parr C^{\perp})$ which is the formula of L $(A \circ\!- B) -\!\circ (A \circ\!- C)$. Thus this CyM-PN corresponds to a proof in L of $B \circ\!\!- C \vdash (A \circ\!\!- B) -\!\!\circ (A \circ\!\!- C)$, i.e. a sequent very similar to the Geach law (GEA1); we propose to name this sequent (GEA1*1).

Panel 9: GEA2*1. The conclusions of this CyM-PN belonging to GEA-L(C, A, B) are: $B^\perp \otimes A$ which is the linear negation of the formula of L $A -\!\!\circ B$, and $(A^\perp ⅋ C) ⅋ (C^\perp \otimes B)$ which is the formula of L $(A -\!\!\circ C) \circ\!\!- (B -\!\!\circ C)$. Thus this CyM-PM corresponds to a proof in L of $A -\!\!\circ B \vdash (A -\!\!\circ C) \circ\!\!- (B -\!\!\circ C)$, i.e. a sequent of L very close to the Geach law (GEA2); we name this sequent (GEA2*1).

Panel 10: GEA2*2. The conclusions of this CyM-PN belonging to GEA-L(C, A, B) are: $B^\perp \otimes A$ which is the linear negation of the formula of L $A -\!\!\circ B$, and $(A^\perp ⅋ C^\perp) ⅋ (C \otimes B)$ which is the formula of L $(C \otimes A) -\!\!\circ (C \otimes B)$. Thus this CyM-PM corresponds to a proof in L of $A -\!\!\circ B \vdash (C \otimes A) -\!\!\circ (C \otimes B)$, a sequent of L very close to the Geach law (GEA2) ; we choose to name this sequent (GEA2*2).

Panel 11. This CyM-PN does not belong to GEA-L(C, A, B) since there is a $\otimes$-link with conclusion $C^\perp \otimes B^\perp$ (no conclusion is the negation of a formula of L).

Panel 12: GEA2*3. The conclusions of this CyM-PN belonging to GEA-L(C, A, B) are $B \otimes A$ and $(A^\perp ⅋ C^\perp) ⅋ (C \otimes B^\perp)$ (the linear negation of $(B \circ\!\!- C) \otimes (C \otimes A)$). This CyM-PM corresponds to a proof in L of $(B \circ\!\!- C) \otimes (C \otimes A) \vdash (B \otimes A)$, a sequent very close to the Geach law (GEA2); we name this sequent (GEA2*3).

Panel 13, 14. These CyM-PN's do not belong to GEA-L(C, A, B) since there is a $\otimes$-link with conclusion $B^\perp \otimes A^\perp$ (no conclusion is the negation of a formula of L).

Panel 15. This CyM-PN does not belong to GEA-L(C, A, B) since there is a $\otimes$-link with conclusion $C^\perp \otimes B^\perp$ (no conclusion is a formula of L).

Panel 16: GEA2. The conclusions of this CyM-PN belonging to GEA-L(C, A, B) are: $B \otimes A^\perp$ which is the linear negation of the formula of L $A \circ\!\!- B$, and $(A ⅋ C^\perp) ⅋ (C \otimes B^\perp)$ which is the formula of L $(A \circ\!\!- C) \circ\!\!- (B \circ\!\!- C)$. Thus this CyM-PM corresponds to a proof in L of $A \circ\!\!- B \vdash (A \circ\!\!- C) \circ\!\!- (B \circ\!\!- C)$, i.e. the Geach law (GEA2).

Therefore, by means of CyM-PN's belonging to GEA-L(C, B, A), we get geometrical representations of proofs in L of the following sequents which have to be considered as *Geach laws*:

- the law (GEA1) $B -\!\!\circ C \vdash (A -\!\!\circ B) -\!\!\circ (A -\!\!\circ C)$,
- the law (GEA2) $A \circ\!\!- B \vdash (A \circ\!\!- C) \circ\!\!- (B \circ\!\!- C)$,
- six other laws strictly related to the previous ones, at least from a geometrical point of view:
  - (GEA1*1) $B \circ\!\!- C \vdash (A \circ\!\!- B) -\!\!\circ (A \circ\!\!- C)$,
  - (GEA1*2) $B \circ\!\!- C \vdash (B \otimes A) \circ\!\!- (C \otimes A)$,
  - (GEA1*3) $(C \otimes A) \otimes (A -\!\!\circ B) \vdash C \otimes B$,
  - (GEA2*1) $A -\!\!\circ B \vdash (A -\!\!\circ C) \circ\!\!- (B -\!\!\circ C)$,
  - (GEA2*2) $A -\!\!\circ B \vdash C \otimes A -\!\!\circ C \otimes B$,
  - (GEA2*3) $(B \circ\!\!- C) \otimes (C \otimes A) \vdash B \otimes A$.

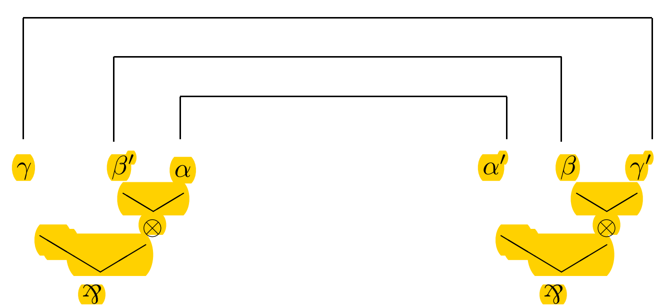
6.3 Switching laws

We introduce a class of graphs $\mathrm{SWI}(C,B,A)$, defined as follows:

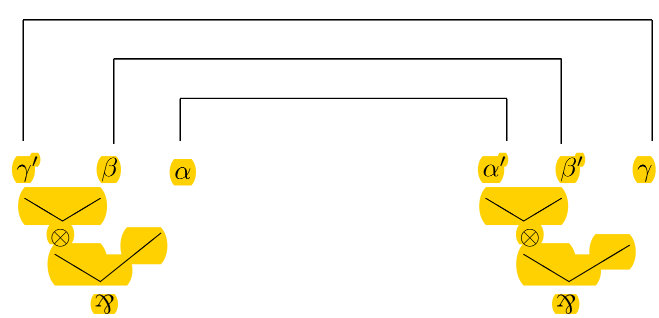
- $\mathrm{SWI}(C,B,A) = \mathrm{SWI1}(C,B,A) \cup \mathrm{SWI2}(C,B,A)$
- A graph belongs to $\mathrm{SWI1}(C,B,A)$ iff it is constructed by starting from $\mathrm{SYL}(C,B,A)$ - case 3 and by adding:
  - a $\parr$-link whose first premise is the conclusion of $\mathrm{ID}({}_C)$ which is not the premise of a $\otimes$-link and the second premise,
  - a $\parr$-link where the first premise is the conclusion of the $\mathrm{ID}({}_A)$ which is not the premise of a $\otimes$-link, and the second premise is the conclusion of the $\otimes$-link connecting one of the conclusions of $\mathrm{ID}({}_C)$
- A graph belongs to $\mathrm{SWI2}(C,B,A)$ iff it is constructed by starting from $\mathrm{SYL}(C,B,A)$ - case 2 and by adding:
  - a $\parr$-link whose first premise is the conclusion of the $\otimes$-link connecting one of the conclusions of $\mathrm{ID}({}_C)$ and the second premise is the conclusion of $\mathrm{ID}({}_A)$ which is not the premise of a $\otimes$-link;
  - a $\parr$-link where the first premise is the conclusion of the $\otimes$-link connecting one of the conclusions of $\mathrm{ID}({}_A)$ and the second premise is the conclusion of $\mathrm{ID}({}_C)$ which is not the premise of a $\otimes$-link.

Remark that, when α, α' are the conclusions of ID_A, β, β' are the conclusions of ID_B and γ, γ' are the conclusions of ID_C

- each graph belonging to $\mathrm{SWI1}(C,B,A)$ looks as follows:



- each graph belonging to $\mathrm{SWI2}(C,B,A)$ looks as follows:



Lemma 6. *every graph belonging to SWI*(C, B, A) *is a CyM-PN.*

Proof. Easy to check

SWI-L(C, B, A) is the subclass of SWI(C, B, A), whose elements are graphs where no $\otimes$-link has as conclusion one of these formulas: $C^\perp \otimes B^\perp$, $B^\perp \otimes A^\perp$, $A^\perp \otimes B^\perp$, $B^\perp \otimes C^\perp$.

The following theorem says that SWI-L(C, B, A) gives the geometrical representations of Switching laws.

Theorem 5. *The elements of SWI*(C, B, A) *(the eight elements of SWI1*(C, B, A) *and the eight elements of SWI2*(C, B, A)*) are:*

1. *the eight elements of SWI-L*(C, B, A) *(four inside SWI1*(C, B, A) *and four inside SWI2*(C, B, A)*) which corresponds to proofs in L of the two Switching laws ((SWI1) and (SWI2), in 2 different formulations) and to the proofs in L of four sequents strictly related to Switching laws,*
2. *eight CyM-PN which do not correspond to proofs in L.*

Proof. We list all the elements of SWI(C, B, A), by showing for each element of SWI-L(C, B, A) the sequent of L which is proved. The first 8 CyM-PN's belong to SWI1(C, B, A) (see Figure 4) and the other 8 belong to SWI2(C, B, A) (see Figure 5).

Panel 1.This graph does not belong to SWI-L(C, B, A), since there is a $\otimes$-link with conclusion $B^\perp \otimes A^\perp$ (no conclusion is a formula of L).

Panel 2. This graph does not belong to SWI-L(C, B, A), since there is a $\otimes$-link with conclusion $B^\perp \otimes A^\perp$ (no conclusion is a formula of L).

Panel 3: SWI1*1. This CyM-PN belongs to SWI-L(C, B, A) and its conclusions are: $C^\perp \parr (B \otimes A^\perp)$ i.e. the linear negation of the formula of L $(A \circ\!\!- B) \otimes C$, $A \parr (B^\perp \otimes C$ i.e. the formula of L $A \circ\!\!- (C \multimap B)$. Thus, this graph corresponds to a proof in L of $(A \circ\!\!- B) \otimes C \vdash A \circ\!\!- (C \multimap B)$, a sequent which is close to the Switching law (SWI1); we call this law (SWI1*1).

Panel 4.This graph does not belong to SWI-L(C, B, A), since there is a $\otimes$-link with conclusion $B^\perp \otimes C^\perp$ (no conclusion is the linear negation of a formula of L).

Panel 5: SWI1. This CyM-PN belongs to SWI-L(C, B, A) and its conclusions are: $C^\perp \parr (B^\perp \otimes A)$ which is the linear negation of the formula of L $(A \multimap B) \otimes C$, $A^\perp \parr B \otimes C$ which is the formula of L $A \multimap B \otimes C$. Thus, this graph corresponds to a proof in L of $(A \multimap B) \otimes C \vdash A \multimap B \otimes C$ i.e. (SWI1).

Panel 6: SWI1*1. This CyM-PN belongs to SWI-L(C, B, A), and its conclusions are: $C \parr (B^\perp \otimes A)$ i.e. the formula of L $C \circ\!\!- (A \multimap B)$, $A^\perp \parr B \otimes C^\perp$ which is the linear negation of the formula of L $(C \circ\!\!- B) \otimes A$. Thus, this graph corresponds to a proof in L of $(C \circ\!\!- B) \otimes A \vdash C \circ\!\!- (A \multimap B)$, another formulation of the new Switching law (SWI1*1).

Panel 7: SWI1. This CyM-PN belongs to SWI-L(C, B, A), and its conclusions are: $C^\perp \parr (B \otimes A)$ which is the formula of L $C \multimap B \otimes A$, $A^\perp \parr B^\perp \otimes C$ which is the linear

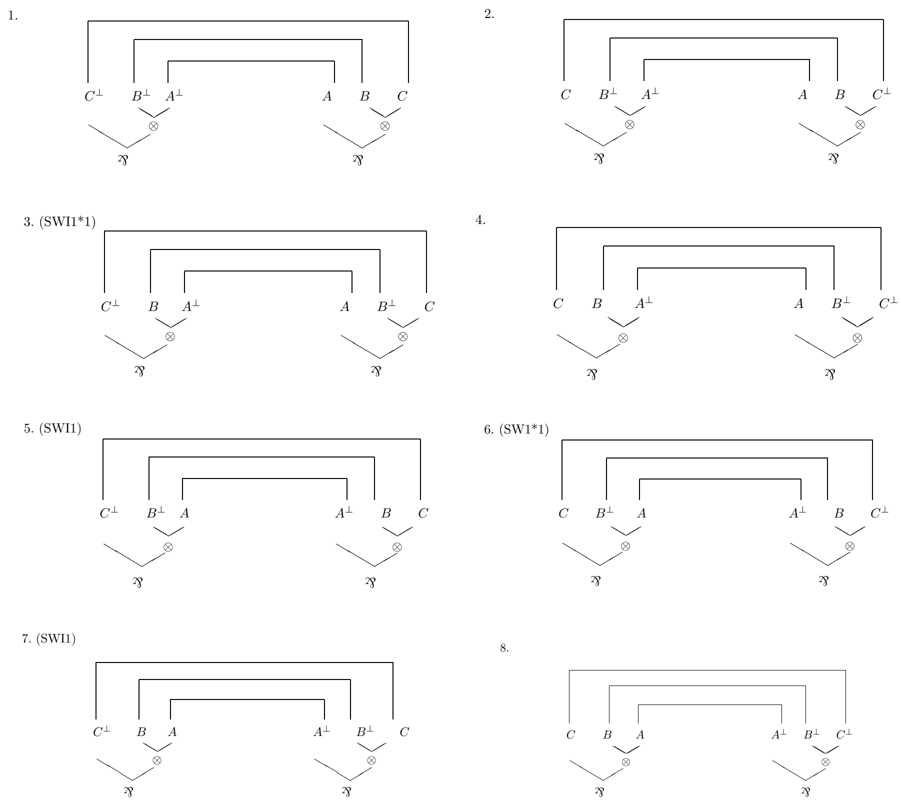


Fig. 4. Switching laws: SWI1(C, B, A)

negation of the formula of L $(C \multimap B) \otimes A$. Thus, this graph corresponds to a proof in L of $(C \multimap B) \otimes A \vdash C \multimap B \otimes A$ i.e. (SWI1).

Panel 8. This graph does not belong to SWI-L(C, B, A), since there is a $\otimes$-link with conclusion $B^{\perp} \otimes C^{\perp}$ (no conclusion is a formula of L).

Panel 9. This CyM-PN does not belong to SWI-L(C, A, B) since it contains a $\otimes$-link with conclusion $C^{\perp} \otimes B^{\perp}$; and no conclusion is a formula of L.

Panel 10: SWI2. This CyM-PN belongs to SWI-L(C, B, A), and its conclusions are: $(C \otimes B^{\perp}) \parr A^{\perp})$ which is the linear negation of the formula of L $A \otimes (B \multimapinv C)$, $(A \otimes B) \parr C^{\perp}$ which is the formula of L $(A \otimes B) \multimapinv C$. Thus, this graph corresponds to a proof in L of $A \otimes (B \multimapinv C) \vdash (A \otimes B) \multimapinv C$ i.e. (SWI2).

Panel 11: SWI2*1. This CyM-PN belongs to SWI-L(C, B, A), and its conclusions are: $(C^{\perp} \otimes B) \parr A^{\perp})$ which is the linear negation of the formula of L $A \otimes (B \multimap C)$, $(A \otimes B^{\perp}) \parr C$ which is the formula of L $(B \multimapinv A) \multimap C$. Thus, this graph corresponds to a proof in L of $A \otimes (B \multimap C) \vdash (B \multimapinv A) \multimap C$ i.e. a sequent very close to the Switching law (SWI2); we will call it (SWI2*1).

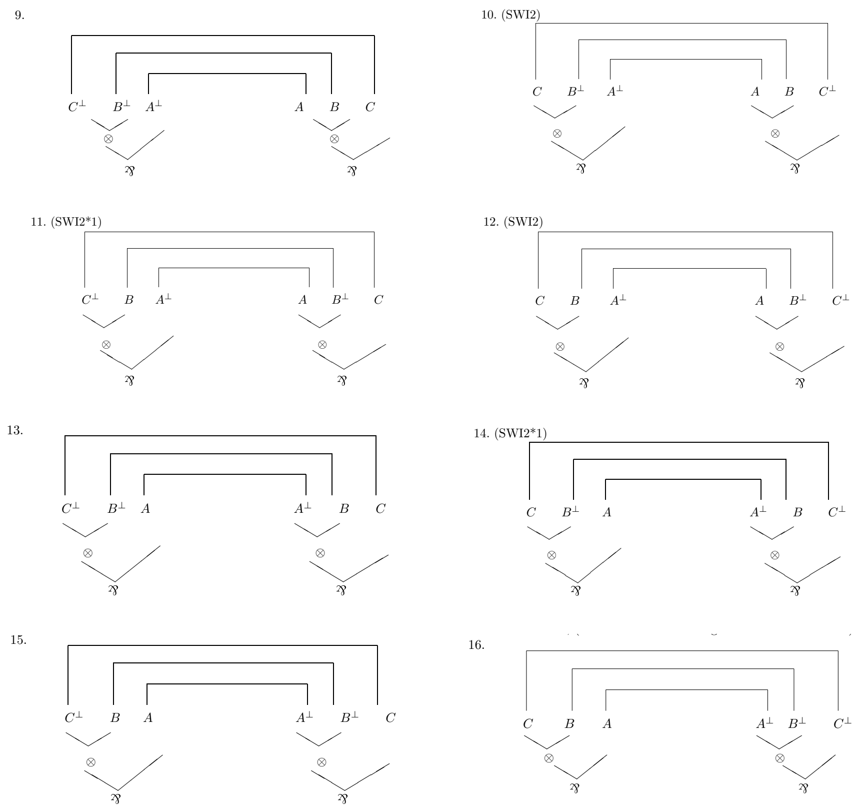


Fig. 5. Switching laws: SWI2(C, B, A)

Panel 12: SWI2. This CyM-PN belongs to SWI-L(C, B, A), and its conclusions are: $(C \otimes B) \parr A^{\perp}$ i.e. the formula of L $C \otimes (B \circ\!- A)$ and $(A \otimes B^{\perp}) \parr C^{\perp}$ i.e. the linear negation of the formula of L $C \otimes (B -\!\circ A)$. Thus, this graph corresponds to a proof in L of $C \otimes (B \circ\!- A) \vdash C \otimes B \circ\!- A$ i.e. (SWI2).

Panel 13. This CyM-PN does not belong to SWI-L(C, A, B) since it contains a $\otimes$-link with conclusion $C^{\perp} \otimes B^{\perp}$ (no conclusion is the negation of a formula of L).

Panel 14: SWI2*1. This CyM-PN belongs to SWI-L(C, B, A), and its conclusions are: $(C \otimes B^{\perp}) \parr A$ which is the formula of L $(B \circ\!- C) -\!\circ A$, $(A^{\perp} \otimes B) \parr C^{\perp}$ which is the linear negation of the formula of L $C \otimes (B -\!\circ A)$. Thus, this graph corresponds to a proof in L of $C \otimes (B -\!\circ A) \vdash (B \circ\!- C) -\!\circ A$ i.e. another formulation of the new Switching law (SWI2*1).

Panel 15. This CyM-PN does not belong to SWI-L(C, A, B) since it contains a $\otimes$-link with conclusion $A^{\perp} \otimes B^{\perp}$ (no conclusion is the negation of a formula of L).

Panel 16. This CyM-PN does not belong to SWI-L(C, A, B) since it contains a $\otimes$-link with conclusion $A^{\perp} \otimes B^{\perp}$ (no conclusion is a formula of L).

Therefore, by means of the CyM-PN's belonging to SWI-L(C, B, A), we get the geometrical representations of proofs in L of the following sequents that are all forms of *Switching* laws:

- the law (SWI1) $(A \multimap B) \otimes C \vdash A \multimap (B \otimes C)$,
- the law (SWI2) $A \otimes (B \multimapinv C) \vdash (A \otimes B) \multimapinv C$,
- two other laws strictly related to the previous ones, from a geometrical point of view:
  - (SWI1*1) $(A \multimapinv B) \otimes C \vdash A \multimapinv (C \multimap B)$,
  - (SWI2*1) $A \otimes (B \multimap C) \vdash (B \multimapinv A) \multimap C$.

7 Conclusions

In this paper we introduce a geometrical representation of the basic laws of categorial grammar by means of the proof nets of Cyclic Multiplicative Linear Logic (CyMLL). By assuming this geometrical point of view, *Residuation* laws are the simplest case of a large class of laws: they are obtained by considering the different readings of a single proof net with three conclusions (Section 3).

Then the family MON of *Monotonicity* laws is taken into consideration (Section 4): they are obtained by taking a proof net with two conclusions and one axiom link. In the particular case in which the proof net is itself an axiom link, we obtain the family ID(A, B) of *Application* laws, *Expansion* laws and *Type Raising* laws (Section 5).

We have then considered the class of graphs consisting of three proof nets: if they are three axioms links, one obtains the class SYL(C, B, A) of Syllogistic laws (Section 6) including the family COM(C, B, A) of *Composition* laws (6.1), GEA(C, B, A) of *Geach* laws (6.2), and SWI(C, B, A) of *Switching* laws (6.3). It remains to be investigated the still more complex cases in which, instead of three axioms links, one or more proof nets are considered.

An interesting result of this geometrical approach is the individuation of a group of particular laws belonging to the family of *Syllogisms*, precisely: COM*1, COM*2 within *Composition* laws, GEA1*1, GEA1*2, GEA1*3, GEA2*1, GEA2*2, GEA2*3 within *Geach* laws, and SWI1*1, SWI1*2 within *Switching* laws. The following are the translations of these laws in Lambek calculus notation:

(COM*1)	$C \otimes A \vdash C \otimes B \multimapinv (A \multimap B)$	$\Longrightarrow$	$C \cdot A \vdash C \cdot B/(A \backslash B)$
(COM*2)	$C \otimes A \vdash (B \multimapinv C) \multimap B \otimes A$	$\Longrightarrow$	$C \cdot A \vdash (B/C) \backslash B \cdot A$
(GEA1*1)	$B \multimapinv C \vdash (A \multimapinv B) \multimap (A \multimapinv C)$	$\Longrightarrow$	$B/C \vdash (A/B) \backslash (A/C)$
(GEA1*2)	$B \multimapinv C \vdash (B \otimes A) \multimapinv (C \otimes A)$	$\Longrightarrow$	$B/C \vdash (B \cdot A)/(C \cdot A)$
(GEA1*3)	$(C \otimes A) \otimes (A \multimap B) \vdash C \otimes B$	$\Longrightarrow$	$(C \cdot A) \cdot (A \backslash B) \vdash (C \cdot B)$
(GEA2*1)	$A \multimap B \vdash (A \multimap C) \multimapinv (B \multimap C)$	$\Longrightarrow$	$A \backslash B \vdash (A \backslash C)/(B \backslash C)$
(GEA2*2)	$A \multimap B \vdash C \otimes A \multimap C \otimes B$	$\Longrightarrow$	$A \backslash B \vdash (C \cdot A) \backslash (C \cdot B)$
(GEA2*3)	$(B \multimapinv C) \otimes (C \otimes A) \vdash B \otimes A$	$\Longrightarrow$	$B/C \cdot (C \cdot A) \vdash (B \cdot A)$
(SWI1*1)	$(A \multimapinv B) \otimes C \vdash A \multimapinv (C \multimap B)$	$\Longrightarrow$	$(A/B) \cdot C \vdash A/(C \backslash B)$
(SWI1*2)	$A \otimes (B \multimap C) \vdash (B \multimapinv A) \multimap C$	$\Longrightarrow$	$A \cdot (B \backslash C) \vdash (B/A) \backslash C$

Giving a careful look at the Lambek formulas, one observes that in fact these rules are obtainable from suitable extension of simplest rules, precisely:

1. From the sequent $A, A{-\circ}B \vdash B$ (that is at the basis of APP1) and the axiom $C \vdash C$, one obtains $C, A, A{-\circ}B \vdash C \otimes B$, then $C \otimes A, A{-\circ}B \vdash C \otimes B$. From this sequent one derives the laws GEA1*3 (by linking the two premises with $\otimes$), COM*1 (by discharging the second premise) and GEA2*2 (by discharging the first premise).
2. From the sequent $B{\circ-}C, C \vdash B$ (at the basis of APP2) and the axiom $A \vdash A$, one obtains $B{\circ-}C, C, A \vdash B \otimes A$, then $B{\circ-}C, C \otimes A \vdash B \otimes A$. From this sequent one derives the laws GEA2*3 (by linking the two premises with $\otimes$), GEA 1*2 (by discharging the second premise) e COM*2 (by discharging the first premise).
3. From the sequent $A \vdash (B{\circ-}A){-\circ}B$, i.e. TYR1, and the axiom $B{-\circ}C \vdash B{-\circ}C$, one obtains $A, B{-\circ}C \vdash ((B{\circ-}A){-\circ}B) \otimes (B{-\circ}C)$, then by COM1 one derives the sequent $A, B{-\circ}C \vdash (B{\circ-}A){-\circ}C$, from which SWI2*1 follows (by linking the two premises with $\otimes$).
4. From the sequent $C \vdash B{\circ-}(C{-\circ}B)$, i.e. TYR 2, and the axiom $A{\circ-}B \vdash A{\circ-}B$, one obtains $A{\circ-}B, C \vdash (A{\circ-}B) \otimes (B{\circ-}(C{-\circ}B))$, then $A{\circ-}B, C \vdash A{\circ-}(C{-\circ}B)$ by COM2, from which SWI1*1 follows (by linking the two premises with $\otimes$).
5. From the sequent $A{-\circ}B, B{-\circ}C \vdash A{-\circ}C$ one obtains COM1 (by linking the two premises with $\otimes$), GEA1 (by discharging the first premise) and GEA2*1 (by discharging the second premise).
6. From the sequent $A{\circ-}B, B{\circ-}C \vdash A{\circ-}C$ one obtains COM2 (by linking the two premises with $\otimes$), GEA2 (by discharging the second premise), and GEA1*1 (by discharging the first premise).

To summarize, the laws in 1. and 2. are related to the family APP, those in 3. and 4. to TYR and COM, those in 5. and 6. belong to the same root from which the COM laws are obtained.

The geometric relation between the usual composition laws and (COM*1), (COM*2) is rather striking, since these last laws, though of course theorems, are not part of the "standard" set of laws normally mentioned in Lambek calculus literature; similar remarks apply to the usual GEA laws, SWI laws and the new GEA* and SWI* laws, respectively. We are grateful to one of the anonymous referees for this observation that is promising in the perspective of future research.

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