

SYSMICS 2016

Syntax Meets Semantics

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Booklet of Abstracts

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Preface

This volume contains the papers presented at SYSMICS 2016: Syntax Meets Semantics 2016 held on September 5-9, 2016 in Barcelona.

There were 37 accepted papers and 7 invited talks.

This conference is the first of a series of meetings planned in the "SYSMICS" RISE project during 2016-2019. The project website can be found at <http://logica.dmi.unisa.it/sysmics/>.

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September 1, 2016
Barcelona, Spain

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Invited Talks

Open Lecture

Ex Falso Veritas

Proof by Reductio ad Absurdum, and beyond

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Abstract

Mathematics has the unique property that from a false proposition a new true proposition can be derived. More precisely, given a proposition P , if the rules and axioms of mathematical logic are able to derive from P a proposition of the form “ Q and not Q ”, then (P is false and) the negation of P is true. In this way the Greeks proved that there are arbitrarily large prime numbers, and that there is no common submultiple of the diagonal of a square and its side. The time-honored rules of mathematical reasoning are made up of simple manipulations of symbols. Starting from the list of symbols needed to write P and the axioms P_1, \dots, P_n which we have already accepted as true, the logical rules produce new lists of symbols P_{n+1}, P_{n+2}, \dots , in a mechanical way, much as we do when we multiply 79 by 841 using the Pythagorean table as our main axiom set. We say that P has a proof from P_1, \dots, P_n if for some t and Q , the proposition P_{n+t} has the form “ Q and not Q ”. While multiplication always terminates producing the desired result in a finite number of steps, nobody tells us how large t may be—if it exists at all. The rules of mathematical reasoning rest on the fundamental principle stating that *a mathematical proposition can only be true or false*. Almost no proposition in real life is either true or false, and yet we often draw reasonable conclusions from it. In this open lecture we will see how this is possible for propositions having more than two truth-values, such as those occurring in the Rényi-Ulam game of Twenty Questions with lies/errors.

Residuated lattices and twist-products

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based on joint works with R. Cignoli.

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Given a lattice $\mathbf{L} = (L, \vee, \wedge)$, the product $\mathbf{L} \times \mathbf{L}$ can be endowed with an structure of involutive lattice $Tw(\mathbf{L}) = (L \times L, \cup, \cap, \sim)$ defining the operations as follows:

$$(a, b) \cup (c, d) = (a \vee c, b \wedge d) \tag{1}$$

$$(a, b) \cap (c, d) = (a \wedge c, b \vee d) \tag{2}$$

$$\sim (a, b) = (b, a). \tag{3}$$

The idea of considering this kind of construction to deal with order involutions on lattices starts with a work of Kalman in 1958, and since then it has been widely used to represent many involutive lattices with additional operations, such as Nelson algebras, involutive residuated lattices, N4-lattices and bilattices.

We are interested in commutative residuated lattices: for each integral commutative residuated lattice \mathbf{L} , it can be defined in $Tw(\mathbf{L})$ a commutative residuated lattice structure in such a way that the negative cone of $Tw(\mathbf{L})$ is isomorphic to \mathbf{L} . We call it twist-product obtained from \mathbf{L} . In the talk we will explore the subvariety of residuated lattices that can be represented by twist-products and we will comment on some interesting subvarieties.

Syntax meets semantics in abstract algebraic logic

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The creation of the Leibniz hierarchy is undoubtedly one of the highest achievements of abstract algebraic logic. Among its salient features is the diversity of the characterizations that most of its classes enjoy, some of a syntactic character, some of a semantic one. Thus, abstract algebraic logic seems to be a field where syntax meets semantics in a fruitful way.

I will review some of these features and expose some recently found ones concerning the order structure of the class of protoalgebraic logics.

Linear Logic Properly Displayed

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Joint work with Giuseppe Greco

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Linear Logic [3] is one of the best known substructural logics, and the best known example of a resource-sensitive logic. The fact that formulas are treated as resources implies that e.g. two copies of a given assumption A guarantee a different proof-power than three copies, or one copy. From an algebraic perspective, this fact translates in the stipulation that linear conjunction and disjunction, denoted \otimes and \wp respectively, are *not* idempotent, in the sense that none of the inequalities composing the identities $a \otimes a = a = a \wp a$ are valid in linear algebras. From a proof-theoretic perspective, this fact translates into the well known stipulation that (left- and right-) weakening and contraction rules cannot be included in Gentzen-style presentations of linear logic.

However, resources might exist which are available *unlimitedly*, i.e. having one or more copies of these special resources guarantees the same proof-power. To account for this difference, the language of linear logic includes, along with the connectives \otimes and \wp (sometimes referred to as the *multiplicative* conjunction and disjunction), also *additive* conjunction and disjunction, respectively denoted $\&$ and \oplus , which are idempotent (i.e. $A \& A = A = A \oplus A$ for every formula A). Moreover, the language of linear logic includes the modal operators $!$ and $?$, called *exponentials*, which respectively govern the controlled application of left and right weakening and contraction rules for formulas under their scope, algebraically encoded by the following identities, which capture the essential properties of the exponentials:

$$!(A \& B) = !A \otimes !B \quad ?(A \oplus B) = ?A \wp ?B.$$

Accounting for the interplay between additive and multiplicative connectives, mediated by exponentials, presents the main hurdles towards a smooth proof-theoretic treatment of linear logic. Indeed, modulo the specific conventions of any given formalism, this interplay is encoded by means of rules the parametric parts (or contexts) of which are not arbitrary, and hence closed under arbitrary substitution, but are *restricted* in some way. These restricted contexts create additional complications in the definition of smooth and general reduction strategies for syntactic cut-elimination.

The present talk reports on a recent paper [2] in which proof calculi for intuitionistic and classical linear logics are introduced in which all parameters in rules occur *unrestricted*. This is made possible thanks to the introduction of a richer, *multi-type* language in which general and unlimited resources are assigned different *types*, each of which is interpreted by a different type of algebra (linear algebras for general resource type terms, and Heyting algebras or Boolean algebras for unlimited resource type terms), and their interaction is mediated by means of pairs of adjoint connectives, the composition of which recreates Girard's exponentials $!$ and $?$ as defined connectives. The proof-theoretic behaviour of the new connectives is that of standard normal modal operators. Moreover, the information capturing the essential properties of the exponentials can be expressed in the new language by means of identities of a syntactic shape called *analytic inductive* (cf. [4]), which guarantees that they can be equivalently encoded into analytic rules. The metatheory of these calculi is smooth and encompassed in a general theory (cf. [4, 5]), so that one obtains soundness, completeness, conservativity and cut-elimination as easy corollaries of general facts.

These calculi are designed according to the *multi-type* methodology, introduced in [11, 8, 7] to provide DEL and PDL with analytic calculi, and further developed in [10, 1, 6]. The multi-type methodology can be understood as an attempt to refine and generalize Belnap's display calculi so as to expand their reach. Technically, this methodology is based on an algebraic and order-theoretic analysis of the semantic environment of a given logical system (linear logic in this case), with the aim of identifying the crucial syntactic interactions. The synergy between syntax and semantics advocated by the multi-type methodology has been key to its success in defining analytic calculi for logics as proof-theoretically impervious as DEL.

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Epimorphisms in Varieties of Residuated Structures

J.G. Raftery

Joint work with Guram Bezhanishvili, and Tommaso Moraschini.

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A homomorphism $h: \mathbf{A} \rightarrow \mathbf{B}$ between algebras in a variety \mathbf{K} is called a (\mathbf{K} -) *epimorphism* provided that, for any two homomorphisms $f, g: \mathbf{B} \rightarrow \mathbf{C}$, with $\mathbf{C} \in \mathbf{K}$, if $f \circ h = g \circ h$, then $f = g$. Surjective homomorphisms are clearly epimorphisms, but the converse may fail. If every \mathbf{K} -epimorphism is surjective, then \mathbf{K} is said to have the *ES property*. This property need not persist in subvarieties and is not generally easy to detect. Here, we shall prove it for a range of varieties of residuated structures. Our main motivation comes from logic, as residuated structures algebraize substructural logics [2].

When a variety \mathbf{K} algebraizes a logic L , then \mathbf{K} has the ES property iff L has the *infinite Beth (definability) property* [1]. The latter signifies that, in L , whenever a set Z of variables is defined *implicitly* in terms of a disjoint set X of variables by means of some set Γ of formulas over $X \cup Z$, then Γ also defines Z *explicitly* in terms of X . The *finite Beth property* makes the same demand, but only when the set Z is finite. A homomorphism $h: \mathbf{A} \rightarrow \mathbf{B}$ between algebras is said to be *almost-onto* if \mathbf{B} is generated by $h[A] \cup \{b\}$ for some $b \in B$. A variety \mathbf{K} is said to have the *weak ES property* if every almost-onto \mathbf{K} -epimorphism is surjective. An algebraizable logic has the finite Beth property iff its algebraic counterpart has the weak ES property (see [1] and its references).

The question of whether the finite Beth property implies the infinite one was posed by Blok and Hoogland [1]. They conjectured a negative answer, which will be confirmed here. All varieties of Heyting algebras have the weak ES property [5], but we exhibit one that lacks the ES property. The counter-example is locally finite. Using Esakia duality, however, we prove that varieties of Heyting algebras of finite depth have the ES property, as do all varieties of Gödel algebras. These facts yield new definability results for various super-intuitionistic logics. With the aid of some recently obtained category equivalences [3, 4], we infer the ES property for certain varieties of non-integral residuated structures as well, including all varieties of Sugihara monoids. Thus, the infinite Beth property obtains in all axiomatic extensions of the relevance logic R-mingle (formulated with Ackermann constants).

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How useful is proof theory for substructural logics?

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There is no doubt that proof theory has been playing certain significant roles in the development of substructural logics. In fact, the basic substructural logics were introduced upon a proof-theoretic intuition. Decidability is often shown first by a proof-theoretic argument, and later followed by an algebraic one. Proof theory also gives an inspiration to the algebraic side. For example, the density elimination theorem in hypersequent calculi for fuzzy logics led to a uniform embedding of given residuated chains into dense ones via construction of residuated frames. Another advantage is that it sometimes, though not always, works very well for first-order logics. A classic example is the decidability of first-order FL, which can be easily shown proof-theoretically, whereas I am not aware of any easy algebraic proof of this fact.

While proof theory is still developing, my personal feeling is that it is getting close to the limitation. The main obstacle is that it does not work so well for proof-theoretically hard logics, which involve so-called N3 axioms in the substructural hierarchy (though some attempts exist). Given this apparent limitation, I personally believe that we would have to look for a “killer application” very seriously, if we are to pursue this approach any further.

In this talk, I will review some successful applications of proof theory in the past, introduce some challenging problems, and then discuss potential future directions of this field.

Lattice-Ordered Groups in Logic: Influence and Centrality

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There have been a number of studies providing compelling evidence of the importance of lattice-ordered groups (ℓ -groups) in the study of residuated lattices and other algebras of logic. For example, a fundamental result [14] in the theory of MV algebras is the categorical equivalence between the category of MV algebras and the category of unital Abelian ℓ -groups. Likewise, the non-commutative generalization of this result in [7] establishes a categorical equivalence between the category of pseudo-MV algebras and the category of unital ℓ -groups. Further, the generalization of these two results in [8] shows that one may view GMV-algebras as ℓ -groups with a suitable modal operator. Likewise, the work in [13] offers a new paradigm for the study of various classes of cancellative residuated lattices by viewing these structures as ℓ -groups with a suitable modal operator (a conucleus). In the first half of my talk, I will discuss these connections and suggest possibilities for future explorations.

While the importance of these connections cannot be overstated, they are just the tip of the iceberg. In the second half of my talk, I will summarize recent work [1, 2, 9–12], which demonstrates that large parts of the Conrad Program [3–6] for ℓ -groups can be profitably extended in the setting of e -cyclic residuated lattices – that is residuated lattices that satisfy the equation $x \setminus e \approx e/x$. The term *Conrad Program* traditionally refers to Paul Conrad’s approach to the study of ℓ -groups, which analyzes the structure of individual ℓ -groups, or classes of ℓ -groups, by primarily using strictly lattice theoretic properties of their lattices of convex ℓ -subgroups. For the purposes of this program, convex ℓ -subgroups play a far more significant role than congruence relations (ℓ -ideals). A byproduct of this work is the introduction of tools and techniques from the theory of ℓ -groups into the study of algebras of logic.

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Axiomatizing modal fixpoint logics

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Modal fixpoint logics are extensions of modal logic with either fixpoint connectives, such as the until operator of temporal logic, or explicit fixpoint operators such as in the modal mu-calculus. These formalisms have important applications ranging from epistemic logic to program specification and verification, and the area of modal fixpoint logic has a mathematically interesting theory.

In this talk we focus on the problem of providing sound and complete axiomatizations for modal fixpoint logics. There are various important results around, such as axiomatizations for individual logics like PDL and CTL, but in contrast to the completeness theory of basic modal logic, the area as a whole seems to lack a systematic approach, and important results such as the Kozen-Walukiewicz completeness theorem for the modal mu-calculus, up to now have remained isolated points.

We will give an overview of the existing results, discuss the obstacles for developing a systematic theory, and provide some recent positive results.

Contributed Talks

A geometrical representation of Categorical Grammar laws

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1 Overview

We propose a geometrical representation of the set of laws that are at the basis of Categorical Grammar, developing our analysis in the framework of *Cyclic Multiplicative Linear Logic*, a purely non-commutative fragment of Linear Logic [4]. The rules we investigate are: *Residuation* laws, *Monotonicity* laws, *Application* laws, *Expansion* laws, *Type-raising* laws, *Composition* laws, *Geach* laws, *Switching* laws [8, 6, 5, 9, 7].

First, we characterize the notion of a *cyclic multiplicative proof-net*. In Linear Logic proof-nets are geometrical representations of proofs [1]. Cyclic multiplicative proof-nets represent proofs in Cyclic Multiplicative Linear Logic (CyM-LL), a purely non-commutative fragment of Linear Logic. The conclusions of a CyM-PN may be described in different ways corresponding to different sequents of CyM-LL. A subset of the sequents of CyM-LL represent the sequents of the Lambek Calculus (L), and a subset of the CyM-PN's represent proofs in L.

Then, we recall the result presented in [2], where is given the geometrical representation of *Residuation* laws and is explained that Residuation laws correspond to different ways to read the conclusions of a *single* CyM-PN.

In the central part, we discuss the results presented in [3], concerning a relevant set of categorical grammar rules, as they are defined within Lambek Calculus (L).

In the final part, we discuss some new laws that are obtained by means of this geometrical representation.

2 Cyclic multiplicative proof-nets

Cyclic multiplicative proof-nets are a subclass of multiplicative proof nets. Multiplicative proof-nets are defined by means of the language of Multiplicative Linear Logic (MLL), a fragment of Linear Logic. Formulas of MLL are defined by using atoms and the binary connectives: \otimes (*multiplicative conjunction*), \wp (*multiplicative disjunction*).

The language of MLL has the following features:

- for each atom X there is another atom which is the dual of X and is denoted by X^\perp , in such a way that for every atom X , $X^{\perp\perp} = X$;
- for each formula A the linear negation A^\perp is defined as follows, in order to satisfy the principle $A^{\perp\perp} = A$:
 - if A is an atom, A^\perp is the atom which is the dual of A ,
 - $(B \otimes C)^\perp = C^\perp \wp B^\perp$
 - $(B \wp C)^\perp = C^\perp \otimes B^\perp$.

Left and *right* residual connectives, i.e. the *left implication* \multimap and the *right implication* \multimap , can be defined by means of the linear negation $(\)^\perp$ and \wp :

$$A \multimap B = A^\perp \wp B \quad ; \quad B \multimap A = B \wp A^\perp$$

3 Categorical Grammar laws

In an algebraic style, the basic laws of Categorical Grammar involve:

- a binary operation on a set M , the *product* or the *residuated* operation, denoted by \cdot ;
- two binary *residual* operations on the same set M : \backslash (the *left residual* operation of the product) and $/$ (the *right residual* operation of the product);
- a partial ordering on the same set M , denoted by \leq .

The following is the algebraic formulation of these laws (cf. [5], pp. 17-19):

(a) *Residuation laws*

- (RES) $a \cdot b \leq c$ iff $b \leq a \backslash c$ iff $a \leq c / b$

(b) *Monotonicity laws*

- (MON1.1) if $a \leq b$ then $a \cdot c \leq b \cdot c$ (MON1.2) if $a \leq b$ then $c \cdot a \leq c \cdot b$
- (MON2.1) if $a \leq b$ then $c \backslash a \leq c \backslash b$ (MON2.2) if $a \leq b$ then $b \backslash c \leq a \backslash c$
- (MON3.1) if $a \leq b$ then $a / c \leq b / c$ (MON3.2) if $a \leq b$ then $c / b \leq c / a$

(c) *Application laws*

- (APP1) $a \cdot a \backslash b \leq b$
- (APP2) $b / a \cdot a \leq b$

(d) *Expansion laws*

- (EXP1) $a \leq b \backslash (b \cdot a)$
- (EXP2) $a \leq (a \cdot b) / b$

(e) *Type-raising laws*

- (TYR1) $a \leq (b / a) \backslash b$
- (TYR2) $a \leq b / (a \backslash b)$

(f) *Composition laws*

- (COM1) $(a \backslash b) \cdot (b \backslash c) \leq (a \backslash c)$
- (COM2) $(a / b) \cdot (b / c) \leq (a / c)$

(g) *Geach laws*

- (GEA1) $b \backslash c \leq (a \backslash b) \backslash (a \backslash c)$
- (GEA2) $a / b \leq (a / c) / (b / c)$

(h) *Switching laws*

- (SWI1) $(a \backslash b) \cdot c \leq a \backslash (b \cdot c)$
- (SWI2) $a \cdot (b / c) \leq (a \cdot b) / c$

4 Geometrical representation

4.1 Residuation laws

In [2] it is studied the question of offering a geometrical representation of the *Residuation* laws of Lambek Calculus (L):

$$A \otimes B \vdash C \quad , \quad B \vdash A \multimap C \quad , \quad A \vdash C \multimap B$$

stating their equivalence in L (i.e. the proof in L of one of these sequents can be transformed into the proof in L of each of the other sequents). Since CyM-PN's are geometrical representations of proofs in L, every possible proof in L of one of these sequents is a CyM-PN with 2 conclusions: the formula which is on the right side of the sequent, and the linear negation of the formula which is on the left side of the sequent.

4.2 Monotonicity, Application, Type-raising, Expansion laws

In the paper we show that:

- (i) the geometrical representation of *Monotonicity* laws is given by the CyM-PN's obtained from an arbitrary CyM-PN (corresponding to the premise of the law) and a single axiom link;
- (ii) the geometrical representations of *Application* laws, *Expansion* laws and *Type-raising* laws, are given by the CyM-PN's obtained from two axiom links, one \otimes -link and one \wp -link.

4.3 Composition laws, Geach laws and Switching laws

We then consider a larger class of proof nets on the basis of which we show that:

- (iii) the geometrical representations of *Composition* laws, *Geach* laws and *Switching* laws are given by the CyM-PN's obtained from three axiom links, two \otimes -links and two \wp -links.

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On linear varieties of MTL-algebras

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Abstract

In this talk we focus on those non-trivial varieties of MTL-algebras whose lattice of subvarieties is totally ordered. Such varieties will be called *linear*. We show that a variety \mathbb{V} of MTL-algebras is linear if and only if each of its subvarieties is generated by a chain, and we provide an equivalent characterization, in purely logical terms. As a further result, we will provide a complete classification of the linear varieties of BL-algebras. The more general case of MTL-algebras is out of reach, nevertheless we will classify all the linear varieties of WNM-algebras.

Extended abstract

MTL-algebras and their corresponding logic MTL were firstly introduced in [6], as a generalization of Hájek's basic logic BL, the logic that was proven in [5] to be the logic of all continuous t-norms and their residua. On the other hand, MTL is the logic of all left-continuous t-norm and their residua [11].

As pointed out in [14], MTL and its axiomatic extensions (logics obtained by adding axioms to it) are all algebraizable in the sense of [4], and their corresponding semantics forms an algebraic variety. The variety of MTL-algebras and its subvarieties forms an algebraic lattice, and during the years many scholars directed their research to analyze and classify parts of such a lattice, as well as the corresponding logics. Given an axiomatic extension L of MTL, we will denote by \mathbb{L} its corresponding variety. Conversely, given a variety \mathbb{L} of MTL-algebras we will call L the corresponding logic. Given an MTL-chain \mathcal{A} , with $\mathbf{V}(\mathcal{A})$ we denote the variety generated by \mathcal{A} . With $\mathbf{2}$ we denote the two-element Boolean algebra.

Recently, in [13] Franco Montagna introduced and studied the notion of single chain completeness (SCC).

Definition 1. *An axiomatic extension L of MTL enjoys the single chain completeness (SCC) if there is an L -chain such that L is complete w.r.t. it.*

In [2] this topic has been further investigated, by finding new results and solving some problems left open in [13]. A notable example of logic having the SCC is given by Gödel logic. However, the variety \mathbb{G} of Gödel-algebras (which coincides with the class of MTL-algebras satisfying the equation $x * x = x$) has also another interesting property: the lattice of its subvarieties forms a chain, as shown in [10]. So, one may ask if there is a general way to classify the varieties of MTL-algebras having this property, and if there is a general relation with the SCC. In this talk we will provide an answer for both the questions.

Definition 2. • *A variety \mathbb{V} of MTL-algebras is said to be linear whenever it is non-trivial, and the lattice of its subvarieties is totally ordered.*

- *A consistent axiomatic extension L of MTL enjoys the extended single chain completeness (ESCC), if L and every of its (consistent) axiomatic extensions have the SCC.*

We have that:

Theorem 3. *Let \mathbb{L} be a non-trivial variety of MTL-algebras. Then \mathbb{L} is linear if and only if L has the ESCC.*

As a first result, we will classify all the linear varieties of \mathbb{BL} .

Theorem 4. *The linear subvarieties of \mathbb{BL} are exactly the following ones.*

- \mathbb{G} and $\{\mathbb{G}_k\}_{k \geq 2}$.
- The family of varieties $\{\mathbb{L}_k : k = h^n + 1, 1 \leq h \text{ is prime and } n \geq 1\}$ and $\{\mathbf{V}(\mathbf{2} \oplus \mathbf{L}_k) : k = h^n + 1, 1 \leq h \text{ is prime and } n \geq 1\}$.
- The variety \mathbb{C} generated by Chang's MV-algebra.
- \mathbb{P} (the variety of product algebras), \mathbb{P}_∞ , and $\{\mathbb{P}_k\}_{k \geq 2}$.

Where:

- The symbols \oplus, \bigoplus denote the ordinal sum construction, introduced in [1, 7].
- \mathbb{P}_∞ is the variety whose class of chains is given by all the chains of the form $\mathbf{2} \oplus \bigoplus_{i \in I} C_i$, where every C_i is a cancellative hoop.
- For $k \geq 2$, \mathbb{P}_k is the variety whose class of chains is given by all the chains of the form $\mathbf{2} \oplus \bigoplus_{i \in I} C_i$, where $|I| \leq k$, and every C_i is a cancellative hoop.

The more general case of MTL is out of reach, due to the lack of a sufficiently strong classification of the structure of these algebras, as the one given for BL-algebras in [1]. Nevertheless, we are able to classify all the linear varieties of WNM-algebras, firstly introduced in [6].

We recall that a negation over an MTL-chain \mathcal{A} is a map $\sim : A \rightarrow A$ such that $\sim 1 = 0$, $\sim \sim x \geq x$, and if $x < y$ then $\sim x \geq \sim y$. A negation fixpoint is an element f such that $\sim f = f$: it is easy to check if such element exists, then it is unique. As shown in [9] the operations $*$, \Rightarrow of a WNM-chain \mathcal{A} are the following ones.

$$x * y = \begin{cases} 0 & \text{if } x \leq \sim y \\ \min\{x, y\} & \text{otherwise.} \end{cases} \quad x \Rightarrow y = \begin{cases} 1 & \text{if } x \leq y \\ \max\{\sim x, y\} & \text{otherwise.} \end{cases}$$

Where \sim is a negation such that $\sim x = x \Rightarrow 0$. The varieties $\mathbb{G}, \mathbb{DP}, \mathbb{NM}^-, \mathbb{F}$ are subvarieties of WNM such that, respectively:

- Every chain in \mathbb{G} is such that $\sim x = 0$, for $x > 0$.
- Every chain in \mathbb{DP} with more than two elements is such that $\sim \sim x = \sim x$, for every $0 < x < 1$ (see [3]).
- Every chain in \mathbb{NM}^- is such that $\sim \sim x = x$ and it has no negation fixpoint (see [9]).
- Every chain in \mathbb{F} with more than two elements has a coatom c with $\sim \sim c = c$, and $\sim c$ is the predecessor of c .

Theorem 5. *The linear subvarieties of WNM are exactly the following ones.*

- \mathbb{G} and its subvarieties.
- \mathbb{DP} and its subvarieties.
- \mathbb{NM}^- and its subvarieties.
- \mathbb{F} and its subvarieties.

In particular, the only proper subvarieties of $\mathbb{L} \in \{\mathbb{G}, \mathbb{DP}, \mathbb{NM}^-, \mathbb{F}\}$ are the ones of the form $\mathbf{V}(\mathcal{A})$, where \mathcal{A} is a finite chain in \mathbb{L} . Moreover, the order type of the lattice of subvarieties of \mathbb{L} is $\omega + 1$.

Notice that the set \mathcal{V}_L of all the non-trivial linear varieties of MTL-algebras forms a downward-closed meet-semilattice, in the lattice of non-trivial subvarieties of \mathbb{MTL} (ordered by inclusion), having the variety of Boolean algebras as minimum.

Even if the result is not new (see [12, 8]), as a corollary of [Theorem 4](#) and [Theorem 5](#) we obtain a classification of the *almost minimal* subvarieties of \mathbb{BL} and \mathbb{WNL} .

Definition 6. A variety of MTL-algebras is said *almost minimal* whenever the variety of Boolean algebras is its only proper non-trivial subvariety.

Clearly, every almost minimal variety \mathbb{L} of MTL-algebras is linear, and hence we have the following Corollary.

Corollary 7.

- The almost minimal varieties in \mathbb{BL} are \mathbb{G}_3 , \mathbb{P} , \mathbb{C} , and $\{\mathbb{L}_k : k > 2 \text{ and } k - 1 \text{ is prime}\}$.
- The almost minimal varieties in \mathbb{WNL} are \mathbb{G}_3 , \mathbb{L}_3 , \mathbb{NM}_4 .

Since every almost minimal variety \mathbb{L} is linear, then by [Theorem 3](#) we have $\mathbb{L} = \mathbf{V}(\mathcal{A})$, for some MTL-chain \mathcal{A} . For the case of almost minimal varieties generated by a finite chain we have the following result.

Definition 8.

- Given an MTL-chain \mathcal{A} , with $\text{Rad}(\mathcal{A})$ we denote the largest proper filter of \mathcal{A} .
- An MTL-chain \mathcal{A} is said to be *bipartite* if $\mathcal{A} = \text{Rad}(\mathcal{A}) \cup \overline{\text{Rad}(\mathcal{A})}$, where $\overline{\text{Rad}(\mathcal{A})} = \{a \in \mathcal{A} : \sim a \in \text{Rad}(\mathcal{A})\}$.

Theorem 9. Given a finite MTL-chain \mathcal{A} , let $\mathbb{L} = \mathbf{V}(\mathcal{A})$. Then \mathbb{L} is almost minimal if and only if:

1. \mathcal{A} is simple or it is bipartite.
2. $|\mathcal{A}| > 2$, and every element $a \in \mathcal{A} \setminus \{0, 1\}$ generates \mathcal{A} .

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IUML-algebras of refinements of orthopairs

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Abstract

Rough sets and orthopairs are commonly used to deal with approximation of sets and to model uncertainty [8]. Several kind of operations have been considered among rough sets ([5]), corresponding to connectives in three-valued logics. In this paper we focus on Sobociński conjunction and its relation with a uninorm logic called IUML.

Given a partition P of a universe U , every subset X of U determines an orthopair ([2, 3]) denoted by $(\mathcal{L}_P(X), \mathcal{E}_P(X))$, where $\mathcal{L}_P(X)$, called *lower approximation* of X , is the union of the blocks of P included in X , and $\mathcal{E}_P(X)$ represents the *impossibility domain*, namely the union of blocks of P that are not contained in X (therefore an approximation of $U \setminus X$) [8].

In [1] the category of finite IUML-algebras with homomorphisms is proved to be dually equivalent to the category of finite forests with open maps. In particular, finite IUML-algebras A in the variety generated by the three element IUML-algebra (so called three-valued IUML-algebras) are in duality with finite sets F : in this case each IUML-algebra A can be recovered from F considering the set of all pairs of disjoint subsets of F and, in turn, this domain can be interpreted as the set of all orthopairs on F while the conjunction operation of the IUML-algebra is the Sobociński conjunction of orthopairs.

We show that not only three-valued IUML-algebras correspond to rough sets and orthopairs, but that each sequence of successive refinements of orthopairs over a finite universe can be represented by a (not necessarily three-valued) finite IUML-algebra.

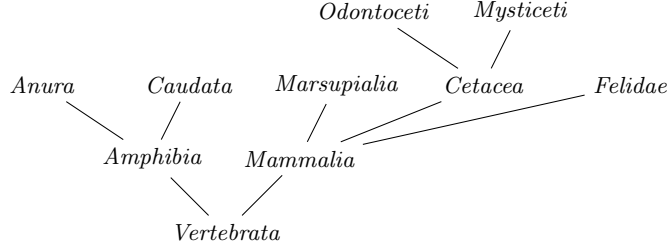
Note that in [6] the category of bounded odd Sugihara monoids (equivalent to the category of IUML-algebras) have been shown to be equivalent to the category of relative Stone algebras.

We call *refinement sequence* a sequence $\mathcal{P} = (P_0, \dots, P_n)$ of partitions of subsets of U such that every block of P_i is contained in a block of P_{i-1} , for each i from 1 to n . Note that in general we do not ask that each P_i is a partition of U , but that it is a partition of a subset of U , meaning that some of the elements of U can be lost during the refinement process.

Example *Refinements of partial partitions can be used to represent classifications in which we want to better specify some classes while ignoring others: suppose to start from animals first classified as Vertebrata. But you are really interested in Amphibia and Mammalia, that do not form a partition of Vertebrata. Then you want to refine such a classification by considering two groups of Amphibia (Anura and Caudata) and three groups of Mammalia (Marsupialia, Cetacea and Felidae) and further, in the group of Cetacea you are interested in Odontoceti and Mysticeti. In this case any set of individuals can be approximated by four orthopairs corresponding to the four partial partitions.*

We associate with \mathcal{P} the forest $F_{\mathcal{P}}$, where the set of the nodes is the set of all subsets of U belonging to $P_0 \cup \dots \cup P_n$ and the partial order relation is the reverse inclusion. We will further assume that each block of P_i has at least two elements.

Example. Referring to the example before, we have the following forest (that in this case is a tree):



For every $X \subseteq U$, a refinement sequence $\mathcal{P} = (P_0, \dots, P_n)$ of U determines the sequence

$$\mathcal{O}_{\mathcal{P}}(X) = ((\mathcal{L}_0(X), \mathcal{E}_0(X)), \dots, (\mathcal{L}_n(X), \mathcal{E}_n(X)))$$

of orthopairs. We can relate this sequence with a pair of disjoint upsets of the forest $F_{\mathcal{P}}$ representing the refinement of partitions; indeed, $\mathcal{O}_{\mathcal{P}}(X)$ corresponds to the pair $(X_{\mathcal{P}}^1, X_{\mathcal{P}}^2)$ such that $X_{\mathcal{P}}^1 = \{N \in F_{\mathcal{P}} : N \subseteq X\}$ and $X_{\mathcal{P}}^2 = \{N \in F_{\mathcal{P}} : N \cap X = \emptyset\}$.

Example. If $U = \{a, b, c, d, e, f, g, h\}$, $X = \{a, b, c, f, g\}$ and $\mathcal{P} = (P_0, P_1)$, where $P_0 = \{\{a, b, c, d, e\}, \{f, g, h\}\}$, and $P_1 = \{\{a, b\}, \{d, e\}, \{f, g\}\}$, then the sequence $\mathcal{O}_{\mathcal{P}}(X)$ of orthopairs of X is $((\emptyset, \emptyset), (\{a, b, f, g\}, \{d, e\}))$ and the corresponding pair (X^1, X^2) of disjoint upsets of $F_{\mathcal{P}}$ is $(\{\{a, b\}, \{f, g\}\}, \{\{d, e\}\})$.

An idempotent uninorm mingle logic algebra (IUML-algebra for short) [7] is an idempotent commutative bounded distributive residuated lattice $A = (A, \wedge, \vee, *, \rightarrow, \perp, \top, e)$, satisfying the following properties:

- (IUML1): $e \leq (x \rightarrow y) \vee (y \rightarrow x)$, and
- (IUML2): $(x \rightarrow e) \rightarrow e = x$.

The set $\{0, 1/2, 1\}$ equipped with the Sobociński operation $*$ defined by $x * 0 = 0 = 0 * y$, $1/2 * 1/2 = 1 * 2$ and $x * y = 1$ otherwise, is a IUML-algebra (with implication given by $\max(1 - x, y)$ if $x \leq y$ and $\min(1 - x, y)$ otherwise). Further, given two orthopairs (i.e. two pairs of disjoint subsets of a universe U) (A, B) and (C, D) we set

$$(A, B) * (C, D) = ((A \cap C) \cup (A \setminus (C \cup D)) \cup (C \setminus (A \cup B)), B \cup D)$$

and this operation equips the set of all orthopairs of a given universe with the structure of IUML-algebra (implication and lattice operations are defined in accordance with the operations on $\{0, 1/2, 1\}$). In general given a forest F , the set $SP(F)$ of all pairs of disjoint upsets (i.e. upward closed subsets) of $F_{\mathcal{P}}$ can be equipped with a structure of IUML-algebra, and in [1] it has been proved that the category of finite IUML-algebras is dually equivalent to the category of finite forests with open maps.

We show that the set $SO(F_{\mathcal{P}}) = \{(X_{\mathcal{P}}^1, X_{\mathcal{P}}^2) : X \subseteq U\}$ (corresponding to sequences of orthopairs $\mathcal{O}_{\mathcal{P}}(X)$) coincides with $SP(F_{\mathcal{P}})$ if and only if every node of $F_{\mathcal{P}}$ is not the union of its successors. Otherwise $SO(F_{\mathcal{P}}) \subset SP(F_{\mathcal{P}})$.

Nevertheless, we provide $SO(F_{\mathcal{P}})$ with a structure of IUML-algebra, and we also find its dual forest that in general will be different from $F_{\mathcal{P}}$. Indeed, we build a forest $F_{\mathcal{P}'}$ assigned to a new refinement sequence \mathcal{P}' of U , by removing from $F_{\mathcal{P}}$ all nodes equal to the union of their successors. In this context, as it is common when dealing with operations among orthopairs ([4, 8]), we suppose that each block of each partition in \mathcal{P} is not a singleton.

Theorem. *Given a refinement sequence \mathcal{P} of a universe U , there exists a refinement sequence of partitions \mathcal{P}' of U such that*

$$SO(F_{\mathcal{P}}) \cong SP(F_{\mathcal{P}'})$$

as IUML-algebras.

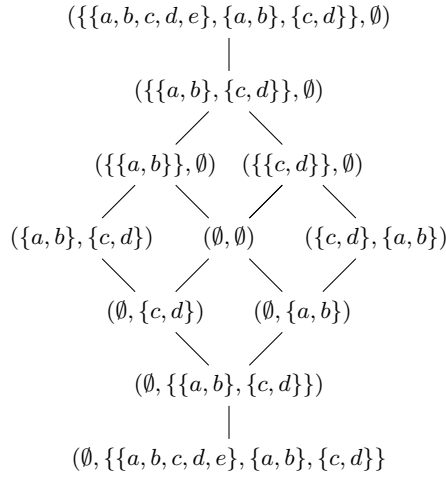
For example, from the sequence $\mathcal{P} = (\{\{a, b, c, d\}, \{e, f, g, h, i\}\}, \{\{a, b\}, \{c, d\}, \{e, f\}, \{g, h\}\})$ we get the sequence $\mathcal{P}' = (\{\{a, b\}, \{c, d\}, \{e, f, g, h, i\}\}, \{\{e, f\}, \{g, h\}\})$.

We note that if \mathcal{P} is a refinement sequence made of total partitions of U , then $F_{\mathcal{P}'}$ is the subforest of $F_{\mathcal{P}}$ made of all leaves of $F_{\mathcal{P}}$ (and the corresponding IUML-algebra is three-valued, cfr. [1]).

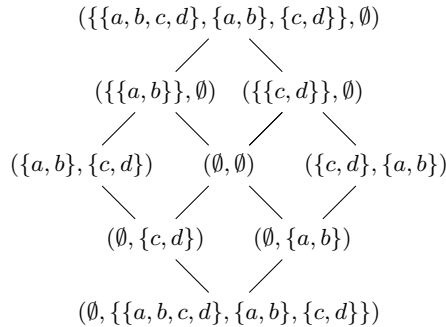
Example. *We consider $U = \{a, b, c, d, e\}$ and $\mathcal{P} = (P_0, P_1)$, where*

- $P_0 = \{\{a, b, c, d, e\}\}$, and
- $P_1 = \{\{a, b\}, \{c, d\}\}$.

Then, $SO(F_{\mathcal{P}}) = SP(F_{\mathcal{P}'})$ and the corresponding IUML-algebra is the following:



On the other hand, if $P_0 = \{\{a, b, c, d\}\}$ and $P_1 = \{\{a, b\}, \{c, d\}\}$, then the IUML-algebra with support $SO(F_{\mathcal{P}})$ is the following:



We observe that its dual forest is not $F_{\mathcal{P}}$, but the forest $F_{\mathcal{P}'}$ that have only the nodes $\{a, b\}$ and $\{c, d\}$.

Theorem. Given an universe U and a refinement sequence $\mathcal{P} = (P_0, \dots, P_n)$, the structure of IUML-algebra on $SO(F\mathcal{P})$ induces on sequences of orthopairs the following operation, for every $X, Y \subseteq U$:

$$\mathcal{O}_{\mathcal{P}}(X) * \mathcal{O}_{\mathcal{P}}(Y) = ((A_0, B_0), \dots, (A_n, B_n))$$

where for each $i = 1, \dots, n$, we firstly set

$$(A'_i, B'_i) = (\mathcal{L}_i(X), \mathcal{E}_i(X)) * (\mathcal{L}_i(Y), \mathcal{E}_i(Y))$$

and then $A_0 = A'_0$ and for $i > 0$:

$$A_{i+1} = \begin{cases} A'_{i+1} & \text{if } A_i = \emptyset \\ A'_{i+1} \cup \{N \in P_{i+1} \mid N \subseteq A_i\} & \text{otherwise} \end{cases}$$

while $B_i = B'_i \setminus A_i$.

In other words, the operation maps each pair of sequences of orthopairs to the sequence of orthopairs given by applying the Sobociński conjunction between orthopairs relative to same partition and then closing with respect to the inclusion in the first component.

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A uniform way to build strongly perfect MTL-algebras via Boolean algebras and prelinear semihoops

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Given an MTL-algebra $\mathbf{A} = (A, *, \rightarrow, \wedge, \vee, 0, 1)$ its *radical* $Rad(\mathbf{A})$ is the intersection of its maximal filters. The *co-radical* of \mathbf{A} , following [3], is defined as $coRad(\mathbf{A}) = \{x \in A \mid \neg x \in Rad(\mathbf{A})\}$ (where $\neg x = x \rightarrow 0$) and an MTL-algebra \mathbf{A} is said to be *perfect* provided that $A = Rad(\mathbf{A}) \cup coRad(\mathbf{A})$. Perfect MTL-algebras do not form a variety. The variety generated by all perfect MTL-algebras has been called \mathbb{BP}_0 in [5], and it can be equationally described as the subvariety of MTL-algebras further satisfying the equation (DL):

$$(2x)^2 = 2(x^2),$$

where $x^2 = x * x$ and $2x = \neg(\neg x * \neg x)$. Let us call \mathbb{SBP}_0 the subvariety of \mathbb{BP}_0 generated by *strongly perfect MTL-algebras*, that is, perfect algebras \mathbf{A} such that $coRad(\mathbf{A}) = \{\neg x \mid x \in Rad(\mathbf{A})\}$. \mathbb{SBP}_0 can be equationally defined as the subvariety of \mathbb{BP}_0 where the following equation holds:

$$\neg(x^2) \rightarrow (\neg\neg x \rightarrow x) = 1. \quad (1)$$

Notice that with Equation (1), we characterize all \mathbb{BP}_0 -algebras \mathbf{A} whose elements of the co-radical are involutive, i.e. for every $x \in coRad(\mathbf{A})$, $x = \neg\neg x$. Relevant subvarieties of \mathbb{SBP}_0 are the variety \mathbb{G} of Gödel algebras, the variety \mathbb{P} of Product algebras and the variety \mathbb{DLMV} generated by perfect MV-algebras.

Triples with prelinear semihoops

In [1], we recently generalised the definition of *triples* given in [4], introducing, for every subvariety \mathbb{H} of the variety \mathbb{PSH} of prelinear semihoops, the category $\mathcal{T}_{\mathbb{H}}$ made of triples $(\mathbf{B}, \mathbf{H}, \vee_e)$ where \mathbf{B} is a Boolean algebra, \mathbf{H} is a prelinear semihoop in \mathbb{H} and $\vee_e : B \times H \rightarrow H$ is a suitably defined map intuitively representing the natural join between the elements of \mathbf{B} and those of \mathbf{H} . If $(\mathbf{B}, \mathbf{H}, \vee_e)$ and $(\mathbf{B}', \mathbf{H}', \vee'_e)$ are two triples, a morphism in $\mathcal{T}_{\mathbb{H}}$ is a pair (f, g) where $f : \mathbf{B} \rightarrow \mathbf{B}'$ is a Boolean homomorphism, $g : \mathbf{H} \rightarrow \mathbf{H}'$ is a hoop homomorphism, and for every $(b, c) \in B \times H$, $g(b \vee_e c) = f(b) \vee'_e g(c)$. These pairs (f, g) have been called *good morphisms pairs* in [4]. In that paper the category of cancellative hoop-triples $(\mathbf{B}, \mathbf{C}, \vee_e)$ (in which \mathbf{C} is a cancellative hoop) has been proved to be equivalent to the algebraic category of product algebras. The main aim of our present work is to generalize this idea, and provide a uniform approach to establish categorical equivalences between relevant subcategories of \mathbb{SBP}_0 -algebras, via the use of a weaker notion of Cignoli and Torrens dl-admissible operators [3], that we will call *wdl-admissible operators*. In particular, we can prove that every directly indecomposable \mathbb{SBP}_0 -algebra \mathbf{A} can be constructed starting from a prelinear semihoop \mathbf{H} and by a wdl-admissible map $\delta : H \rightarrow H$.

Thus, for every subvariety \mathbb{H} of the variety PSH of prelinear semihoops, for every triple $(\mathbf{B}, \mathbf{H}, \vee_e) \in \mathcal{T}_{\mathbb{H}}$ and for every wdl-admissible operator $\delta : H \rightarrow H$, we can define the algebra $\mathbf{B} \otimes_e^\delta \mathbf{H}$ by suitably generalising the construction of $\mathbf{B} \otimes_e \mathbf{H}$ provided in [4] and prove the following key Lemma.

Lemma 1. *For every triple $(\mathbf{B}, \mathbf{H}, \vee_e)$ and for every wdl-admissible operator $\delta : H \rightarrow H$, $\mathbf{B} \otimes_e^\delta \mathbf{H}$ belongs to SBP_0 . Moreover, $\mathbf{B} \otimes_e^\delta \mathbf{H}$ is directly indecomposable iff $\mathbf{B} = \mathbf{2}$.*

In particular, if we choose $\delta_L : x \in H \mapsto 1 \in H$ as admissible operator, $\mathbf{B} \otimes_e^{\delta_L} \mathbf{H}$ is an SMTL-algebra (i.e. a *pseudocomplemented* MTL-algebra), while if we choose the identity map $\delta_D : x \in H \mapsto x \in H$, $\mathbf{B} \otimes_e^{\delta_D} \mathbf{H}$ is an IDL-algebra (i.e. an *involutive* SBP_0 -algebra). Let us denote, for every \mathbb{H} subvariety of PSH , $\text{SMTL}_{\mathbb{H}}$ and $\text{IDL}_{\mathbb{H}}$ the full subcategories of respectively SMTL and IDL-algebras such that each $\mathbf{A} \in \text{SMTL}_{\mathbb{H}} \cup \text{IDL}_{\mathbb{H}}$ has its largest prelinear sub-semihoop in \mathbb{H} . We hence proved the following.

Theorem 2. *For every subvariety \mathbb{H} of PSH , the category $\mathcal{T}_{\mathbb{H}}$ is equivalent to $\text{SMTL}_{\mathbb{H}}$ and to $\text{IDL}_{\mathbb{H}}$. Hence, in particular, $\text{SMTL}_{\mathbb{H}}$ and $\text{IDL}_{\mathbb{H}}$ are equivalent categories for every \mathbb{H} .*

The following are relevant examples of varieties which are categorically equivalent by consequence of Theorem 2 above.

- (i) The variety SMTL of SMTL-algebras and the variety IDL of IDL-algebras, which in turn are equivalent to the category \mathcal{T}_{PSH} .
- (ii) The variety \mathbb{P} of product algebras and the variety DLMV generated by perfect MV-algebras, which in turn are equivalent to the category \mathcal{T}_{CH} where CH is the variety of cancellative hoops.
- (iii) The variety \mathbb{G} of Gödel algebras and the variety NM^- of Nilpotent Minimum algebras generated by chains with no negation fixpoint, which in turn are equivalent to the category \mathcal{T}_{GH} , where GH is the variety of Gödel hoops.

From triples to quadruples

To construct all algebras in SBP_0 we shall use all wdl-admissible operators. In order to cope with generic wdl-admissible operators, we take another step of generalisation and introduce *prelinear-semihoop-based quadruples*. These are defined in the following way. Fix again any subvariety \mathbb{H} of PSH and let $\mathcal{Q}_{\mathbb{H}}$ be the following category:

- The objects of $\mathcal{Q}_{\mathbb{H}}$ are quadruples $(\mathbf{B}, \mathbf{H}, \vee_e, \delta)$ where $\mathbf{H} \in \mathbb{H}$, $(\mathbf{B}, \mathbf{H}, \vee_e) \in \mathcal{T}_{\mathbb{H}}$ and $\delta : H \rightarrow H$ is wdl-admissible.
- The morphisms are pairs $(f, g) : (\mathbf{B}_1, \mathbf{H}_1, \vee_e^1, \delta_1) \rightarrow (\mathbf{B}_2, \mathbf{H}_2, \vee_e^2, \delta_2)$, such that (f, g) is a good morphism pair from $(\mathbf{B}_1, \mathbf{H}_1, \vee_e^1)$ to $(\mathbf{B}_2, \mathbf{H}_2, \vee_e^2)$, and $g(\delta_1(x)) = \delta_2(g(x))$ for all $x \in H_1$.

Let $\text{SBP}_{0\mathbb{H}}$ be the full subcategory of SBP_0 consisting of those algebras whose largest prelinear sub-semihoop belongs to \mathbb{H} . Then the following holds.

Theorem 3. *Given any subvariety \mathbb{H} of PSH , the categories $\text{SBP}_{0\mathbb{H}}$ and $\mathcal{Q}_{\mathbb{H}}$ are equivalent. In particular \mathcal{Q}_{PSH} and SBP_0 are equivalent categories.*

Boolean products

If we focus on our very construction, any SBP_0 -algebra results to be an exemplification of a weak Boolean product. We recall the definition from [2].

Definition 4. A *weak Boolean product* of an indexed family $(\mathbf{A}_x)_{x \in X}$, $X \neq \emptyset$, of algebras is a subdirect product $\mathbf{A} \leq \prod_{x \in X} \mathbf{A}_x$, where X can be endowed with a Boolean space topology such that:

1. The equalizer $\llbracket x = y \rrbracket$ is open for $x, y \in \mathbf{A}$.
2. If $x, y \in \mathbf{A}$ and N is a clopen subset of X , then $x|_N \cup y|_{X \setminus N} \in \mathbf{A}$.

If $\llbracket x = y \rrbracket$ is clopen, $\mathbf{A} \leq \prod_{x \in X} \mathbf{A}_x$ is a *Boolean product*.

We are able to exhibit an explicit proof of the following standard result.

Theorem 5. *Every SBP_0 -algebra \mathbf{A} is a weak Boolean product of the indexed family $\mathcal{B}(\mathbf{A})/\mathfrak{p} \otimes_e^\delta \mathcal{H}(\mathbf{A})/\mathfrak{p}$, for some wdl-admissible operator δ , and for $\mathfrak{p} \in \text{Max } \mathcal{B}(\mathbf{A})$.*

In particular, in the proof of Theorem 5 above, it results that if the Boolean skeleton of \mathbf{A} is complete, then we actually prove that $\llbracket x = y \rrbracket$ is clopen. Thus, the following holds.

Theorem 6. *Let \mathbf{A} a SBP_0 -algebra whose Boolean skeleton is complete. Then \mathbf{A} is a Boolean product of the indexed family $\mathcal{B}(\mathbf{A})/\mathfrak{p} \otimes_e^\delta \mathcal{H}(\mathbf{A})/\mathfrak{p}$, for some wdl-admissible operator δ , and for $\mathfrak{p} \in \text{Max } \mathcal{B}(\mathbf{A})$.*

Note that this does not characterize SBP_0 -algebras with complete Boolean skeleton. Indeed, we obtain that the equalizer is clopen also in the case that, for instance, a SBP_0 -algebra \mathbf{A} is isomorphic to the direct product of the family $\mathcal{B}(\mathbf{A})/\mathfrak{p} \otimes_e^\delta \mathcal{H}(\mathbf{A})/\mathfrak{p}$, for $\mathfrak{p} \in \text{Max } \mathcal{B}(\mathbf{A})$, where $\mathcal{B}(\mathbf{A})$ need not be complete.

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Embedding Superintuitionistic Logics into Extensions of Lax Logic via Canonical Formulas

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1 Introduction

The propositional lax logic **PLL**, is an intuitionistic modal logic axiomatized by the formulas $p \rightarrow \circ p$, $\circ \circ p \rightarrow \circ p$, and $\circ(p \wedge q) \leftrightarrow (\circ p \wedge \circ q)$. \circ , called the lax modality, is quite peculiar, as it has features of both, the box and the diamond modality from classical modal logic. The lax modality arises in several contexts. From an algebraic point of view, \circ corresponds to a nucleus on a Heyting algebra [7]. Goldblatt noticed that the axioms of \circ also describe the logic of Grothendieck topologies [6]. The motivation for Fairtlough and Mendler [5] comes from formal verification of computer hardware. Wolter and Zakharyashev [8] studied **PLL** in the context of preservation results between superintuitionistic to intuitionistic modal logics. **PLL** is Kripke complete, has the finite model property and is decidable [6, 5, 8].

In this paper, we make a first attempt to study extensions of **PLL**. In particular, we will investigate extensions of **PLL** that arise by equipping a superintuitionistic logic L (i.e., an extension of the intuitionistic propositional calculus **IPC**) with the \circ modality. These are extensions of the shape **PLL** + Γ , where Γ is a set of formulas in the language of intuitionistic logic.

We will show several preservation results. If $L = \mathbf{IPC} + \Gamma$ has the finite model property, is tabular or Kripke complete, then **PLL** + Γ enjoys the same property. Moreover, if L is Kripke complete and decidable, then also **PLL** + Γ is decidable (Theorem 4.2). Similar results for extensions of the basic intuitionistic modal logic **IntK** have been obtained by Wolter and Zakharyashev [8]. Their proofs are based on embedding intuitionistic modal logics into fusions of classical modal logics.

We are using different methods. In order to obtain our preservation results, we develop the machinery of Zakharyashev's canonical formulas (see, e.g., [4]) for **PLL**. We are taking an algebraic approach to this problem. Our formulas can be seen as a combination of $(\wedge, \rightarrow, \perp)$ -formulas of [1] and stable canonical formulas of [2]. The main preservation results are then obtained using canonical formulas. This is done by replacing each formula of **PLL** with the corresponding canonical formula (Theorem 3.3) and using the fact that a refutation of the canonical formula of a finite nuclear Heyting algebra (A, j) in some nuclear Heyting algebra (B, j) is equivalent to the existence of a special embedding of (A, j) into a homomorphic image of (B, j) (Theorem 3.2).

2 Propositional Lax Logic

Recall that a unary operation j on a Heyting algebra A is called a *nucleus* if for each $a, b \in A$ we have (1) $a \leq j(a)$, (2) $j(a \wedge b) = j(a) \wedge j(b)$, (3) $j(j(a)) = j(a)$. A pair (A, j) is called a *nuclear Heyting algebra*, if A is a Heyting algebra and j a nucleus on A . By the standard

Lindenbaum-Tarski construction one can show that every extension of **PLL** is complete with respect to nuclear Heyting algebras.

An *intuitionistic modal frame* is a triple (X, \leq, R) , where (X, \leq) is a poset and $R \subseteq X^2$ a binary relation such that $\leq \circ R = R \circ \leq = R$ [8]. A **PLL**-frame is an intuitionistic Kripke frame (X, \leq, R) such that (1) xRy implies $x \leq y$, and (2) xRy implies that there is $z \in X$ such that xRz and zRy [6, 5]. Let (X, \leq, R) be a **PLL**-frame. For each $x \in X$ let $R[x] = \{y \in X : xRy\}$ and for $U \subseteq X$ let $\Box_R(U) = \{x \in X : R[x] \subseteq U\}$.

There is a duality between nuclear Heyting algebras and the so-called nuclear spaces, which can be seen as particular examples of **PLL**-frames, see [3] for details. Here we only note that if (X, \leq, R) is a **PLL**-frame, then $(\text{Up}(X), \Box_R)$ is a nuclear Heyting algebra, where $\text{Up}(X)$ is the Heyting algebra of all \leq -upsets of (X, \leq, R) , and if (A, j) is a nuclear Heyting algebra, then (X^*, \subseteq, R^*) is a **PLL**-frame, where X^* is the set of prime filters of A and for $x, y \in X^*$ we have xR^*y if $j^{-1}(x) \subseteq y$.

The truth and validity of modal formulas in **PLL**-frames, tabularity, the finite model property (the fmp), and Kripke completeness in extensions of **PLL** are defined in a standard way, see, e.g., [6, 5, 8].

Theorem 2.1 ([6, 5, 8]). ***PLL** has the fmp and is decidable.*

3 Canonical Formulas for PLL

Zakharyashev's canonical formulas are a powerful tool for studying the lattice of superintuitionistic and transitive modal logics. Every superintuitionistic logic is axiomatizable by these formulas [4, Thm. 9.44]. Zakharyashev's method was model theoretic. In [1] an algebraic approach to canonical formulas was developed. For a subdirectly irreducible (s.i.) Heyting algebra A and $D \subseteq A^2$, the $(\wedge, \rightarrow, \perp)$ -canonical formula $\beta(A, D)$ of (A, D) encodes fully the $(\wedge, \rightarrow, 0)$ -structure of A and it encodes the behaviour of \vee only partially, for the pairs in D . Using the fact that $(\wedge, \rightarrow, 0)$ -algebras (implicative semilattices) are locally finite (Diego's theorem) [1] shows that every intermediate logic is axiomatizable by $(\wedge, \rightarrow, \perp)$ -canonical formulas.

In [2] stable canonical rules and formulas for modal algebras are defined. Using stable maps between modal algebras it is shown that every modal logic is axiomatizable by stable canonical rules and that every transitive modal logic is axiomatizable by stable canonical formulas. We will adopt the notion of a stable map to the setting of nuclear Heyting algebras and we will incorporate appropriate parts of stable canonical formulas and of $(\wedge, \rightarrow, \perp)$ -canonical formulas into the definition of canonical formulas for nuclear Heyting algebras.

Let (A, j) and (B, j) be nuclear Heyting algebras. A map $f : A \rightarrow B$ is called a $(\wedge, \rightarrow, 0)$ -morphism if for each $a, b \in A$ we have $f(0) = 0$, $f(a \wedge b) = f(a) \wedge f(b)$ and $f(a \rightarrow b) = f(a) \rightarrow f(b)$.

It is well known that there is a one-to-one correspondence between the congruences and filters of a given Heyting algebra and that a Heyting algebra A is subdirectly irreducible iff A has the second largest element. Let (A, j) be a nuclear Heyting algebra. Then since $a \leq j(a)$ for each $a \in A$, it is easy to show that there is a one-to-one correspondence between the j -congruences (congruences of nuclear Heyting algebras) and filters of A . Therefore, (A, j) is subdirectly irreducible nuclear Heyting algebra iff A is a subdirectly irreducible Heyting algebra.

Definition 3.1. *Let (A, j) , (B, j) be nuclear Heyting algebras, $D^\vee \subseteq A^2$ and $D^\circ \subseteq A$. Let $f : A \rightarrow B$ be a $(\wedge, \rightarrow, 0)$ -morphism.*

- *If $j(f(a)) \leq f(j(a))$ for all $a \in A$, then we call f stable.*

- If $f(a \vee b) = f(a) \vee f(b)$ for every $(a, b) \in D^\vee$ and $f(j(a)) = j(f(a))$ for every $a \in D^\circ$, we say that f is (D^\vee, D°) -stable.

Let (A, j) be a finite s.i. nuclear Heyting algebra, let $D^\vee \subseteq A^2$, $D^\circ \subseteq A$. And let $s \in A$ be the second largest element of (A, j) . For $a \in A$ let p_a be a propositional letter. Define the canonical formula of (A, D^\vee, D°) as

$$\begin{aligned} \beta(A, D^\vee, D^\circ, \perp) := & \{p_0 \leftrightarrow 0\} \wedge \bigwedge \{p_{a*b} \leftrightarrow (p_a * p_b) \mid a, b \in A, * \in \{\wedge, \rightarrow\}\} \wedge \\ & \bigwedge \{\circ p_a \rightarrow p_{j(a)} \mid a \in A\} \wedge \\ & \bigwedge \{p_{a \vee b} \leftrightarrow (p_a \vee p_b) \mid (a, b) \in D^\vee\} \wedge \\ & \bigwedge \{p_{j(a)} \rightarrow \circ p_a \mid a \in D^\circ\} \\ & \rightarrow p_s. \end{aligned}$$

The key property of canonical formulas is formulated in the next theorem.

Theorem 3.2. *For every nuclear Heyting algebra, the following are equivalent:*

1. $(B, j) \not\models \beta(A, D^\vee, D^\circ)$
2. *There is a s.i. homomorphic image (C, k) of (B, j) and a (D^\vee, D°) -stable embedding from (A, j) into (C, k) .*

The proof of the next theorem essentially uses the local finiteness of implicative semilattices.

Theorem 3.3. *For every PLL-formula φ , there are $((A_1, j_1), D_1^\vee, D_1^\circ), \dots, ((A_n, j_n), D_n^\vee, D_n^\circ)$, such that (A_i, j_i) is a finite s.i. nuclear Heyting algebra, $D_i^\vee \subseteq A_i^2$ and $D_i^\circ \subseteq A_i$, for each $i = 1, \dots, n$, and for each nuclear Heyting algebra (B, j) , TFAE:*

1. $(B, j) \not\models \varphi$.
2. *There is $1 \leq i \leq n$ and a s.i. homomorphic image (C, k) of (B, j) and a (D_i^\vee, D_i°) -stable embedding from (A_i, j_i) into (C, k) .*

Corollary 3.4. *Every extension M of PLL is axiomatizable by canonical formulas.*

4 Preservation results

We are ready to formulate our main preservation results. Let $L = \mathbf{IPC} + \Gamma$ be a superintuitionistic logic. Define $\sigma(L) := \mathbf{PLL} + \Gamma$. Then $\sigma(L)$ is an extension of **PLL**. Let **ExtIPC** denote the lattice of superintuitionistic logics and **ExtPLL** the lattice of (normal) extensions of **PLL**.

Theorem 4.1. $\sigma : \mathbf{ExtIPC} \rightarrow \mathbf{ExtPLL}$ *is an embedding.*

The proof of the next theorem essentially uses canonical formulas.

Theorem 4.2. *Let L be a superintuitionistic logic. If L has one of the properties*

- *tabularity,*
- *the fmp,*

- *Kripke completeness*,
- *decidability and Kripke completeness*,

then $\sigma(L)$ also enjoys the same property.

As a corollary of Theorem 4.2 we obtain (an alternative proof of the fact) that **PLL** has the fmp and is decidable.

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Logics of knowledge and belief of sceptical agents

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To model knowledge or belief of (or within groups of) rational agents logically, one has to start with specifying what kind of agents, and consequently what notion of knowledge, we have in mind. A prototypical agent for this line of work is a scientist (cf. e.g. with the notion of scientific or rational scepticism), working with collections of data — and those, in contrast with complete and consistent descriptions of a state of the world, might be *incomplete* and *inconsistent*. The agent (e.g. by weighting the evidence supported by the available data) eventually accepts some of the available data as knowledge or belief. But only *confirmed data* might be accepted. The background propositional logic we use to model collections of data is therefore a particular substructural logic of information states, where collections of data are modeled as (not necessarily consistent) theories. We allow for some information states to act as reliable sources of confirmation of data available at the current state. The modal part of the logic then consists of epistemic operators of knowledge and belief confirmed by a source, which are, in contrast to standard approaches, *diamond-like* operators. Such logics have been studied in [1], based on distributive non-associative commutative Lambek calculus with a negation as a basic propositional logic, allowing for further modularity by varying the propositional base.

In this paper, we present a multiagent extension of a substructural epistemic logic introduced in [1]. The framework is, as in [1], based on the relational semantics for distributive substructural logics as presented in Restall's book [5], interpreting the elements of a relational frame as information states, where the principal epistemic relation between the states (which replaces the accessibility relation of normal modal logics) is the one of being a reliable source of information. From this point of view it is natural to define the epistemic operator existentially as a (backward-looking) diamond modality. More technically — a piece of data φ is known in an information state w , if φ holds in a state v which is a source for w (where being a source implies v precedes the current state w and is compatible with it).

Basic logic for sceptical agents

The language is that of a full Lambek logic extended with modalities:

$$\varphi ::= p \mid t \mid \varphi \otimes \varphi \mid \varphi \rightarrow \varphi \mid \top \mid \perp \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \neg \varphi \mid \langle k \rangle \varphi \mid \langle b \rangle \varphi$$

where $\langle k \rangle$ is the confirmed knowledge operator, $\langle b \rangle$ is the confirmed belief operator, $p \in \mathbf{Prop}$ is a propositional atom.

A *frame* is a tuple $F = (X, \leq, R, L, C, S^k, S^b)$, where (X, \leq) is a poset of information states, R is a ternary monotone relation¹ on X , L a nonempty upwards closed set of *logical states*, C is a binary *compatibility* monotone relation on X . We moreover consider R to be commutative and C to be symmetric. For the interpretation of the propositional part consult [5, 1].

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¹What we call monotone here corresponds to respective plump conditions in [5].

The epistemic relations S^k and S^b are binary monotone relations on (X, \leq) ($\leq \circ S^k \circ \leq \subseteq S^k$ and $\leq \circ S^b \circ \leq \subseteq S^b$), satisfying (all or some of) the conditions:

$$sS^b x \text{ and } s'S^b x \quad \text{implies} \quad sCs' \quad (1)$$

$$sS^k x \quad \text{implies} \quad sS^b x \quad (2)$$

$$sS^k x \quad \text{implies} \quad s \leq x \quad (3)$$

$$sS^b x \quad \text{implies} \quad xCs \quad (4)$$

We read $sS^k x$ as *s is a reliable source confirming knowledge in x*, and similarly for belief. The modalities are interpreted as follows:

$$- x \Vdash \langle k \rangle \varphi \text{ iff } \exists s (sS^k x \wedge s \Vdash \varphi) \text{ (confirmed knowledge)}$$

$$- x \Vdash \langle b \rangle \varphi \text{ iff } \exists s (sS^b x \wedge s \Vdash \varphi) \text{ (confirmed belief)}$$

Properties of the source relations:

- sources for belief are mutually compatible
- $S^k \subseteq S^b$ (as well as $S^k \subseteq \leq \cap C$) implies that sources for knowledge are mutually compatible as well
- sources are self-compatible (and therefore *consistent*)
- $S^k \subseteq \leq$ implies that what is known is satisfied in the current state

To sum up properties of the resulting notions of knowledge and belief, *knowledge implies belief, it is strongly consistent and factive. Belief is consistent.*

Axioms and completeness

The basic epistemic logic introduced in [1] extends the propositional base with the following axioms, first two of which are widely accepted as intuitive properties of knowledge:

$$\langle k \rangle \varphi \rightarrow \varphi \text{ (factivity)} \quad \neg \varphi \wedge \langle k \rangle \varphi \rightarrow \perp \text{ (strong consistency)}$$

$$\frac{\varphi \rightarrow \psi}{\langle k \rangle \varphi \rightarrow \langle k \rangle \psi} \text{ (monotonicity)} \quad \langle k \rangle (\varphi \vee \psi) \rightarrow \langle k \rangle \varphi \vee \langle k \rangle \psi$$

Factivity and consistency are standard properties of knowledge. The monotonicity rule is considered to be a weak form of logical omniscience, but the system avoids its stronger forms (the analogue of the necessitation rule and the K-axiom of normal modal logics) as well as other closure properties discussed and often accepted in normal epistemic logics (like the positive and negative introspection axioms). For these properties we provided characteristic frame conditions, so that they can be present in the system if they are considered to be appropriate for some specific epistemic context. The system is modular in the sense that the axiomatization of the epistemic operator is sound and complete over a wide class of background propositional logics, which makes the system potentially applicable to a wide class of epistemic contexts.

We can naturally extend this system with the belief operator with (at least) the following axioms:

$$\langle k \rangle \varphi \rightarrow \langle b \rangle \varphi \quad \langle b \rangle \neg \varphi \wedge \langle b \rangle \varphi \rightarrow \perp \text{ (consistency)}$$

$$\frac{\varphi \rightarrow \psi}{\langle b \rangle \varphi \rightarrow \langle b \rangle \psi} \text{ (monotonicity)} \quad \langle b \rangle (\varphi \vee \psi) \rightarrow \langle b \rangle \varphi \vee \langle b \rangle \psi$$

The proof of completeness based on canonical models from [1] can easily be extended to obtain the following correspondence (and canonicity and completeness) results, including some additional axioms, e.g. introspections and Stalnaker's axiom $\langle b \rangle \varphi \rightarrow \langle b \rangle \langle k \rangle \varphi$:

Axiom or rule	condition
$\langle k \rangle (\varphi \vee \psi) \rightarrow \langle k \rangle \varphi \vee \langle k \rangle \psi$	\top
$\langle b \rangle (\varphi \vee \psi) \rightarrow \langle b \rangle \varphi \vee \langle b \rangle \psi$	\top
$\langle k \rangle \varphi \rightarrow \varphi$ $\langle k \rangle \varphi \rightarrow \langle b \rangle \varphi$ $\langle b \rangle \varphi \wedge \langle b \rangle \neg \varphi \rightarrow \perp$ $\langle k \rangle \varphi \wedge \neg \varphi \rightarrow \perp$	$sS^k x \rightarrow s \leq x$ $sS^k x \rightarrow sS^b x$ $sS^b x \wedge s' S^b x \rightarrow sC s'$ $sS^k x \rightarrow sC x$
$\langle k \rangle \varphi \rightarrow \langle k \rangle \langle k \rangle \varphi$ $\langle b \rangle \varphi \rightarrow \langle b \rangle \langle b \rangle \varphi$ $\langle b \rangle \varphi \rightarrow \langle b \rangle \langle k \rangle \varphi$ $\langle k \rangle \varphi \wedge \langle k \rangle \psi \rightarrow \langle k \rangle (\varphi \wedge \psi)$ $\vdash \varphi / \vdash \langle k \rangle \varphi$	$sS^k x \rightarrow \exists s' (sS^k s' S^k x)$ $sS^b x \rightarrow \exists s' (sS^b s' S^b x)$ $sS^b x \rightarrow \exists s' (sS^k s' S^b x)$ $sS^k x \wedge tS^k x \rightarrow \exists v (vS^k x \wedge s, t \leq v)$ $(\forall x \in L)(\exists s \in L) sS^k x$

To illustrate a further modularity of the setting, consider a few additional properties of negation: in presence of $\top \rightarrow \varphi \vee \neg \varphi$ (condition $xCy \rightarrow y \leq x$), the factivity scheme $\langle k \rangle \varphi \rightarrow \varphi$ is derivable from a stronger consistency scheme $\langle k \rangle \varphi \wedge \neg \varphi \rightarrow \perp$. In presence of $\varphi \wedge \neg \varphi \rightarrow \perp$ (condition xCx), it is the other way round.

Display proof theory

We have decided to employ a format of display calculi to obtain a nice structural proof theory for the logics, mainly for its modularity and a general Belnap's cut-elimination theorem. For simplicity and a clear reference, consider the display calculus for bi-intuitionistic logic as the base (as presented with all necessary definitions in [3] Definition 21) and extend it with *logical* and *display* rules for negation (which is a *g*-type connective and its structural counterpart is denoted \sharp) and our two epistemic modalities:

$$\frac{X \vdash \varphi}{\bullet^b X \vdash \langle b \rangle \varphi} \quad \frac{\bullet^b \varphi \vdash X}{\langle b \rangle \varphi \vdash X} \quad \frac{\bullet^b X \vdash Y}{X \vdash \circ^b Y} \quad \frac{X \vdash \varphi}{\bullet^k X \vdash \langle k \rangle \varphi} \quad \frac{\bullet^k \varphi \vdash X}{\langle k \rangle \varphi \vdash X} \quad \frac{\bullet^k X \vdash Y}{X \vdash \circ^k Y}$$

and *structural* rules (with the corresponding epistemic axioms mentioned on the side):

$$\frac{X \vdash Y}{\bullet^k X \vdash Y} \langle k \rangle \varphi \rightarrow \varphi \quad \frac{\bullet^b X \vdash Y}{\bullet^k X \vdash Y} \langle k \rangle \varphi \rightarrow \langle b \rangle \varphi \quad \frac{\bullet^b \bullet^k X \vdash Y}{\bullet^b X \vdash Y} \langle b \rangle \varphi \rightarrow \langle b \rangle \langle k \rangle \varphi$$

$$\frac{X \vdash \sharp Y}{X \vdash \circ^b(\bullet^b Y > I)} \langle b \rangle \varphi \wedge \langle b \rangle \neg \varphi \rightarrow \perp \quad \frac{X \vdash \sharp Y}{X \vdash \bullet^k Y > I} \langle k \rangle \varphi \wedge \neg \varphi \rightarrow \perp$$

This results in a complete and cut-free calculus for the logic given by a choice of the epistemic axioms and their respective structural rules.

A multiagent extension by common knowledge

A challenge is how to deal with group attitudes like group knowledge and belief in this setting. We fix a finite set of indexes $I = \{1, \dots, k\}$ to refer to agents in a group and consider set of indexed modalities for knowledge (or belief or both). For example, the common knowledge of φ can be expressed as the following greatest fixed point

$$C\varphi \equiv \nu x. \bigwedge_{i \in I} \langle k \rangle_i (\varphi \wedge x).$$

As noted in [1] (and it is not surprising), the resulting logic is not compact, and therefore not strongly complete. While proving weak completeness of this particular fixed point extension of the logic seems to be as hard as proving completeness of the full fixed point logic, we aim at a stronger, *infinitary* Hilbert-style proof system for the logic, and prove *strong* completeness of this stronger axiomatization via a canonical model, as was done for PDL and epistemic logic with common knowledge in [4]. Indeed, if the infinitary rule for $C\varphi$ is formulated using unfoldings of the fixed point, and closed under conditionalisation (cf. pseudo-modalities in 3.3 of [4]), we may prove the infinitary version of the pair-extension theorem (which we found of independent interest) and construct the canonical model similarly as in [5, 1].

Extending the logic with common knowledge with further dynamic modalities (like public announcement), and providing it with a structural proof theory is still work in progress.

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On Paraconsistent Weak Kleene Logic and Involutive Bisemilattices

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Extended abstract

In his *Introduction to Metamathematics* [13, § 64], S.C. Kleene distinguishes between a “strong sense” and a “weak sense” of propositional connectives when partially defined predicates are present. Each of these meanings is made explicit via certain 3-valued truth tables, which have become widely known as *strong Kleene tables* and *weak Kleene tables*, respectively. If the elements of the base set are labelled as $0, \frac{1}{2}, 1$, the strong tables for conjunction, disjunction and negation are given by $a \wedge b = \min\{a, b\}$, $a \vee b = \max\{a, b\}$, $\neg a = 1 - a$. The weak tables for the same connectives, on the other hand, are given by:

\wedge	0	$\frac{1}{2}$	1	\vee	0	$\frac{1}{2}$	1	\neg	
0	0	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	1	1	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
1	0	$\frac{1}{2}$	1	1	1	$\frac{1}{2}$	1	0	1

Each set of tables naturally gives rise to two options for building a many-valued logic, depending on whether we choose to consider only 1 as a designated value, or 1 together with the “middle” value $\frac{1}{2}$. Thus, there are four logics in the Kleene family:

- Strong Kleene logic [13, § 64], given by the strong Kleene tables with 1 as a designated value;
- The Logic of Paradox, LP [17], given by the strong Kleene tables with $1, \frac{1}{2}$ as designated values;
- Bochvar’s logic [4], given by the weak Kleene tables with 1 as a designated value;
- Paraconsistent Weak Kleene logic, PWK [12, 19], given by the weak Kleene tables with $1, \frac{1}{2}$ as designated values.

The first three logics have all but gone unnoticed by mathematicians, philosophers, and computer scientists. Strong Kleene logic has applications in artificial intelligence as a model of partial information [1] and nonmonotonic reasoning [22], and in philosophy as a bedrock logic for Kripke’s theory of truth and other related proposals [10]; the theory of *Kleene algebras*, moreover, has stirred a considerable amount of interest in general algebra [14]. LP has been fervently supported by Graham Priest in the context of a dialethic approach to the truth-theoretical and set-theoretical paradoxes, and has enjoyed an enduring popularity that made it the object of intense study both on the proof-theoretical and on the semantical level [18]. And even Bochvar’s logic, while not the biggest game in the 3-valued town, is still touched on in several papers and books (see e.g. [3, Ch. 5]).

In terms of sheer impact, PWK is the “ugly duckling” in the family of Kleene logics. Essentially introduced by Halldén [12] and, in a completely independent way, by Prior [19], it is often passed over in silence in the main reviews on finite-valued logics. Most of the extant literature concerns the philosophical interpretation of the third value [2, 5, 9, 12, 23] and a discussion of the so-called *contamination principle* (any sentence containing a subsentence evaluated at $\frac{1}{2}$ is itself evaluated at $\frac{1}{2}$), as well as proof systems of various kinds [2, 7, 8, 11]. An important study on PWK as a consequence relation is [6], to be analysed later in this paper. It has also been noticed early on that the negation-free reduct of element algebra **WK** defined by the weak Kleene tables is an instance of a *distributive bisemilattice*, a notion on which there is a burgeoning literature — actually, the variety of distributive bisemilattices is generated by this reduct. Yet, despite this intriguing connection to algebra, virtually no paper has viewed PWK in the perspective of Algebraic Logic. This makes a sharp contrast with LP, which has been thoroughly studied under this aspect [20, 21]. A partial exception is [11], but a careful assessment of the results in this paper is made difficult by issues with the similarity type of the algebras and logics it considers, and by the authors’ failure to adopt the language and concepts of mainstream Abstract Algebraic Logic (AAL).

The aim of this work is to give a contribution towards filling this gap, so as to surmise that the ugly duckling might actually be a gorgeous swan. Firstly, we give a Hilbert-style system for PWK. Actually, we prove the following:

Theorem A. *Given a Hilbert system $\langle AX, MP \rangle$ for CL, where AX is a set of axioms and MP is the only rule, the Hilbert system $\langle AX, RMP \rangle$ is a Hilbert system of PWK, where RMP is the Restricted Modus Ponens given by*

$$[RMP] \frac{\alpha \quad \alpha \rightarrow \beta}{\beta} \quad \text{provided that } \text{var}(\alpha) \subseteq \text{var}(\beta).$$

Next, we introduce some algebraic structures for PWK, called *involutive bisemilattices*, which are algebras $\langle A, \wedge, \vee, \neg, 0, 1 \rangle$ such that $\langle A, \wedge, 0 \rangle$ and $\langle A, \vee, 1 \rangle$ are a meet and a join semilattices with lower and upper bound, respectively, and \neg is an idempotent operation, satisfying the De Morgan identities, and moreover the equation:

$$x \wedge (\neg x \vee y) \approx x \wedge y.$$

Among other results, we show that involutive bisemilattices are always distributive as bisemilattices and that **WK** generates the variety \mathcal{IBSL} of involutive bisemilattices. More in detail, we prove the following:

Theorem B. *The only nontrivial subdirectly irreducible bisemilattices are **WK**, the 2-element semilattice \mathbf{S}_2 , and the 2-element Boolean algebra \mathbf{B}_2 , up to isomorphism.*

Finally, we use the algebraic construction of Plonka sums, which was introduced in [15, 16], and prove the following representation theorem for involutive bisemilattices:

Theorem C. *Every involutive bisemilattice is representable as the Plonka sum over a direct system of Boolean algebras.*

As a consequence, we obtain that the equation satisfied by all the involutive bisemilattices are exactly the regular equations satisfied by all the Boolean algebras. We then axiomatise relative to \mathcal{IBSL} its nontrivial subvarieties, namely, Boolean algebras and lower-bounded semilattices.

In the second part, we study PWK by recourse to the toolbox of Abstract Algebraic Logic. It is not inappropriate to wonder whether the variety \mathcal{IBSL} is the actual algebraic counterpart of the logic PWK. Such a guess stands to reason, for PWK is the logic defined by the matrix \mathbf{PWK} with \mathbf{WK} as an underlying algebra, and \mathcal{IBSL} is the variety generated by \mathbf{WK} . We show though that \mathcal{IBSL} is not the equivalent algebraic semantics of any algebraisable logic, and furthermore, PWK is not algebraisable, since it is not even protoalgebraic. We also show that PWK is not selfextensional either.

We start by characterising the Leibniz congruence of the models of PWK, what allows us to prove that the class $\text{Alg}^*(\text{PWK})$ of the algebraic reducts of the reduced models of PWK is a subclass of \mathcal{IBSL} . As a consequence we obtain the following:

Theorem D. *The intrinsic variety of PWK is $\mathbb{V}(\text{Alg}^*(\text{PWK})) = \mathcal{IBSL}$.*

Next, we fully characterise the deductive PWK-filters on members of \mathcal{IBSL} and the reduced matrix models of PWK. More in detail, given an involutive bisemilattice \mathbf{B} , we define the set of *positive* elements of \mathbf{B} as the set $P(\mathbf{B}) = \{c \in B : 1 \leq c\}$, that is, those elements that are above of 1 in the order given by \vee , and we prove the following:

Theorem E. *$\mathbf{B} \in \text{Alg}^*(\text{PWK})$ if and only if \mathbf{B} is an involutive bisemilattice and for every $a < b$ positive elements, there is $c \in B$ such that*

$$1 \leq \neg b \vee c \quad \text{but} \quad 1 \not\leq \neg a \vee c.$$

Moreover, $\langle \mathbf{B}, F \rangle \in \text{Mod}^(\text{PWK})$ if and only if \mathbf{B} is an involutive bisemilattice satisfying the above condition and $F = P(\mathbf{B})$.*

As a consequence, we obtain that PWK is truth-equational, since for every involutive bisemilattice \mathbf{B} , $P(\mathbf{B})$ can be equationally described by the equation $1 \vee x \approx x$. This result finishes the exact location of PWK within the Leibniz hierarchy. Interestingly enough, we show that the class $\text{Alg}^*(\text{PWK})$ is not even a generalised quasivariety, since it is not closed under quotients nor subalgebras.

Next, we investigate PWK with the methods of second-order AAL. We prove that the classes $\text{Alg}^*(\text{PWK})$ and $\text{Alg}(\text{PWK})$, are different. Furthermore, the class $\text{Alg}(\text{PWK})$ is formed by the involutive bisemilattices with at most one *fix* element (i.e., $\neg c = c$), which can be expressed by a quasiequation. Hence, we obtain the following:

Theorem F. *$\text{Alg}(\text{PWK})$ is the quasivariety of involutive bisemilattices satisfying the quasiequation*

$$\neg x \approx x \ \& \ \neg y \approx y \Rightarrow x \approx y.$$

We remark the fact that PWK is one of the few natural examples of a logic whose algebraic counterpart, i.e. $\text{Alg}(\text{PWK})$, is a quasivariety but not a variety. Finally, using the representation of an involutive bisemilattice as a Płonka sum of Boolean algebras, we prove that every $\mathbf{B} \in \text{Alg}(\text{PWK})$ is a subalgebra of a power of \mathbf{WK} , which entails the following:

Theorem G. *$\text{Alg}(\text{PWK})$ is the quasivariety generated by \mathbf{WK} .*

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A representation for the n -generated free algebra in the subvariety of BL-algebras generated by $[0, 1]_{\mathbf{MV}} \oplus [0, 1]_{\mathbf{G}}$

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Abstract

We give a representation for a subvariety of BL-algebras generated by the ordinal sum of the standard MV algebra on $[0,1]$ and the Gödel hoop on $[0,1]$. In this representation there is a clear insight of the role of the regular and dense elements. Using the representation we give a simple characterization of the maximal filters in the free algebra.

BL-algebras were introduced by Hájek (see [1]) to formalize fuzzy logics in which the conjunction is interpreted by continuous t-norms over the real interval $[0, 1]$. These algebras form a variety, usually called \mathcal{BL} . Important examples of its proper subvarieties are the variety \mathcal{MV} of MV-algebras, \mathcal{P} of product algebras and \mathcal{G} of Gödel algebras.

For each integer $n \geq 0$, we will write $Free_{\mathcal{BL}}(n)$ to refer to the free n -generated BL-algebra, which is generated by the algebra $(n+1)[0, 1]_{\mathbf{MV}}$, that is, the ordinal sum of $n+1$ copies of the standard MV-algebra. This fact allows us to characterize the free n -generated BL-algebra $Free_{\mathcal{BL}}(n)$ as the algebra of functions $f : (n+1)[0, 1]_{\mathbf{MV}}^n \rightarrow (n+1)[0, 1]_{\mathbf{MV}}$ generated by the projections. Using this, in [3] and [2] there is a representation of the free- n -generated BL-algebra in terms of elements of free Wajsberg hoops (\perp -free subreducts of Wajsberg algebras), organized in a structure based on the ordered partitions of the set of generators and satisfying certain geometrical constraints.

In this work we will concentrate in the subvariety $\mathcal{V} \subseteq \mathcal{BL}$ generated by the ordinal sum of the algebra $[0, 1]_{\mathbf{MV}}$ and the Gödel hoop $[0, 1]_{\mathbf{G}}$, that is, generated by $\mathbf{A} = [0, 1]_{\mathbf{MV}} \oplus [0, 1]_{\mathbf{G}}$. Though it is well-known that $[0, 1]_{\mathbf{G}}$ is decomposable as an infinite ordinal sum of two-elements Boolean algebra, the idea is to treat it as a whole block. The elements of this block are the dense elements of the generating chain and the elements in $[0, 1]_{\mathbf{MV}}$ are usually called regular elements of \mathbf{A} . The main advantage of this approach, is that unlike the work done in [3] and [2], when the number n of generators of the free algebra increase the generating chain remains fixed. This provides a clear insight of the role of the two main blocks of the generating chain in the description of the functions in the free algebra: the role of the regular elements and the role of the dense elements.

We give a functional representation for the free algebra $Free_{\mathcal{V}}(n)$. To define the functions in this representation we need to decompose the domain $[0, 1]_{\mathbf{MV}} \oplus [0, 1]_{\mathbf{G}}$ in a finite number of pieces. In each piece a function $F \in Free_{\mathcal{V}}(n)$ coincides either with McNaughton functions or functions on the free algebra in the variety of Gödel hoops (which we define using a base different from the one given by Gerla in [4]) in the following way:

- For every $\bar{x} \in ([0, 1]_{\mathbf{MV}})^n$, $F(\bar{x}) = f(\bar{x})$, where f is a function of $Free_{\mathcal{MV}}(n)$.

For the rest of the domain, the functions depend on this function $f : ([0, 1]_{\mathbf{MV}})^n \rightarrow [0, 1]_{\mathbf{MV}}$:

- On $([0, 1]_{\mathbf{G}})^n$: If $f(\bar{1}) = 0$, then $F(\bar{x}) = 0$ for every $\bar{x} \in ([0, 1]_{\mathbf{G}})^n$, and if $f(\bar{1}) = 1$, then $F(\bar{x}) = g(\bar{x})$, for a function $g \in Free_{\mathcal{G}}(n)$, for every $\bar{x} \in ([0, 1]_{\mathbf{G}})^n$.

Let $B = \{x_{\sigma(1)}, \dots, x_{\sigma(m)}\}$ be a non empty proper subset of the set of variables $\{x_1, \dots, x_n\}$ and R_B be the subset of $([0, 1]_{MV} \oplus [0, 1]_G)^n$ where $x_i \in B$ if and only if $x_i \in [0, 1]_G$. For every $\bar{x} \in R_B$ we also define \tilde{x} as:

$$\tilde{x}_i = \begin{cases} x_i & \text{if } x_i \notin B \\ 1 & \text{if } x_i \in B \end{cases}$$

- On R_B : If $f(\tilde{x}) < 1$ then $F(\bar{x}) = f(\tilde{x})$, and if $f(\tilde{x}) = 1$, then there is a regular triangulation Δ of $f^{-1}(1) \wedge R_B$ which determines the simplices S_1, \dots, S_n and l Gödel functions h_1, \dots, h_n in $n - m$ variables $x_{\sigma(m+1)}, \dots, x_{\sigma(n)}$ such that $F(\bar{x}) = h_i(x_{\sigma(m+1)}, \dots, x_{\sigma(n)})$ for each point $(x_{\sigma(1)}, \dots, x_{\sigma(m)})$ in the interior of S_i .

This representation allows us to give a simple characterization of the maximal filters in this free algebra.

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Kleene algebras with implication

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Abstract

Inspired by an old construction due to J. Kalman that relates distributive lattices and centered Kleene algebras, in this paper we study an equivalence for a category whose objects are algebras with implication $(H, \wedge, \vee, \rightarrow, 0, 1)$ which satisfies the following property for every $a, b, c \in H$: if $a \leq b \rightarrow c$ then $a \wedge b \leq c$.

1 Introduction

Motivated by results due to Kalman [6], R. Cignoli proved in [5] that the construction of [6] induces a functor K from the category of bounded distributive lattices into the category of centered Kleene algebras. It was also shown in [5] that K has a left adjoint [5, Theorem 1.7]. He also determined an equivalence between the category of bounded distributive lattices and a full subcategory of centered Kleene algebras [5, Theorem 2.4]. In particular, there exists an equivalence between the category of Heyting algebras and the category of centered Nelson algebras [5, Theorem 3.14]. Later these results were extended in the context of residuated lattices [2, 3].

A possible generalization of Heyting algebras is provided by the notion of algebras with implication, *DLI*-algebras for short [4]. Then the natural question arises if it is possible to consider some category of *DLI*-algebras and some category of centered Kleene algebras with implication in order to obtain an equivalence between them. In this article we answer this question by considering *DLI*-algebras $(H, \wedge, \vee, \rightarrow, 0, 1)$ which satisfy the following property for every $a, b, c \in H$: if $a \leq b \rightarrow c$ then $a \wedge b \leq c$ (or, equivalently, the inequality $a \wedge (a \rightarrow b) \leq b$ for every $a, b \in H$).

2 Basic results

In the process of our research on this topic we have found it useful to place our problems in the following general context.

We assume the reader is familiar with bounded distributive lattices and Heyting algebras [1]. A *De Morgan* algebra is an algebra $(H, \wedge, \vee, \sim, 0, 1)$ of type $(2, 2, 1, 0, 0)$ such that $(H, \wedge, \vee, 0, 1)$ is a bounded distributive lattice and \sim fulfills the equations $\sim\sim x = x$ and $\sim(x \vee y) = \sim x \wedge \sim y$. An operation \sim which satisfies the previous two equations is called an *involution*. A *Kleene algebra* is a De Morgan algebra in which the inequality $x \wedge \sim x \leq y \vee \sim y$ holds. A *centered Kleene algebra* is a Kleene algebra with an element c such that $c = \sim c$. It follows from the distributivity of the lattice that c is necessarily unique. We write BDL for the category of bounded distributive lattices and Kl_c for the category of centered Kleene algebras.

If H is a bounded distributive lattice, we define

$$K(H) := \{(a, b) \in H \times H : a \wedge b = 0\}.$$

We have that $(K(H), \wedge, \vee, c, \sim, 0, 1)$ is a centered Kleene algebra by defining the following operations:

$$\begin{aligned} (a, b) \vee (d, e) &:= (a \vee d, b \wedge e), \\ (a, b) \wedge (d, e) &:= (a \wedge d, b \vee e), \\ \sim (a, b) &:= (b, a), \end{aligned}$$

$0 := (0, 1)$, $1 := (1, 0)$ and $c := (0, 0)$. Moreover, if $f : H \rightarrow G$ is a morphism in BDL, then $K(f) : K(H) \rightarrow K(G)$ given by $K(f)(a, b) = (f(a), f(b))$ is a morphism in Kl_c . Then there is a functor K from BDL to Kl_c .

Let $(T, \wedge, \vee, \sim, c, 0, 1)$ be a centered Kleene algebra. We define

$$C(T) := \{x \in T : x \geq c\}.$$

We have that $(C(T), \wedge, \vee, c, 1)$ is a bounded distributive lattice. Moreover, if $g : T \rightarrow U$ is a morphism in Kl_c then $C(g) : C(T) \rightarrow C(U)$ given by $C(g)(x) = g(x)$ is a morphism in BDL. Then we have a functor C from Kl_c to BDL. See [2, 3, 5].

Remark 1. *If H is in BDL then the map $\alpha_H : H \rightarrow C(K(H))$ given by $\alpha_H(a) = (a, 0)$ is an isomorphism in BDL. If T is in Kl_c then $\beta_T : T \rightarrow K(C(T))$ given by $\beta_T(x) = (x \vee c, \sim x \vee c)$ is an injective map which is a morphism in Kl_c .*

Theorem 2. *With the notation above we have that the functor K is the right adjoint of C .*

Let $T \in \text{Kl}_c$. We consider the following algebraic condition:

(CK) For every $x, y \geq c$ such that $x \wedge y = c$ there exists z such that $z \vee c = x$ and $\sim z \vee c = y$.

Remark 3. *If $H \in \text{BDL}$ then $K(H)$ satisfies (CK).*

The condition (CK) is not necessarily verified for every centered Kleene algebra. Moreover, T satisfies (CK) if and only if β_T is a surjective map. We write Kl'_c for the full subcategory of Kl_c whose objects satisfy (CK).

Theorem 4. *There is a categorical equivalence $K \dashv C$ between BDL and Kl'_c , whose unit is α and whose counit is β .*

3 Kalman's functor

Recall from [4] that an algebra $(H, \wedge, \vee, \rightarrow, 0, 1)$ of type $(2, 2, 2, 0, 0)$ is a *DLI*-algebra if $(H, \wedge, \vee, 0, 1)$ is a bounded distributive lattice and the following conditions are satisfied for every $a, b, d \in H$:

- (I1) $(a \rightarrow b) \wedge (a \rightarrow d) = a \rightarrow (b \wedge d)$,
- (I2) $(a \rightarrow d) \wedge (b \rightarrow d) = (a \vee b) \rightarrow d$,
- (I3) $0 \rightarrow a = 1$,
- (I4) $a \rightarrow 1 = 1$.

We write DLI^+ for the variety of *DLI*-algebras whose algebras satisfy the following equation:

$$(I5) \quad a \wedge (a \rightarrow b) \leq b.$$

Remark 5. *In any DLI-algebra the equation (I5) is equivalent to the following condition: for every a, b, d , if $a \leq b \rightarrow d$ then $a \wedge b \leq d$.*

In every bounded distributive lattice H , if we define a binary operation \rightarrow by $a \rightarrow b = 1$ for every a, b then (H, \rightarrow) is a *DLI*-algebra. Consider the chain of two elements $\{0, 1\}$. With the implication before defined we have that $1 \wedge (1 \rightarrow 0) = 1 \not\leq 0$. Hence, DLI^+ is a proper subvariety of the variety of *DLI*-algebras. Furthermore, in every bounded distributive lattice H is possible to define a binary operation \rightarrow with the property that $(H, \rightarrow) \in \text{DLI}^+$ as we show in the following example.

Example 6. *Let H be a bounded distributive lattice. Then $(H, \rightarrow) \in \text{DLI}^+$ by defining*

$$a \rightarrow b = \begin{cases} 1, & \text{if } a = 0; \\ b, & \text{if } b \neq 0. \end{cases}$$

The fact that Kalman's construction can be extended consistently to Heyting algebras led us to believe that some of this picture could be lifted to the variety DLI^+ . In this work we study this level of generality.

Let $H \in \text{DLI}^+$. Write \rightarrow for the implication of H . We define a binary operation on $\text{K}(H)$ (also written \rightarrow) by

$$(a, b) \rightarrow (d, e) := ((a \rightarrow d) \wedge (e \rightarrow b), a \wedge e). \quad (1)$$

Since $a \wedge b = d \wedge e = 0$ then $(a \rightarrow d) \wedge (e \rightarrow b) \wedge a \wedge e = 0$. Hence, \rightarrow is well defined in $\text{K}(H)$. Thus we can consider algebras $(\text{K}(H), \wedge, \vee, \rightarrow, \sim, (0, 0), (0, 1), (1, 0))$ of type $(2, 2, 2, 1, 0, 0, 0)$ where the reduct $(\text{K}(H), \wedge, \vee, \sim, (0, 0), (0, 1), (1, 0))$ is a centered Kleene algebra. The next definition is motivated by the original construction of Kalman.

Definition 7. *We write KLI for the category whose objects are algebras $(T, \wedge, \vee, \rightarrow, \sim, c, 0, 1)$ of type $(2, 2, 2, 1, 0, 0, 0)$ such that $(T, \wedge, \vee, \sim, c, 0, 1)$ is a centered Kleene algebra and \rightarrow is a binary operation on T which satisfies the following conditions:*

- (KI1) $(T, \wedge, \vee, \rightarrow, 0, 1)$ is a *DLI*-algebra.
- (KI2) $(x \wedge (x \rightarrow y)) \vee c \leq y \vee c$ for every x, y .
- (KI3) $c \rightarrow c = 1$.
- (KI4) $(x \rightarrow y) \wedge c = (\sim x \vee y) \wedge c$ for every x, y .
- (KI5) $(x \rightarrow \sim y) \vee c = ((x \rightarrow (\sim y \vee c)) \wedge (y \rightarrow (\sim x \vee c)))$, for every x, y .

The objects of this category are called Kleene lattices with implication.

Proposition 8. *Let $H \in \text{DLI}^+$. Then $\text{K}(H)$ admits a structure of Kleene lattice with the implication given in (1). Furthermore, K extends to a functor from DLI^+ to KLI, which we also will write K .*

Proposition 9. *If $(T, \wedge, \vee, \sim, \rightarrow, c, 0, 1) \in \text{KLI}$ then $(\text{C}(T), \wedge, \vee, \rightarrow, c, 1) \in \text{DLI}^+$. Furthermore, C extends to a functor from KLI to DLI^+ , which we also will write C .*

Remark 10. *For $H \in \text{DLI}^+$ and $a, b \in H$ we have that $\alpha_H(a \rightarrow b) = \alpha_H(a) \rightarrow \alpha_H(b)$, where α_H is the map defined in Remark 1. Then α_H is an isomorphism in DLI^+ . For $T \in \text{KLI}$ the map β_T is a morphism in KLI, where β_T is the map defined in Remark 1.*

For every $(T, \wedge, \vee, \sim, c, 0, 1) \in \text{Kl}_c$ it is possible to define a binary operation \rightarrow such that $(T, \wedge, \vee, \rightarrow, \sim, c, 0, 1) \in \text{KLI}$. We do this in the following example.

Example 11. Let $(T, \wedge, \vee, \sim, c, 0, 1) \in \text{Kl}_c$. Define in T the following binary map:

$$x \rightarrow y = \begin{cases} 1, & \text{if } x \leq c \text{ and } y \geq c; \\ \sim x, & \text{if } x \leq c \text{ and } y \not\geq c; \\ y, & \text{if } x \not\leq c \text{ and } y \geq c; \\ ((y \vee c) \wedge \sim x) \vee ((\sim x \vee c) \wedge y), & \text{if } x \not\leq c \text{ and } y \not\geq c. \end{cases}$$

It can be seen that $(T, \wedge, \vee, \rightarrow, \sim, c, 0, 1) \in \text{KLI}$.

There are examples of algebras of KLI which does not satisfy the condition (CK). We write KLI' for the full subcategory of KLI whose objects satisfy the condition (CK).

Theorem 12. There exists a categorical equivalence $\text{K} \dashv \text{C} : \text{DLI}^+ \rightarrow \text{KLI}'$.

Remark 13. The previous theorem can be seen as a generalization from [5, Theorem 3.14], which establishes an equivalence for the category of Heyting algebras.

It is possible to establish an order isomorphism between the lattice of congruences of any $H \in \text{DLI}^+$ and the lattice of congruences of $\text{K}(H) \in \text{KLI}'$. We can also restrict all the mentioned results to certain subcategories of DLI^+ .

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Modal operators for meet-complemented lattices

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Abstract

We investigate some modal operators of necessity and possibility in the context of meet-complemented (not necessarily distributive) lattices. We proceed in stages. We compare our operators with others in the literature.

Extended abstract

We investigate certain new modal operators of necessity and possibility in the context of a (not necessarily distributive) meet-complemented lattice. Our operators are univocal and satisfy many good modal properties in the sense of [12]. They are sort of relatives of modal operators to our best knowledge first studied by Moisil in 1942 (see [7] or [8]). He worked in a logical context where he had both intuitionistic negation \neg and its dual, which we annotate D . He used DD for necessity and $\neg\neg$ for possibility. This choice we have found somewhat intriguing as, for instance, the normal modal inequality is not the case, that is, we do not have $DD(x \rightarrow y) \leq DDx \rightarrow DDy$. Also, because he could have chosen $\neg D$ and $D\neg$ for necessity and possibility, respectively, $\neg D$ satisfying the just given normal modal inequality. In 1974, Rauszer (see [9]) considered lattices expanded with both the meet and the join relative complements. In those algebras both \neg and D are easily definable. However, she does not mention Moisil. Also, she does not seem to be interested in necessity or possibility. Later on, in 1985, López-Escobar (see [6]), who seems not to have been acquainted with Moisil's paper, considered, in the context of Beth structures, modal operators of necessity and possibility, $\neg D$ and $D\neg$, respectively. Operators of this form were also studied (although with a different motivation) in [1, 10, 11]. For a more recent paper on intuitionistic modal logic, see [2].

In the context of a meet-complemented lattice $\mathbf{A} = (A; \wedge, \vee, \neg)$, we define our necessity operator as $\Box a := \max\{b \in A : a \vee \neg b = 1\}$, for any $a \in A$. Somehow dually, we define our possibility operator as $\Diamond a = \min\{b \in A : \neg a \vee b = 1\}$, for any $a \in A$. It is clear that our operators are univocal, i.e., when they exist, there cannot be two different operations satisfying their definitions. It is also the case that their definition does not require D . Note, also, that the relative meet-complement, i.e. the algebraic counterpart of intuitionistic conditional, is also not needed, not forcing us to work in a distributive context.

The modalities \Box and \Diamond are similar to the mentioned $\neg D$ and $D\neg$, respectively. In fact, in any lattice where \neg and D exist, $\Box = \neg D$ and $\Diamond = D\neg$. However, \Box and \Diamond exist in some algebras without D .

We also compare \Box with the unary operator B , which, in the context of a meet-complemented lattice with universe A , is defined as the maximum Boolean below, i.e. for any $x \in A$, $Bx = \max\{y \in A : y \leq x \text{ and } y \vee \neg y = 1\}$. For a study of B in the context of residuated lattices, see [4].

In what follows, we talk of *extensions* of a class of algebras when we only add some (new) property to the operations in the given class, for instance, distributivity. On the other hand,

we talk of *expansions* when adding a (new) operation to the class, for example, when we add necessity \Box to meet-complemented lattices.

Since in the context of meet-complemented lattices, as already noted by Frink (see [5]), distributivity is not forced, we will only assume it when necessary. We expand meet-complemented lattices with necessity \Box and prove that the expansion is an equational class. Then, we expand meet-complemented lattices with possibility \Diamond and prove that the expansion is not an equational class. In the next stage, we expand meet-complemented lattices with both \Box and \Diamond and prove that the expansion is, again, an equational class.

Call the S -extension of the equational class of meet-complemented lattices with necessity \Box , the extension given by the modal logic $S4$ -Schema, that is, $\Box\Box = \Box$. If we extend this class with both distributivity and the $S4$ -Schema, then \Diamond is definable as $\neg\Box\neg$.

We also consider the number of modalities, that is, finite combinations of \neg , \Box , and \Diamond in any algebra of the corresponding class. It is the case that there are a finite number of them in the case of the S -extension, even not having distributivity.

We also consider two logics involved, both of them expanding intuitionistic logic with the obvious translations of the equations for \Box and \Diamond . Also, one of them with rules for both \Box and \Diamond , the first being $\alpha/\Box\alpha$, and the second, $\alpha \rightarrow \beta/\Diamond\alpha \rightarrow \Diamond\beta$. The other logic corresponds to the S -extension, where we only need the given rule for \Box .

For some further details, the reader may see [3].

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A representation theorem for integral rigs and its applications to residuated lattices

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Abstract

We generalize the Dubuc-Poveda representation theorem for MV-algebras so that it applies to other algebraic categories of residuated join-semilattices. In particular, as a corollary, we obtain a representation result for pre-linear residuated join-semilattices in terms of totally ordered fibers. The main result is analogous to the Zariski representation of (commutative) rings and it is proved using tools from topos theory.

From theories of representation of rings by sheaves, due to Grothendieck [12], Pierce [19] and Dauns and Hoffman [4], general constructions of sheaves to universal algebras [5] evolved. All these representations were developed using toposes of local homeos over adequate spaces. A concrete example of the method employed in the case of bounded distributive lattices can be found in [1].

In this line, Filipoiu and Georgescu find in [11] an equivalence between MV-algebras and certain type of sheaves of MV-algebras over compact Hausdorff spaces. In the same line, a presentation closer to the construction given by Davey in [5], is given by Dubuc and Poveda in [9]. They find an adjunction between MV-algebras and another version of MV-spaces. In [10] it is proposed a third kind of representation for MV-algebras using fibers that are certain local MV-algebras.

As examples of representation by sheaves of other classes of residuated structures, we can quote the Grothendieck-type duality for Heyting algebras proposed in [6] and the one given by Di Nola and Leuştean in [7] for BL-algebras.

A more explicit use of topos theory is exemplified by the representation theorems for rigs and lattices proved by Johnstone [13] and Coste [3]. See also [2].

The present work is motivated by the Dubuc-Poveda representation theorem for MV-algebras [9, 8] and Lawvere's strategic ideas about the topos-theoretic analysis of coextensive algebraic categories hinted at in page 5 of [15] and also in [17].

A *rig* in **Set**, as introduced in [20], is a commutative “ring without negatives”, that is, having two commutative monoid structures $(0, +)$ and $(1, \cdot)$ related by the distributive laws $0 = a \cdot 0$ and $a \cdot b + a \cdot c = a \cdot (b + c)$. If the equation $1 + x = 1$ is satisfied it will be said that the rig is *integral*. In any integral rig the additive monoid defines a semilattice and so, an underlying partial order. We write **iRig** to denote the algebraic category of integral rigs in **Set**.

Recall that a commutative ring with unit R is called local if it possesses a unique maximal ideal. It is well known in literature that such definition is equivalent to the following conditions: (i) R is not trivial and (ii) if the sum of two elements in R is invertible then some of them is invertible. In [16] Lawvere applied the last observation to the context of rigs and announced it in terms of positive quantities (having in mind that a rig is a “ring without negatives”) introducing in this way the concept of *really local*. In words of Lawvere himself: “*The preservation*

of addition is a strengthening, possible for positive quantities, of the usual notion of localness (which on truth values was only an inequality)". Besides, it is clarified that the word "really" refer to two ideas: (1) a strengthening of the fact of localness and (2) a concept appropriate to a real (as opposed to complex) environment.

Let \mathcal{E} be a category with finite limits. For any rig A in \mathcal{E} we define the subobject $\text{Inv}(A) \rightarrow A \times A$ by declaring that the diagram below

$$\begin{array}{ccc} \text{Inv}(A) & \xrightarrow{!} & 1 \\ \downarrow & & \downarrow 1 \\ A \times A & \longrightarrow & A \end{array}$$

is a pullback. The two projections $\text{Inv}(A) \rightarrow A$ are mono in \mathcal{E} and induce the same subobject of A . Of course, the multiplicative unit $1 : 1 \rightarrow A$ always factors through $\text{Inv}A \rightarrow A$. In particular, if A is a distributive lattice then the factorization $1 : 1 \rightarrow \text{Inv}A$ is an iso.

Definition 1. A rig morphism $f : A \rightarrow B$ between rigs in \mathcal{E} is *local* if the following diagram

$$\begin{array}{ccc} \text{Inv}A & \longrightarrow & \text{Inv}B \\ \downarrow & & \downarrow \\ A & \xrightarrow{f} & B \end{array}$$

is a pullback.

If \mathcal{E} is a topos with subobject classifier $\top : 1 \rightarrow \Omega$ then there exists a unique map $\iota : A \rightarrow \Omega$ such that the square below

$$\begin{array}{ccc} \text{Inv}(A) & \xrightarrow{!} & 1 \\ \downarrow & & \downarrow \top \\ A & \xrightarrow{\iota} & \Omega \end{array}$$

is a pullback. It is well-known that the object Ω is an internal Heyting algebra and so, in particular, a distributive lattice. The basic properties of invertible elements imply that $\iota : A \rightarrow \Omega$ is a morphism of multiplicative monoids. The following definition is borrowed from [16].

Definition 2. The rig A in \mathcal{E} is *really local* if $\iota : A \rightarrow \Omega$ is a morphism of rigs.

In other words, A is really local if ι is a map of additive monoids. For example, a distributive lattice D in **Set** is really local if and only if it is non-trivial and, for any $x, y \in D$, $x \vee y = \top$ implies $x = \top$ or $y = \top$.

Let D be a distributive lattice seen as a coherent category (A1.4 in [14]). Its *coherent coverage* (A2.1.11(b) loc. cit.) is the function that sends each $d \in D$ to the set of finite families $(d_i \leq d \mid i \in I)$ such that $\bigvee_{i \in I} d_i = d$. (These will be called *covering* families or simply *covers*.) The resulting topos of sheaves will be denoted by $\mathbf{Shv}(D)$. Let \widehat{D} the category of presheaves in D . We write Λ to denote the object in \widehat{D} that sends $d \in D$ to $(\downarrow d)$. For any $d \in D$, $x \in \Lambda d$

and $c \leq d$, we have that $x \cdot c = x \wedge c \in \Lambda c$. It follows that Λ is a sheaf.

Definition 3. A *representation (of an integral rig)* is a pair (D, X) consisting of a distributive lattice D and an integral rig X in $\mathbf{Shv}(D)$ such that there exists a local morphism of rigs $\chi : X \rightarrow \Lambda_D$.

We now define a category \mathfrak{I} whose objects are representations in the sense above. To describe the arrows in \mathfrak{I} first recall that any rig map $f : D \rightarrow E$ between distributive lattices induces a functor $f_* : \mathbf{Shv}(E) \rightarrow \mathbf{Shv}(D)$ that sends Y in $\mathbf{Shv}(E)$ to the composite

$$D^{\text{op}} \xrightarrow{f^{\text{op}}} E^{\text{op}} \xrightarrow{Y} \mathbf{Set}$$

which lies in $\mathbf{Shv}(D)$. See Theorem VII.10.2 in [18]. Moreover, the functor f_* is the direct image of a geometric morphism $\mathbf{Shv}(E) \rightarrow \mathbf{Shv}(D)$ so it sends integral rigs in the domain to integral rigs in the codomain. Observe that the map $f : D \rightarrow E$ also determines a morphism $f : \Lambda_D \rightarrow f_*\Lambda_E$ of lattices in $\mathbf{Shv}(D)$ such that for each $d \in D$, $f_d : \Lambda_D d = (\downarrow d) \rightarrow (f_*\Lambda_E)d = \Lambda_E(fd) = (\downarrow fd)$ sends $a \leq d$ to $fa \leq fd$.

For representations (D, X) and (E, Y) , a map $(D, X) \rightarrow (E, Y)$ in \mathfrak{I} is a pair (f, ϕ) with $f : D \rightarrow E$ and $\phi : X \rightarrow f_*Y$ rig maps such that the following diagram

$$\begin{array}{ccc} X & \xrightarrow{\phi} & f_*Y \\ \chi \downarrow & & \downarrow f_*\chi \\ \Lambda_D & \xrightarrow{f} & f_*\Lambda_E \end{array}$$

commutes in $\mathbf{Shv}(D)$. We emphasize that $f : D \rightarrow E$ is a morphism in $\mathbf{dLat}(\mathbf{Set})$ and $\phi : X \rightarrow f_*Y$ is a morphism in $\mathbf{iRig}(\mathbf{Shv}(D))$.

For each (D, X) in \mathfrak{I} define $\Gamma(D, X) = X\top$ and, for $(f, \phi) : (D, X) \rightarrow (E, Y)$ in \mathfrak{I} , define $\Gamma(f, \phi) = \phi_\top : D\top \rightarrow (f_*Y)\top = Y(f\top) = Y\top$. It follows that $\Gamma : \mathfrak{I} \rightarrow \mathbf{iRig}$ is a functor.

The main result of this paper is a representation theorem for integral rigs as internal really local integral rigs in toposes of sheaves over bounded distributive lattices.

Theorem 1. The functor $\Gamma : \mathfrak{I} \rightarrow \mathbf{iRig}$ has a full and faithful left adjoint.

A rig will be called *residuated* if each monotone map $a \cdot (_)$ has right adjoint with respect to the order induced by the sum. We also show that Theorem 1 may be lifted to residuated integral rigs and then restricted to varieties of these.

In particular, as a corollary, we obtain a representation theorem for pre-linear residuated join-semilattices in terms of totally ordered fibers. The restriction of this result to the level of MV-algebras coincides with the Dubuc-Poveda representation theorem.

We stress that our main results do not use topological spaces. Instead, we use the well-known equivalence between the topos of sheaves over a topological space X and the topos of local homeos over X in order to translate our results to the language of bundles. This translation allows us to compare our results with related work.

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Hypersequents and Systems of Rules: An Embedding

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Proof theory provides a constructive approach for the investigation of meta-logical and computational properties of a logic through the design and study of suitable proof systems. An essential feature of such proof systems is analyticity. A proof system is analytic if its proofs only contain subformulae of the formula to be proved.

Sequent calculus has been extensively and successfully used in the definition of analytic proof systems since its introduction [7]. Unfortunately it is not powerful enough to capture many non-classical logics. Hence, many variants and extensions of the framework of sequents have been introduced; these include the labelled calculus [6, 8] and the hypersequent calculus [1]. The labelled calculus consists of sequent rules acting on labelled formulae and relations on labels. The hypersequent calculus consists of rules acting on multisets of sequents, *i.e.* objects of the form

$$\Gamma_1 \Rightarrow \Delta_1 \mid \dots \mid \Gamma_n \Rightarrow \Delta_n$$

where $\Gamma_i \Rightarrow \Delta_i$, for $1 \leq i \leq n$, are sequents, and the symbol \mid is usually interpreted disjunctively.

The multitude and diversity of the introduced formalisms has made it increasingly important to identify their interrelationships and relative expressive power. Embeddings between formalisms, *i.e.*, functions that take any calculus in some formalism and yield a calculus for the same logic in another formalism, are useful tools to prove that a formalism subsumes another one in terms of expressiveness or—when the embedding is bidirectional—that two formalisms are equally expressive. Such embeddings can also provide useful reformulations of known calculi and allow the transfer of proof-theoretical results.

Using propositional intermediate logics as a case study, we present a bidirectional embedding between two formalisms for the proof theory of non-classical logics: hypersequents and two-level systems of rules [5].

Systems of Rules

The formalism of systems of rules was introduced [9] to define analytic labelled calculi for logics semantically characterised by frame conditions. A system of rules is a set of sequent rules reciprocally related by conditions on their applicability. For example, the system of rules $Sys_{(com)}$ corresponding to the linearity axiom $(\varphi \supset \psi) \vee (\psi \supset \varphi)$ is the following:

$$\frac{\frac{\varphi, \Gamma_1 \Rightarrow \Pi_1}{\psi, \Gamma_1 \Rightarrow \Pi_1} (com_1) \quad \frac{\psi, \Gamma_2 \Rightarrow \Pi_2}{\varphi, \Gamma_2 \Rightarrow \Pi_2} (com_2)}{\frac{\Gamma \Rightarrow \Pi \quad \Gamma \Rightarrow \Pi}{\Gamma \Rightarrow \Pi} (com_{end})}$$

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where φ, ψ are metavariables for formulae; $\Gamma, \Gamma_1, \Gamma_2$ for multisets of formulae; and Π, Π_1, Π_2 for multisets of formulae with at most one element. By this schema we represent the following conditions:

- (com_1) and (com_2) can only be applied above different premisses of (com_{end}) ,
- the metavariables φ and ψ are shared by the two applications.

System of rules are quite powerful and can be used in labelled calculi to define analytic proof systems for all the modal logics characterised by frame properties that correspond to formulae in the Sahlqvist fragment. The downside of this great expressivity is the non-locality of rules in this framework, which appears at two levels: horizontally, because of the dependency between rules occurring in disjoint branches; and vertically, because of rules that can only be applied above other rules.

The Embedding

A “possible connection” between hypersequents and systems of rules has been hinted at [9]. We formalised in full this intuition defining a bidirectional embedding, w.r.t. intermediate propositional logics, between hypersequents and a proper fragment of the full formalism of systems of rules, *i.e.*, two-level systems of rules. An example of this kind of system is $Sys_{(com)}$ above—indeed only one application of (com_1) or (com_2) (the rules of the second level) can occur above each premiss of (com_{end}) (the rule of the first level).

The specific outcomes of the embedding are

- (i) a local representation of two-level systems of rules by hypersequent rules (*e.g.*, for intermediate logics characterised by Hilbert axioms within the class \mathcal{P}_3 of the substructural hierarchy [4]),
- (ii) the transfer of analyticity results from the hypersequent formalism to the formalism of two-level systems of rules (this is achieved translating the two-level system of rules into a hypersequent rule, constructing a version of the latter that preserves cut-elimination [4], and translating the rule back),
- (iii) the definition of new cut-free proof systems with two-level systems of rules,
- (iv) a reformulation of hypersequent calculi which may be of independent interest due to its close relation to natural deduction systems.

The connection between hypersequents and two-level systems suggests a promising approach to the problem of extracting the computational content of logics formalised by hypersequent proof systems. Indeed, translating a hypersequent proof system into a suitable natural deduction system it is possible to establish a Curry–Howard correspondence (see, *e.g.*, [2] for an attempt in this direction with Gödel logic).

Furthermore, the fact that all propositional axiomatisable intermediate logics are definable by adding suitable formulae (*canonical formulae*) to intuitionistic logic [3] points at another research direction. Indeed, these formulae belong to a class which is immediately above the class for which hypersequents can provide analytic rules [4], and therefore *three-level* systems seem a suitable choice to transform formulae in the higher class into analytic rules.

Finally, the embedding does not essentially depend on the specific rules of the considered calculus and can be naturally extended to other classes of propositional logics, *e.g.*, substructural or modal logics.

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An abstract approach to consequence relations

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Given a class \mathbf{K} of commutative residuated lattices, we can define a deductive relation $\vdash_{\mathbf{K}}$ between *finite multisets* of formulas as follows:

$$\langle \varphi_1, \dots, \varphi_n \rangle \vdash_{\mathbf{K}} \langle \psi_1, \dots, \psi_m \rangle \iff \mathbf{K} \models \varphi_1 \cdots \varphi_n \leq \psi_1 \cdots \psi_m. \quad (1)$$

In this contribution we develop a general study of these deductive relations between finite multisets, based on the notion of multiset *inclusion* and multiset *join*.

More precisely, we say that a finite multiset $\langle \varphi_1, \dots, \varphi_n \rangle$ is *included* into a finite multiset $\langle \psi_1, \dots, \psi_m \rangle$, in symbols $\langle \varphi_1, \dots, \varphi_n \rangle \leq \langle \psi_1, \dots, \psi_m \rangle$, when from every formula γ , then the number of occurrences of γ in $\langle \varphi_1, \dots, \varphi_n \rangle$ is smaller or equal than the number of its occurrences in $\langle \psi_1, \dots, \psi_m \rangle$. Moreover, the *join* of finite multisets is defined as the concatenation:

$$\langle \varphi_1, \dots, \varphi_n \rangle \uplus \langle \psi_1, \dots, \psi_m \rangle := \langle \varphi_1, \dots, \varphi_n \psi_1, \dots, \psi_m \rangle.$$

With this ingredients at hand, we can abstract the situation in (1) as follows [2]. Let Fm be the set of formulas over a fixed algebraic language. Moreover, let Fm^* be the set of finite multisets of formulas in Fm . A *structural deductive relation* between finite multisets based on Fm is a substitution-invariant reflexive and transitive relation $\vdash_{\subseteq} Fm^* \times Fm^*$ that satisfies the following additional postulates:

1. If $\langle \varphi_1, \dots, \varphi_n \rangle \leq \langle \psi_1, \dots, \psi_m \rangle$, then $\langle \psi_1, \dots, \psi_m \rangle \vdash \langle \varphi_1, \dots, \varphi_n \rangle$.
2. If $\langle \psi_1, \dots, \psi_m \rangle \vdash \langle \varphi_1, \dots, \varphi_n \rangle$, then

$$\langle \gamma_1, \dots, \gamma_m \rangle \uplus \langle \psi_1, \dots, \psi_m \rangle \vdash \langle \gamma_1, \dots, \gamma_m \rangle \uplus \langle \varphi_1, \dots, \varphi_n \rangle.$$

The deductive expressiveness of each of these structural deductive relations can be reproduced in a suitable Gentzen system and, in this format, can be studied with the tools of Abstract Algebraic Logic [10, 11, 9] and its categorical generalizations such as [1, 3, 7, 6, 4, 5, 8]. Nevertheless, these abstractions blur the motivating relation between structural deductive relations and the *multiset operations* of inclusion and join. To overcome this problem, we propose a new framework that embraces the main categorical abstractions and in which the importance of the multiset-operations can be recovered. This new approach is based on the observation that structural deductive relations can be studied in suitable categories of modules over semirings, whose underlying semiring plays the role of a set of finite multisets of substitutions.

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Neighborhood semantics for non-classical logics with modalities*

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The field of Mathematical Fuzzy Logic (MFL) has recently seen an increased interest on propositional systems expanded with modal operators, with works like [1, 2, 3, 7, 12] that follow the steps of the initial developments in [5, 6]. In these studies, modal fuzzy logics are endowed with a Kripke-style semantics which generalizes the classical one by allowing a fuzzy scale for either (or for both) the truth-values of propositions at each possible world and for the degree of accessibility from one world to another. However, axiomatizing such semantics over a given algebra (or a class of algebras) of truth-values is in general a difficult problem. Also, conversely, proof systems with natural syntactic conditions may fail to be complete with any such Kripke-style semantics.

Neighborhood semantics [8, 11] (also known as Scott–Montague semantics) was proposed, in the study of modal systems built over classical logic, as a more general framework that allows to prove completeness for non-normal modal logics, where the Kripke-style semantics would not work. In the context of MFL it has been considered for some particular cases in [9, 10] and, more generally, for axiomatic extensions of MTL in [4].

In this talk we will extend it, beyond fuzzy logics, to a wider framework of algebraizable non-classical logics, thus encompassing most of substructural logics. Given an algebra \mathbf{A} for such a logic, an *\mathbf{A} -neighborhood frame* (short: $\text{SM}(\mathbf{A})$ -frame) is defined as a pair $\langle W, N \rangle$ such that W is a non-empty (classical) set of worlds while N is a function $N: W \rightarrow A^{A^W}$ that assigns to each world $x \in W$ a fuzzy set of fuzzy subsets of W , called the *\mathbf{A} -neighborhood* of $x \in W$.

We define an *\mathbf{A} -neighborhood model* (short: $\text{SM}(\mathbf{A})$ -model) to be a triple $\langle W, N, V \rangle$, where $\langle W, N \rangle$ is an $\text{SM}(\mathbf{A})$ -frame and V is an evaluation $V: \text{Var} \times W \rightarrow A$ that is extended to formulas $\varphi \in Fm_{\square}$ (the expanded language with an arbitrary unary modality) inductively as follows: the non-modal connectives are interpreted locally at each world as the corresponding operations of \mathbf{A} , while for a modal formula one defines:

$$V(\Box\varphi, x) = ([\varphi]_{\mathfrak{M}} \in N(x)),$$

where for any formula $\varphi \in Fm_{\square}$, $[\varphi]_{\mathfrak{M}}$ denotes the fuzzy subset of W to which y belongs to the degree $V(\varphi, y)$, i.e., the fuzzy subset $\{y \in W \mid V(\varphi, y)\}$.

We will explain the relationship between Kripke and neighborhood semantics and give Hilbert-style axiomatizations for logics semantically given by certain classes of neighborhood frames. Finally, we will discuss possible generalizations of the framework by considering frames with different algebras of truth-values in each possible world, more (non necessarily unary) modalities and their interplay, and the relationship with the theory of vague quantifiers.

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A Sequent Calculus for Minimal and Subminimal Logic of Negation*

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In 1937, I. Johansson [3] developed a system, named ‘minimal logic’, obtained by discarding *ex falso quodlibet* from the standard axioms for intuitionistic logic. Minimal logic can be seen as the *paraconsistent* analogue of intuitionistic logic, and has been studied in its two equivalent formulations. The one used nowadays conceives the negation operator as a derived connective, while the original one proposed by Johansson (even before, by Kolmogorov [5]) assumes the negation as a primitive operator and its behavior is ruled by an axiom of negation. Keeping the negation operator as the focus of interest, this work aims to study some intriguing subminimal systems defined by means of axioms of negation. The basic logic among the ones we study is the system in which the unary operator \neg has no properties at all, except the property of being functional. We treat the considered systems as paraconsistent logics. The main part of the work is proof-theoretic. Sequent calculi for the main subminimal systems are developed. Rules for negation are defined, resembling the Hilbert-style axiomatization of each system. The behavior of the sequent calculi is further studied by means of some examples and significant results.

Further connections of this piece of work with related literature (see e.g., [4] and its bibliography for additional references) are not pursued here and are left for future research. Particular emphasis is deserved by the approach to the subject matter presented in [1], which could lead to a more uniform account of the logical systems of our interest.

Preliminaries

Given a countable set P of propositional variables, let \mathcal{L}_\neg be the set of connectives $\{\wedge, \vee, \rightarrow, \neg\}$. The systems we are going to present are obtained by adding axioms in the language \mathcal{L}_\neg to the positive fragment of intuitionistic logic (e.g., see [6]). Let us introduce the significant axioms.

Axioms.

- (1) $(p \leftrightarrow q) \rightarrow (\neg p \leftrightarrow \neg q)$,
- (2) $(p \wedge \neg p) \rightarrow \neg q$,
- (3) $(p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p)$,
- (4) $(p \rightarrow q) \wedge (p \rightarrow \neg q) \rightarrow \neg p$,
- (5) $(p \rightarrow \neg p) \rightarrow \neg p$.

*Joint work with Marta Bílková (Charles University in Prague) and Dick de Jongh (Institute for Logic, Language and Computation, University of Amsterdam).

The basic logic (**N**) of the unary operator \neg is axiomatized by (1) together with the axioms of the positive fragment of intuitionistic logic. The other systems in this work are extensions of **N**. Negative ex falso logic and contraposition logic are the systems obtained, respectively, by adding the axiom (2) to the system **N**, and by substituting the basic axiom with the ‘weak’ direction of contraposition, (3). Finally, minimal propositional logic is axiomatized by substituting the basic axiom (1) with the axiom (4). The four systems introduced here form a chain within the lattice of the logics in the language \mathcal{L}_{\neg} .

Proposition 1. *Minimal logic can be alternatively axiomatized by the weak contraposition axiom, together with (5). A third alternative axiomatization can be obtained by $(p \rightarrow \neg q) \rightarrow (q \rightarrow \neg p)$.*

Kripke-style Semantics. A propositional Kripke frame for the system is a triple $\langle W, R, N \rangle$, where $\langle W, R \rangle$ is an intuitionistic frame [2] and N is a function $N : \mathcal{U}(W) \rightarrow \mathcal{U}(W)$, where $\mathcal{U}(W)$ denote the set of all upward closed subsets (*upsets*) of W . Let $U, V \subseteq W$ denote arbitrary upsets. Consider the following properties:

- For every world $w \in W$, $w \in N(U) \Leftrightarrow w \in N(U \cap R(w))$, where $R(w)$ is the upset generated by w .

A frame satisfying this property is an **N**-frame.

- $U \cap N(U) \subseteq N(V)$.

This additional property gives us a frame for negative ex falso logic.

- $U \subseteq V \Rightarrow N(V) \subseteq N(U)$.

Adding this property to the first one gives us a contraposition logic frame. A Kripke model is obtained from a frame by adding a persistent (i.e., intuitionistic) valuation map V . Each subminimal system is sound and complete with respect to the respective class of frames.

Sequent Calculus Systems

The sequent calculi presented here are extensions of the positive fragment¹ of the system **G3i** for intuitionistic logic as defined in [7]. Following the notation in [7], let us denote this fragment as **G3m**. A sequent consists of a finite multiset of formulas on the left and a single formula on the right. Let us introduce the four rules we are going to add to such system.

$$\begin{array}{cc} \mathbf{N} \frac{\Gamma, \neg\alpha, \alpha \Rightarrow \beta \quad \Gamma, \neg\alpha, \beta \Rightarrow \alpha}{\Gamma, \neg\alpha \Rightarrow \neg\beta} & \mathbf{NeF} \frac{\Gamma, \neg\alpha \Rightarrow \alpha}{\Gamma, \neg\alpha \Rightarrow \neg\beta} \\ \\ \mathbf{CoPC} \frac{\Gamma, \neg\beta, \alpha \Rightarrow \beta}{\Gamma, \neg\beta \Rightarrow \neg\alpha} & \mathbf{An} \frac{\Gamma, \alpha \Rightarrow \neg\alpha}{\Gamma \Rightarrow \neg\alpha} \end{array}$$

The sequent calculus obtained by adding the rule **N** to **G3m** is a sound and complete system for the basic logic **N**. By adding the rule **NeF** to this system, we obtain a similar sequent calculus for negative ex falso logic. On the other hand, the rule **CoPC** together with the system **G3m** gives us a sound and complete sequent calculus for contraposition logic; extending the latter by

¹We basically discard the axiom $\perp \Rightarrow \varphi$.

means of An , we obtain a sequent calculus system for minimal logic. It is worth emphasizing that, although different versions of sequent calculi for minimal logic already exist (see e.g., [7]), the one proposed here makes use of Proposition 1 and keeps negation as the focus of interest.

The rules introduced here are the only available rules for negation, which is treated as a ‘modality’. The only right rule among the ones presented is An , which is also the only depth-preserving invertible rule. The other rules introduce principal formulas both on the left and on the right-hand side of the conclusion sequent. If not differently specified, the results we present here hold for all the systems we are considering.

Theorem 1. *Weakening on the left and Contraction on the left are admissible rules.*

Theorem 2. *The cut rule is admissible.*

The proof of Theorem 2 has the same structure as its intuitionistic analogue [7].

Applications

The result claimed in Theorem 2 allows us to use the defined sequent calculi to search for cut-free proofs. In order to understand whether the sequent calculi are nicely behaved, we are going to see some results proven exploiting the defined sequent systems. The following result holds for the systems containing the rule CoPC , i.e., contra-position and minimal logic.

Theorem 3. *Let $n \in \mathbb{N}$ be an arbitrary natural number such that $n \geq 1$. Then, we have that $\neg^{(2n+1)}p \rightarrow \neg p$ and $\neg^{(2n)}p \leftrightarrow \neg\neg p$ are theorems of contra-position logic, where $\neg^{(m)}$ denotes m nested application of the negation operator for any natural number m .*

Proof. The proof goes by induction on n . For the base case, we search for a cut-free proof of the basic result $\neg\neg p \rightarrow \neg p$, which, together with an application of the rule CoPC , gives us a proof of $\neg\neg\neg p \leftrightarrow \neg\neg p$ as well. \square

Craig’s Interpolation Theorem. A proof of Craig’s Interpolation Theorem via sequent calculus gives us further information about the sequent system. As a matter of fact, the shape of the interpolants strongly depend on the rules we are considering at a certain step of the proof.

Theorem 4. *Let Γ, Γ' denote arbitrary finite multisets of formulas and let φ be an arbitrary formula such that the common language of Γ and Γ', φ is not empty. If $\vdash \Gamma, \Gamma' \Rightarrow \varphi$, there exists a formula σ such that:*

1. *the propositional variables contained in σ are in the common variables of Γ and Γ', φ ,*
2. *$\vdash \Gamma \Rightarrow \sigma$ and $\vdash \Gamma', \sigma \Rightarrow \varphi$.*

Proof. An induction on the depth of the proof of $\vdash \Gamma, \Gamma' \Rightarrow \varphi$ is sufficient. For the rules that introduce a principal formula on the left, we need to distinguish two cases: the one in which the principal formula is an element of Γ , and the one in which it is an element of Γ' . The proof proceeds in a straightforward way, with an exception: the case in which the considered rule is N and the principal left formula is an element of Γ . Given the interpolants σ' and σ'' , whose existence is ensured by the induction hypothesis, the interpolant for this case is of the form

$$(\sigma' \rightarrow \sigma'') \rightarrow ((\sigma'' \rightarrow \sigma') \wedge \neg\sigma').$$

This result is not surprising. As a matter of fact, the negation operator in N has no properties at all, and the only available rule for negation is N . \square

Translation. It is well-known that there exists a *negative translation* from classical propositional logic into intuitionistic logic. We want here to use the defined sequent calculi for contraposition logic and minimal logic to prove the existence of a sound and truthful translation from the latter into the former. The idea behind the definition of the translation is to ‘add’ to contraposition logic what it lacks from minimal logic: absorption of negation.

Definition 1. Let φ be an arbitrary formula in MPC. We define a translation φ^\sim by recursion over the complexity of φ , as follows:

- $p^\sim := p$
- $\top^\sim := \top$
- $(\varphi \circ \psi)^\sim := \varphi^\sim \circ \psi^\sim$, where $\circ \in \{\wedge, \vee, \rightarrow\}$
- $(\neg\varphi)^\sim := \varphi^\sim \rightarrow \neg\varphi^\sim$

As in the case of classical logic and intuitionistic logic, we need a standard result in order to prove the main theorem.

Lemma 1. For every formula φ , $\text{MPC} \vdash \varphi \leftrightarrow \varphi^\sim$.

Proof. The proof of this result is an induction on the structure of the formula φ . The negation step underlines how the translation is somehow ‘built into’ the rule An. \square

Theorem 5. The sequent $\Gamma \Rightarrow \varphi$ is provable in minimal logic if and only if the sequent $\Gamma^\sim \Rightarrow \varphi^\sim$ is provable in contraposition logic, where $\Gamma^\sim := \{\gamma^\sim \mid \gamma \in \Gamma\}$.

Proof. For the right-to-left direction, it is sufficient to apply Lemma 1, since contraposition logic is a subsystem of minimal logic. For the missing direction, an induction on the depth of the derivation is necessary. \square

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On 3-valued Paraconsistent Logic Programming

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With the advent of digital computers, there was some expectation on the development of the use of some logical formalism as a tool to “do” math, through automatic deductions of mathematical theorems, performed by such devices. In this context, the classic first-order logic strongly influenced demonstrations automation area since its beginning, and then the Logic Programming. One of the reasons that made the first-order logic as the main target in the development of automatic deductions was his syntactic formalism universally recognized, while the use of the Tarskian conception of truth for formalized languages was consensually adopted for its semantical interpretation. Another factor that contributed to the adoption of the first-order logic was the possibility of reducing all sentences to the conjunctive form. This feature was heavily used by an important method in the automatic deduction area: the resolution procedure introduced by J. Robinson in 1965, an inference rule specially developed for use in computers. Such refutation method took advantage of the capabilities of computer calculations based on Herbrand theorem.

Motivated with applications to Artificial Intelligence and databases, logic programming began to be developed for different types of non-classical logic. There are several proposals in the literature for many-valued logic programming, such as annotated logic programming [9] and bilattice-based systems [8, 7]. In particular, 3-valued logic programming was considered [10, 5]. Additionally, paraconsistent logic programming was also investigated [1, 4].

In this paper a resolution calculus for a 3-valued paraconsistent logic called MPT0 will be proposed, as well as a framework for logic programming based on it. As it will be observed, the immediate transfer of the notions and results of classical logic programming is not so linear, and some technical adaptations are necessary.

Consider firstly the 3-valued matrix logic MPT0 introduced in [2] and defined over the propositional signature $\{\wedge, \vee, \rightarrow, \neg, \sim\}$ over the domain $\{1, B, 0\}$, where $D = \{1, B\}$ is the set of designated values (following Dunn and Belnap, ‘B’ stands for ‘both’). The connectives are interpreted by means of the following tables:

\wedge	1	B	0	\vee	1	B	0	\rightarrow	1	B	0	p	$\neg p$	p	$\sim p$
1	1	B	0	1	1	1	1	1	1	B	0	1	0	1	0
B	B	B	0	B	1	B	B	B	1	B	0	B	B	B	0
0	0	0	0	0	1	B	0	0	1	1	1	0	1	0	1

Observe that \vee can be defined from \wedge and \neg by De Morgan. As shown in [2] the above tables are functionally equivalent to that for LPT ([3]) and J3 ([6]), and so the three logics are the same, but presented in different signatures. A sound and complete Hilbert calculus for MPT0 can be found in [2]. The equivalence connective \equiv is defined in MPT0 as $(\alpha \equiv \beta) =_{def} (\alpha \rightarrow \beta)$

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$\beta) \wedge (\beta \rightarrow \alpha)$. Note that \neg represents a weak (paraconsistent) negation, while \sim represents a strong (classical) negation. It is easy to see that, by combining iterations of negations in MPT0, there are just four inequivalent formulas:

P	\neg P	\sim P	$\sim\neg$ P
1	0	0	1
B	B	0	0
0	1	1	0

This motivates the following definitions:

- (i) A *literal* in MPT0 is a formula of the form A , $\neg A$, $\sim A$ or $\sim\neg A$, where A is an atomic formula. In each case it is said that the literal contains the atomic formula A .
- (ii) Literals of form A or $\neg A$ are called *positive*, while the others are called *negative* (thus, negative literals are of the form $\sim L$).
- (iii) A *clause* of MPT0 is a formula of the form $L_1 \vee \dots \vee L_k \vee \sim L_{k+1} \vee \dots \vee \sim L_{k+m}$ such that each L_i is a positive literal in MPT0.
- (iv) A clause is *positive* (*negative*) if only contains positive (negative) literals.
- (v) A set S of clauses is *satisfiable* if there is a valuation on MPT0 such that $v(K) \in \{1, B\}$ for every clause K in S . In that case v is called a *model* of S .
- (vi) A clause K in MPT0 is a consequence in MPT0 of a set of clauses S (denoted by $S \models_{MPT0} K$), if for every valuation v : $v[S] \subseteq D$ implies that $v(K) \in D$.

Thus, there are just four inequivalent literals in MPT0; two of them are positive while the others are negative.

The calculus for the logic MPT0 can be easily extended to a first-order calculus called QMPT0, by adding the usual axioms and inference rules for quantifiers, such that $\neg\forall x\varphi$ is equivalent to $\exists x\neg\varphi$, and $\neg\exists x\varphi$ is equivalent to $\forall x\neg\varphi$. The semantics for QMPT0 is given by *pragmatic structures* (see [3, 2]), which are Tarskian structures $\mathfrak{A} = \langle D, (\cdot)^{\mathfrak{A}} \rangle$ where any n -ary predicate symbol p is interpreted as a triple $p^{\mathfrak{A}} = \langle p_+^{\mathfrak{A}}, p_-^{\mathfrak{A}}, p_B^{\mathfrak{A}} \rangle$ such that p_+ , p_- and p_B are pairwise disjoint sets with $p_+ \cup p_- \cup p_B = D^n$. Elements in p_+ , p_- and p_B are the n -uples which satisfy p , do not satisfy p , and simultaneously satisfy p and not satisfy p , respectively. Thus, by defining inductively a triple $\varphi^{\mathfrak{A}} = \langle \varphi_+^{\mathfrak{A}}, \varphi_-^{\mathfrak{A}}, \varphi_B^{\mathfrak{A}} \rangle$ as above for every formula α with n free variables (see [3, 2]), it holds that $\varphi_+^{\mathfrak{A}} = \{\vec{a} \in D^n : \mathfrak{A} \models \varphi[\vec{a}]\}$ and $\mathfrak{A} \not\models \neg\varphi[\vec{a}]\}$; $\varphi_-^{\mathfrak{A}} = \{\vec{a} \in D^n : \mathfrak{A} \not\models \varphi[\vec{a}]\}$ and $\mathfrak{A} \models \neg\varphi[\vec{a}]\}$; and $\varphi_B^{\mathfrak{A}} = \{\vec{a} \in D^n : \mathfrak{A} \models \varphi[\vec{a}]\}$ and $\mathfrak{A} \models \neg\varphi[\vec{a}]\}$. Hence $\varphi_+^{\mathfrak{A}} \cup \varphi_B^{\mathfrak{A}} = \{\vec{a} \in D^n : \mathfrak{A} \models \varphi[\vec{a}]\}$ and $\varphi_-^{\mathfrak{A}} \cup \varphi_B^{\mathfrak{A}} = \{\vec{a} \in D^n : \mathfrak{A} \models \neg\varphi[\vec{a}]\}$. That is, $\mathfrak{A} \models \varphi[\vec{a}]$ if and only if $\vec{a} \in \varphi_+^{\mathfrak{A}} \cup \varphi_B^{\mathfrak{A}}$.

A *clause* in QMPT0 is a closed formula of the form

$$\forall x_1 \dots \forall x_n (L_1 \vee \dots \vee L_k \vee \sim L_{k+1} \vee \dots \vee \sim L_{k+m})$$

such that each L_i is a positive literal. Such a clause will be written as

$$L_1, \dots, L_k \leftarrow L_{k+1}, \dots, L_{k+m}.$$

Now, a resolution calculus for QMPT0 will be proposed. For this, just a *basic resolution rule* will be considered as an inference rule, as can be seen below. Since the clauses are implicitly universally quantified, an auxiliary concept will be necessary: two literals L_1 and L_2 of MPT0 are said to be *complementary* if one of the following conditions holds:

- (1) L_1 is positive and L_2 is $\sim L_1$; or
- (2) L_2 is positive and L_1 is $\sim L_2$.

Let $K_1 = L_{1,1} \vee \dots \vee L_{1,n}$ and $K_2 = L_{2,1} \vee \dots \vee L_{2,r}$ be two clauses. A clause K is obtained from K_1 and K_2 through a *basic resolution step* if there are literals L_1 and L_2 , with L_i occurring in K_i ($i = 1, 2$), and a substitution σ such that $\sigma(L_1)$ and $\sigma(L_2)$ are complementary literals, being σ the most general unifier with this property. In this case, the resolvent is $K = \sigma(K_0)$, where K_0 is the disjunction of literals that appear in K_1 (unless all instances of L_1 as literal in K_1) or in K_2 (unless all instances of L_2 as literal in K_2). We say that K is a *basic resolvent* of K_1 and K_2 . From K_1 and K_2 we obtain K by a *general resolution step* if there are renaming substitutions μ_1 and μ_2 such that K can be obtained from $\mu_1(K_1)$ and $\mu_2(K_2)$ by a basic resolution step. The closure under resolution of a set S of clauses will be denoted by $Res(S)$. Below are displayed two typical examples of resolution rules (here, $n, m > 0$):

$\frac{L \vee \bigvee_{i=1}^n A \quad K \vee \bigvee_{j=1}^m \sim A}{L \vee K}$	$\frac{L \vee \bigvee_{i=1}^n \neg A \quad K \vee \bigvee_{j=1}^m \sim \neg A}{L \vee K}$
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In order to obtain completeness of the resolution calculus for QMPT0, the paraconsistent third-excluded law (TEL) $\alpha \vee \neg\alpha$, which is valid in QMPT0, must be taken into account. This forces to develop non-trivial technical adjustments, as will be shown below. In order to see why (TEL) need to be considered, let $S = \{\sim K \vee L, \sim \neg K \vee L\}$ where K and L are atoms. In order to derive $L \vee L$ by resolution from S (note that $L \vee L$ does not reduce to L) it is necessary to add the (valid) clause $L \vee \neg L$ to S . That is, the given set of clauses must be expanded to another one in which some (relevant) instances of (TEL) are included. This motivates the following: given a set of clauses S , the *support* of S is the set $SUP(S) = \{A : A \text{ is an atomic formula and there exists } K, K' \in S, \text{ and a substitution } \theta \text{ such that } A \text{ occurs in } K\theta \text{ and } \neg A \text{ occurs in } K'\theta\}$. Finally, S_+ is the set $S \cup \{L(x_1, \dots, x_n) \vee \neg L(x_1, \dots, x_n) : L \text{ is a predicate symbol occurring in } SUP(S)\}$. That is, S_+ adds to S all the potentially relevant instances of (TEL). Hence:

Theorem 0.1 (Completeness of Clausal Resolution in QMPT0, version 1). *Let S be a set of clauses. Then, S_+ is satisfiable in QMPT0 iff the empty clause does not belong to $Res(S_+)$.*

Theorem 0.2 (Completeness of Clausal Resolution in QMPT0, version 2). *Let S be a set of clauses which is satisfiable in QMPT0, and let L be a ground literal (that is, without variables). Then, $S \models_{QMPT0} L$ iff $\bigvee_{i=1}^k L \in Res(S_+)$ for some $k \geq 1$.*

Corolary 0.3 (Completeness of Clausal Resolution in QMPT0, version 3). *Let S be a set of clauses in QMPT0, and let L be a ground literal. Then, $S \models_{QMPT0} L$ iff the empty clause belongs to $Res(S_+ \cup \{\sim L\})$.*

Let \mathcal{L} be a first-order language. The *Herbrand Universe* $U_{\mathcal{L}}$ for \mathcal{L} is the set of ground terms (that is, without variables) of \mathcal{L} (if \mathcal{L} does not have constants, a constant is added in order to generate the ground terms). The *Herbrand base* $B_{\mathcal{L}}$ is the set of ground positive literals of \mathcal{L} , and $B_{\mathcal{L}}^+$ is the subset of $B_{\mathcal{L}}$ formed by atomic formulas. A *Herbrand (pragmatic) interpretation* for \mathcal{L} is any subset \mathcal{I} of $B_{\mathcal{L}}$ with the following property: for every $A \in B_{\mathcal{L}}^+$, either $A \in \mathcal{I}$ or $\neg A \in \mathcal{I}$. It generates a pragmatic structure \mathfrak{A} as follows: for every n -ary predicate symbol p ,

- (i) $p_{\mathfrak{A}}^+ = \{(t_1, \dots, t_n) \in U_{\mathcal{L}}^n : p(t_1, \dots, t_n) \in \mathcal{I} \text{ and } \neg p(t_1, \dots, t_n) \notin \mathcal{I}\}$;
- (ii) $p_{\mathfrak{A}}^- = \{(t_1, \dots, t_n) \in U_{\mathcal{L}}^n : p(t_1, \dots, t_n) \notin \mathcal{I} \text{ and } \neg p(t_1, \dots, t_n) \in \mathcal{I}\}$;
- (iii) $p_{\mathfrak{A}}^{\mathfrak{A}} = \{(t_1, \dots, t_n) \in U_{\mathcal{L}}^n : p(t_1, \dots, t_n) \in \mathcal{I} \text{ and } \neg p(t_1, \dots, t_n) \in \mathcal{I}\}$.

Proposition 0.4. *Let S be a set of clauses and suppose that S has a model \mathfrak{A} . Then, S has a Herbrand model \mathcal{I} defined as follows: $\mathcal{I} = \{L \in B_{\mathcal{L}} : \mathfrak{A} \models L\}$.*

A *program clause in QMPT0* is a clause of the form $L \leftarrow K_1, \dots, K_n$. A *program in QMPT0* is a set of program clauses. A *goal* is a clause of the form $\leftarrow K_1, \dots, K_n$, for $n \geq 1$. Given a program \mathcal{P} in QMPT0, a subset \mathcal{I} of $B_{\mathcal{P}}$ is called a *partial Herbrand interpretation*. If \mathcal{I} contains all the ground literals which are derivable from \mathcal{P} in QMPT0, then \mathcal{I} is a *partial Herbrand model* of \mathcal{P} . The *Least partial Herbrand model* of \mathcal{P} is the set $M_{\mathcal{P}} = \{L \in B_{\mathcal{P}} : \mathcal{P} \models_{QMPT0} L\}$.

Finally, the Least partial Herbrand model $M_{\mathcal{P}}$ of \mathcal{P} will be characterized in terms of the fixed points of monotonic operators, in a similar way to the classical case. Because of (TEL), all the suitable extensions of \mathcal{P} must be considered, which are defined by using its support.

Given a program \mathcal{P} let $\bar{\mathcal{P}}$ be the program formed by all the ground instances of the clauses in \mathcal{P} . Let $\lambda(\mathcal{P})$ be the cardinal of $\text{SUP}(\bar{\mathcal{P}})$, and assume that $\text{SUP}(\bar{\mathcal{P}}) = \{A_i : i < \lambda(\mathcal{P})\}$. Let $\mathcal{P}^+ = \mathcal{P} \cup \{A_i \vee \neg A_i : i < \lambda(\mathcal{P})\}$. For $\gamma \in 2^{\lambda(\mathcal{P})}$ let

$$L_i^\gamma = \begin{cases} A_i & \text{if } \gamma(i) = 0 \\ \neg A_i & \text{if } \gamma(i) = 1 \end{cases} .$$

Let $I_\gamma^{\mathcal{P}} = \{L_i^\gamma : i < \lambda(\mathcal{P})\}$. Define $\mathcal{P}_\gamma = \mathcal{P} \cup I_\gamma^{\mathcal{P}}$, for every $\gamma \in 2^{\lambda(\mathcal{P})}$. The program \mathcal{P}_γ contains a possible extension of \mathcal{P} w.r.t. its support. By considering all that extensions, the set $M_{\mathcal{P}}$ can now be characterized. Observe that the continuous and monotonic operator $T_{\mathcal{P}} : 2^{B_{\mathcal{P}}} \rightarrow 2^{B_{\mathcal{P}}}$ can be defined as in the classical case. Then, we arrive to the following:

Theorem 0.5 (Fixed-Point characterization of the Least partial Herbrand model in QMPT0). *Let \mathcal{P} be a program in QMPT0, and let L be a positive ground literal. Then,*

$$\mathcal{P} \models_{QMPT0} L \quad \text{iff} \quad L \in \bigcap_{\gamma \in 2^{\lambda(\mathcal{P})}} T_{\mathcal{P}_\gamma} \uparrow \omega .$$

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Multialgebraizing logics by swap structures

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1 Introduction

Multialgebras (a.k.a. hyperalgebras) have been very much studied in the literature since 1934 when the French mathematician Frédéric Marty published the first paper about hypergroups.

Several logics in the hierarchy of the so-called *Logics of Formal Inconsistency* (in short **LFI**s) cannot be semantically characterized by a single finite matrix. Moreover, they lie outside the scope of the usual techniques of algebraization of logics such as Blok and Pigozzi's method. Several alternative semantical tools were introduced in the literature in order to deal with such systems. In particular, a special kind of multialgebra, called *swap structures*, was proposed in [3], which generalizes the well-known semantical characterization results of **LFI**s by means of finite non-deterministic matrices (Nmatrices) due to Avron and his collaborators. Additionally, it was proved in [4] that the swap structures semantics allows a Soundness and Completeness theorem by means of a very natural generalization of the well-known Lindenbaum-Tarski process.

In multialgebras the generalization of even basic concepts such as homomorphism, subalgebras and congruences is far to be obvious, and several different alternatives were proposed in the literature. In particular, some results related with Birkhoff's theorem were already obtained, but the proofs are either incomplete or the notions on which they are based are not suitable for our purposes. In this paper, some preliminary results towards the possibility of defining an algebraic theory of swap structures semantic will be shown, by adapting concepts of universal algebra to multialgebras in a suitable way. As a first step, we will concentrate our efforts on the algebraic theory of $\mathbb{K}_{\mathbf{mbC}}$, the class of swap structures for the logic **mbC** (which is the weakest system in the hierarchy of **LFI**s).

It will be shown that the class $\mathbb{K}_{\mathbf{mbC}}$ is closed under sub-swap-structures and products, but it is not closed under homomorphic images, hence it is not a variety in the usual sense. Nevertheless, it is possible to give a representation theorem for $\mathbb{K}_{\mathbf{mbC}}$ with a similar use of the Birkhoff's theorem in traditional algebraic logics. Finally, it is proved that, under the present approach, the classes of swap structures for some axiomatic extensions of **mbC** found in [3] are subclasses of $\mathbb{K}_{\mathbf{mbC}}$. They are obtained by requiring that its elements satisfy precisely the additional axioms. This allow a modular treatment of the algebraic theory of swap structures, as happens in the traditional algebraic setting.

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2 The category $\mathbf{MAlg}(\Sigma)$ of multialgebras over Σ

Definition 2.1. Let Σ be a signature. A *multialgebra* (or *hyperalgebra*) over Σ is a pair $\mathcal{A} = (A, \sigma_{\mathcal{A}})$ such that A is a nonempty set and $\sigma_{\mathcal{A}}$ is a mapping assigning, to each $c \in \Sigma_n$, a function (called *multioperation* or *hyperoperation*) $c^{\mathcal{A}} : A^n \rightarrow \wp(A)_+$. In particular, $\emptyset \neq c^{\mathcal{A}} \subseteq A$ if $c \in \Sigma_0$. When there is no risk of confusion, we write $c^{\mathcal{A}}$ instead of $c^{\mathcal{A}}$.

Definition 2.2. Let $\mathcal{A} = (A, \sigma_{\mathcal{A}})$ and $\mathcal{B} = (B, \sigma_{\mathcal{B}})$ be two multialgebras, and let $f : A \rightarrow B$ be a function. Then f is said to be a *homomorphism* from \mathcal{A} to \mathcal{B} if $f[c^{\mathcal{A}}(\vec{a})] \subseteq c^{\mathcal{B}}(f(\vec{a}))$, for every $c \in \Sigma_n$ and $\vec{a} \in A^n$. In particular, $f[c^{\mathcal{A}}] \subseteq c^{\mathcal{B}}$ for every $c \in \Sigma_0$.

Definition 2.3. $\mathbf{MAlg}(\Sigma)$ is the category formed by multialgebras over Σ and their homomorphisms.

Proposition 2.4. *The category $\mathbf{MAlg}(\Sigma)$ has arbitrary products.*

Definition 2.5. Let $\mathcal{A} = (A, \sigma_{\mathcal{A}})$ and $\mathcal{B} = (B, \sigma_{\mathcal{B}})$ be two multialgebras over Σ such that $B \subseteq A$. Then \mathcal{B} is said to be a *submultialgebra* of \mathcal{A} , denoted by $\mathcal{B} \subseteq \mathcal{A}$, if $c^{\mathcal{B}}(\vec{a}) \subseteq c^{\mathcal{A}}(\vec{a})$, for every $c \in \Sigma_n$ and $\vec{a} \in B^n$; in particular, $c^{\mathcal{B}} \subseteq c^{\mathcal{A}}$ if $c \in \Sigma_0$.

Definition 2.6. Let $\mathcal{A} = (A, \sigma_{\mathcal{A}})$ and $\mathcal{B} = (B, \sigma_{\mathcal{B}})$ be two multialgebras over Σ , and let $f : \mathcal{A} \rightarrow \mathcal{B}$ be a homomorphism in $\mathbf{MAlg}(\Sigma)$. The *direct image* of f is the submultialgebra $f(\mathcal{A}) = (f[A], \sigma_{f(\mathcal{A})})$ of \mathcal{B} such that $c^{f(\mathcal{A})}(\vec{b}) = \bigcup \{f[c^{\mathcal{A}}(\vec{a})] : \vec{a} \in f^{-1}(\vec{b})\}$ for every $c \in \Sigma_n$ and $\vec{b} \in f[A]$. In particular, $c^{f(\mathcal{A})} = f[c^{\mathcal{A}}]$ for every $c \in \Sigma_0$.

3 Swap structures for \mathbf{mbC}

Definition 3.1. The logic \mathbf{mbC} , defined over the signature $\Sigma = \{\wedge, \vee, \rightarrow, \neg, \circ\}$, is obtained from the positive classical propositional logic \mathbf{CPL}^+ by adding the axiom schemes **[Ax10]**: $\alpha \vee \neg\alpha$ and **[bc1]**: $\circ\alpha \rightarrow (\alpha \rightarrow (\neg\alpha \rightarrow \beta))$.

From now on, Σ will denote the signature for \mathbf{mbC} . Let $\mathcal{A} = \langle A, \wedge, \vee, \rightarrow, 0, 1 \rangle$ be a Boolean algebra and let $\pi_{(j)} : A^3 \rightarrow A$ be the canonical projections, for $1 \leq j \leq 3$. If $z \in A^3$ and $z_j = \pi_{(j)}(z)$ for $1 \leq j \leq 3$ then $z = (z_1, z_2, z_3)$.

Definition 3.2. Let \mathcal{A} be a Boolean algebra with domain A . The universe of swap structures over \mathcal{A} is the set $B_{\mathcal{A}} = A^3$.

Definition 3.3. Let $\mathcal{A} = \langle A, \wedge, \vee, \rightarrow, 0, 1 \rangle$ be a Boolean algebra. A *swap structure over \mathcal{A}* is any multialgebra $\mathcal{B} = \langle B, \wedge, \vee, \rightarrow, \neg, \circ \rangle$ over Σ such that $B \subseteq B_{\mathcal{A}}$ and the multioperations satisfy the following, for every z and w in B :

- (i) $z \# w \subseteq \{u \in B : u_1 = z_1 \# w_1\}$, for each $\# \in \{\wedge, \vee, \rightarrow\}$;
- (ii) $\neg z \subseteq \{u \in B : u_1 = z_2\}$;
- (iii) $\circ z \subseteq \{u \in B : u_1 = z_3\}$.

Let $\mathbb{K} = \{\mathcal{B} : \mathcal{B} \text{ is a swap structure over } \mathcal{A}, \text{ for some } \mathcal{A}\}$ be the class of swap structures. If $z \in \mathbb{K}$, for $\mathcal{B} \in \mathbb{K}$, then z_1, z_2 and z_3 represent a possible value of the formulas $\varphi, \neg\varphi$ and $\circ\varphi$, respectively. The full subcategory in $\mathbf{MAlg}(\Sigma)$ of swap structures will be denoted by \mathbf{SW} . Thus, the class of objects of \mathbf{SW} is \mathbb{K} , and the morphisms between two given swap structures are just the homomorphisms between them (as multialgebras over Σ). A special subclass of swap structures is formed by the swap structures for \mathbf{mbC} , defined as follows:

Definition 3.4. The *universe* of swap structures for \mathbf{mbC} over a Boolean algebra \mathcal{A} is the set $B_{\mathcal{A}}^{\mathbf{mbC}} = \{z \in B_{\mathcal{A}} : z_1 \vee z_2 = 1 \text{ and } z_1 \wedge z_2 \wedge z_3 = 0\}$.

Definition 3.5. Let \mathcal{A} be a Boolean algebra. A swap structure over \mathcal{A} is said to be a *swap structure for \mathbf{mbC} over \mathcal{A}* if its domain is included in $B_{\mathcal{A}}^{\mathbf{mbC}}$. Let $\mathcal{B}_{\mathcal{A}}^{\mathbf{mbC}}$ be the unique swap structure for \mathbf{mbC} with domain $B_{\mathcal{A}}^{\mathbf{mbC}}$ such that the equality holds in Definition 3.3(i)-(iii).

Let $\mathbb{K}_{\mathbf{mbC}} = \{\mathcal{B} \in \mathbb{K} : \mathcal{B} \text{ is a swap structure for } \mathbf{mbC}\}$ be the class of swap structures for \mathbf{mbC} . The full subcategory in \mathbf{SW} of swap structures for \mathbf{mbC} will be denoted by $\mathbf{SW}_{\mathbf{mbC}}$. Clearly, $\mathbf{SW}_{\mathbf{mbC}}$ is a full subcategory in $\mathbf{MAlg}(\Sigma)$. Thus, the class of objects of $\mathbf{SW}_{\mathbf{mbC}}$ is $\mathbb{K}_{\mathbf{mbC}}$, and the morphisms between two given swap structures for \mathbf{mbC} are the homomorphisms between them as multialgebras over Σ . If $\{\mathcal{A}_i : i \in I\}$ is a family of Boolean algebras then $\prod_{i \in I} B_{\mathcal{A}_i}^{\mathbf{mbC}}$ is isomorphic to $B_{\prod_{i \in I} \mathcal{A}_i}^{\mathbf{mbC}}$ in $\mathbf{MAlg}(\Sigma)$. Using this, and the fact that $\mathbf{SW}_{\mathbf{mbC}}$ is a subcategory of $\mathbf{MAlg}(\Sigma)$, and the latter has arbitrary products, it follows:

Proposition 3.6. *The category $\mathbf{SW}_{\mathbf{mbC}}$ has arbitrary products.*

Definition 3.7. For each $\mathcal{B} \in \mathbb{K}_{\mathbf{mbC}}$ let $D_{\mathcal{B}} = \{z \in |\mathcal{B}| : z_1 = 1\}$. The *Nmatrix associated to \mathcal{B}* is $\mathcal{M}(\mathcal{B}) = (\mathcal{B}, D_{\mathcal{B}})$. Let $Mat(\mathbb{K}_{\mathbf{mbC}}) = \{\mathcal{M}(\mathcal{B}) : \mathcal{B} \in \mathbb{K}_{\mathbf{mbC}}\}$.

Definition 3.8. Let $\mathcal{B} \in \mathbb{K}_{\mathbf{mbC}}$ and $\mathcal{M}(\mathcal{B})$ as above.

- (i) A *valuation* over $\mathcal{M}(\mathcal{B})$ is a valuation over a Nmatrix (see [1]);
- (ii) Let $\Gamma \cup \{\varphi\} \subseteq For(\Sigma)$ be a set of formulas of \mathbf{mbC} . We say that φ is a *consequence* of Γ in the class $Mat(\mathbb{K}_{\mathbf{mbC}})$ of Nmatrices, denoted by $\Gamma \models_{Mat(\mathbb{K}_{\mathbf{mbC}})} \varphi$, if $\Gamma \models_{\mathcal{M}(\mathcal{B})} \varphi$ for every $\mathcal{M}(\mathcal{B}) \in Mat(\mathbb{K}_{\mathbf{mbC}})$ (for semantic consequence in a Nmatrix, see [1]).

Theorem 3.9 (Carnielli-Coniglio [3]). *Let $\Gamma \cup \{\varphi\} \subseteq For(\Sigma)$ be a set of formulas of \mathbf{mbC} . Then, $\Gamma \vdash_{\mathbf{mbC}} \varphi$ iff $\Gamma \models_{Mat(\mathbb{K}_{\mathbf{mbC}})} \varphi$.*

A similar result can be obtained for positive classical logic \mathbf{CPL}^+ .

Definition 3.10. For each $\mathcal{B} \in \mathbb{K}$ let $D_{\mathcal{B}} = \{z \in |\mathcal{B}| : z_1 = 1\}$. The *Nmatrix associated to \mathcal{B}* is $\mathcal{M}(\mathcal{B}) = (\mathcal{B}, D_{\mathcal{B}})$. Let $Mat(\mathbb{K}) = \{\mathcal{M}(\mathcal{B}) : \mathcal{B} \in \mathbb{K}\}$.

Let \mathbf{CPL}_e^+ be the linguistic extension of \mathbf{CPL}^+ obtained by adding \neg and \circ (without any additional axioms). Then:

Theorem 3.11. *Let $\Gamma \cup \{\varphi\} \subseteq For(\Sigma)$ be a set of formulas of \mathbf{CPL}_e^+ . Then, $\Gamma \vdash_{\mathbf{CPL}_e^+} \varphi$ iff $\Gamma \models_{Mat(\mathbb{K})} \varphi$.*

The Nmatrix $\mathcal{M}_{\mathbb{A}_2}^{\mathbf{mbC}}$ induced by the swap structure $\mathcal{B}_{\mathbb{A}_2}^{\mathbf{mbC}}$ defined over the two-element Boolean algebra \mathbb{A}_2 was originally introduced by A. Avron in [1], in order to semantically characterize the logic \mathbf{mbC} . Avron's result means that the Nmatrix induced by the swap structure $\mathcal{B}_{\mathbb{A}_2}^{\mathbf{mbC}}$ defined over the two-element Boolean algebra \mathbb{A}_2 is sufficient for characterizing the logic \mathbf{mbC} , and so it represents, in a certain way, the whole class $\mathbb{K}_{\mathbf{mbC}}$ of swap structures for \mathbf{mbC} . One of the main purposes of the present study is to prove that the 5-element multialgebra $\mathcal{B}_{\mathbb{A}_2}^{\mathbf{mbC}}$ generates (in some sense) the class $\mathbb{K}_{\mathbf{mbC}}$, in analogy to the fact that the 2-element Boolean algebra \mathbb{A}_2 generates the class of Boolean algebras.

Theorem 3.12 (Representation Theorem for $\mathbb{K}_{\mathbf{mbC}}$). *Let \mathcal{B} be a swap structure for \mathbf{mbC} . Then, there exists a set I and a monomorphism of multialgebras $\hat{h} : \mathcal{B} \rightarrow \prod_{i \in I} \mathcal{B}_{\mathbb{A}_2}^{\mathbf{mbC}}$.*

A question arises: in analogy to Birkhoff's theorem, is the class $\mathbb{K}_{\mathbf{mbC}}$ a variety of multialgebras, that is, a class closed under products, subalgebras and homomorphic images? We know that it is closed under products and subalgebras. However, it will be shown now that it is not closed under homomorphic images:

Proposition 3.13. *The class $\mathbb{K}_{\mathbf{mbC}}$ of multialgebras is closed under submultialgebras and (direct) products, but it is not closed under homomorphic images.*

4 Extension to other systems

It is possible to show that the class of swap structures for some axiomatic extensions of \mathbf{mbC} are subclasses of $\mathbb{K}_{\mathbf{mbC}}$. In order to do this, let \mathbf{mbCciw} and \mathbf{mbCci} be the logics obtained from \mathbf{mbC} by adding the axioms schema $[\mathbf{ciw}]$: $\circ\alpha \vee (\alpha \wedge \neg\alpha)$ and $[\mathbf{ci}]$: $\neg\circ\alpha \rightarrow (\alpha \wedge \neg\alpha)$ respectively. Finally, let \mathbf{CPL}_e be the extension of \mathbf{mbC} obtained by adding the axiom schema $[\mathbf{cons}]$: $\circ\alpha$. As proved in [2], \mathbf{CPL}_e is nothing more than classical propositional logic \mathbf{CPL} expanded with a connective \circ such that $\circ\alpha$ is a theorem for every α (and so, from the semantic point of view, the formula $\circ\alpha$ is always true).

Let \mathcal{A} be a Boolean algebra with domain A . The universe of swap structures for \mathbf{mbCciw} over \mathcal{A} is the set $B_{\mathcal{A}}^{\mathbf{mbCciw}} = \{z \in A^3 : z_1 \vee z_2 = 1 \text{ and } z_3 = \sim(z_1 \wedge z_2)\}$ and a swap structure for \mathbf{mbCciw} is any $\mathcal{B} \in \mathbb{K}_{\mathbf{mbC}}$ such that $|\mathcal{B}| \subseteq B_{\mathcal{A}}^{\mathbf{mbCciw}}$. The class of swap structures for \mathbf{mbCciw} will be denoted by $\mathbb{K}_{\mathbf{mbCciw}}$. A swap structure for \mathbf{mbCci} is any $\mathcal{B} \in \mathbb{K}_{\mathbf{mbCciw}}$ such that $\circ(x) = \{(\sim(x_1 \wedge x_2), x_1 \wedge x_2, 1)\}$ and the class of swap structures for \mathbf{mbCci} will be denoted by $\mathbb{K}_{\mathbf{mbCci}}$. The universe of swap structures for \mathbf{CPL}_e over \mathcal{A} is the set $B_{\mathcal{A}}^{\mathbf{CPL}_e} = \{z \in B_{\mathcal{A}}^{\mathbf{mbCciw}} : z_2 = \sim z_1\} = \{z \in A^3 : z_2 = \sim z_1 \text{ and } z_3 = 1\}$ and a swap structure for \mathbf{CPL}_e is any $\mathcal{B} \in \mathbb{K}_{\mathbf{mbCci}}$ such that $|\mathcal{B}| \subseteq B_{\mathcal{A}}^{\mathbf{CPL}_e}$. The class of swap structures for \mathbf{CPL}_e will be denoted by $\mathbb{K}_{\mathbf{CPL}_e}$. Observe that $\mathbb{K}_{\mathbf{CPL}_e} \subset \mathbb{K}_{\mathbf{mbCci}} \subset \mathbb{K}_{\mathbf{mbCciw}} \subset \mathbb{K}_{\mathbf{mbC}} \subset \mathbb{K}$ while $\mathbf{CPL}_e^+ \subset \mathbf{mbC} \subset \mathbf{mbCciw} \subset \mathbf{mbCci} \subset \mathbf{CPL}_e$. Moreover:

(i) $\mathbb{K}_{\mathbf{mbC}} = \{\mathcal{B} \in \mathbb{K} : \models_{\mathcal{M}(\mathcal{B})} (\mathbf{Ax10}) \wedge (\mathbf{bc1})\}$; (ii) $\mathbb{K}_{\mathbf{mbCciw}} = \{\mathcal{B} \in \mathbb{K}_{\mathbf{mbC}} : \models_{\mathcal{M}(\mathcal{B})} (\mathbf{ciw})\}$, (iii) $\mathbb{K}_{\mathbf{mbCci}} = \{\mathcal{B} \in \mathbb{K}_{\mathbf{mbCciw}} : \models_{\mathcal{M}(\mathcal{B})} (\mathbf{ci})\}$; (iv) $\mathbb{K}_{\mathbf{CPL}_e} = \{\mathcal{B} \in \mathbb{K}_{\mathbf{mbCci}} : \models_{\mathcal{M}(\mathcal{B})} (\mathbf{cons})\}$. So, by an analysis similar to the one presented above, we have that $\mathbb{K}_{\mathbf{mbCciw}}$ is generated by a 3-valued multialgebra which is a submultialgebra of the 5-valued generator of $\mathbb{K}_{\mathbf{mbC}}$ and $\mathbb{K}_{\mathbf{mbCci}}$ is generated by a 3-valued multialgebra which is a submultialgebra of the 3-valued generator of $\mathbb{K}_{\mathbf{mbCciw}}$. By its turn, $\mathbb{K}_{\mathbf{CPL}_e}$ is generated by a 2-valued algebra which is a submultialgebra of the 3-valued generator of $\mathbb{K}_{\mathbf{mbCci}}$. Theorem 3.12 can be adapted to these systems and, additionally:

Theorem 4.1. *For $L \in \{\mathbf{CPL}_e^+, \mathbf{mbC}, \mathbf{mbCciw}, \mathbf{mbCci}, \mathbf{CPL}_e\}$: $\Gamma \vdash_L \varphi$ iff $\Gamma \models_{Mat(\mathbb{K}_L)} \varphi$.*

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Constructive canonicity for lattice-based fixed point logics

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The present contribution lies at the crossroads of at least three active lines of research in nonclassical logics: the one investigating the semantic and proof-theoretic environment of fixed point expansions of logics algebraically captured by varieties of (distributive) lattice expansions [1, 17, 21, 2, 14]; the one investigating constructive canonicity for intuitionistic and substructural logics [15, 22]; the one uniformly extending the state-of-the-art in Sahlqvist theory to families of nonclassical logics, and applying it to issues both semantic and proof-theoretic [6], known as ‘unified correspondence’.

We prove the algorithmic canonicity of two classes of μ -inequalities in a constructive meta-theory of normal lattice expansions. This result simultaneously generalizes Conradie and Craig’s canonicity results for μ -inequalities based on a bi-intuitionistic bi-modal language [3], and Conradie and Palmigiano’s constructive canonicity for inductive inequalities [8] (restricted to normal lattice expansions). Besides the greater generality, the unification of these strands smoothes the existing proofs for the canonicity of μ -formulas and inequalities. Specifically, the two canonicity results proven in [3], namely, the tame and proper canonicity, fully generalize to the constructive setting and normal LEs. Remarkably, the rules of the algorithm ALBA used for this result have exactly the same formulation as those of [8], with no additional rule added specifically to handle the fixed point binders. Rather, fixed points are accounted for by certain restrictions on the application of the rules, concerning the order-theoretic properties of the term functions associated with the formulas to which the rules are applied.

The contributions reported on in the proposed talk pertain to *unified correspondence theory* [6], a line of research which applies duality-theoretic insights to Sahlqvist theory (cf. [10]), with the aim of uniformly extending the benefits of Sahlqvist theory from modal logic to a wide range of logics which include, among others, intuitionistic and distributive lattice-based (modal) logics [7], regular modal logics [20], substructural logics [9], hybrid logics [12], and mu-calculus [3, 5]. Applications of unified correspondence are very diverse, and include the understanding of the relationship between different methodologies for obtaining canonicity results [19], the phenomenon of pseudocorrespondence [4], the dual characterizations of classes of finite lattices [13], the identification of the syntactic shape of axioms which can be translated into structural rules of a properly displayable calculus [16], the definition of cut-free Gentzen calculi for subintuitionistic logics [18], and the investigation of the extent to which the Sahlqvist theory of classes of normal distributive lattice expansions can be reduced to the Sahlqvist theory of normal Boolean expansions, by means of Gödel-type translations [11].

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On Minimal Models for Horn Clauses over Predicate Fuzzy Logics

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Since their introduction in [13], Horn clauses have shown to have good logic properties and have proven to be of importance for many disciplines, ranging from logic programming, abstract specification of data structures and relational data bases, to abstract algebra and model theory. Several authors have contributed to the study of Horn clauses over fuzzy logic. In [5, 4, 3, 1, 2, 15] Bělohlávek and Vychodil study fuzzy equalities, adopting a Pavelka style, and work with theories that consist of formulas that are implications between identities with premises weighted by truth degrees. They prove the main logical properties of varieties of algebras with fuzzy equalities and obtain completeness results for fuzzy equational logic. From a different perspective, in a series of papers [11, 10, 9] Gerla proposes to base fuzzy control on fuzzy logic programming, and observes that the class of fuzzy Herbrand interpretations gives a semantics for fuzzy programs.

Following the above-mentioned works, our contribution is a first step towards a systematic model-theoretic account of Horn clauses in the framework introduced by Hájek in [12]. Our study of the model theory of Horn clauses is focused on the basic predicate fuzzy logic MTLV and some of its extensions based on propositional core fuzzy logics in the sense of [7]. We refer to [6, Ch.1] for a complete and extensive presentation of these logics. Our approach differs from the one of Bělohlávek and Vychodil because we do not restrict to fuzzy equalities. Another difference is that, unlike these authors and Gerla, our structures are not necessarily over the same complete algebra, because we work in the general semantics of [12]. The results of this talk are part of a PhD thesis in progress and they extend previous work by the authors presented in the 18th International Conference of the Catalan Association for Artificial Intelligence (CCIA 2015) [8].

In this talk we define the notion of *term structure associated to a set of formulas* Φ in the fuzzy context (denoted by $\langle \mathbf{F}_{\text{MTL}}(\bar{0}), \mathbf{T}^\Phi \rangle$) where $\mathbf{F}_{\text{MTL}}(\bar{0})$ is the free algebra for the variety of MTL-algebras MTL . We show that this structure is safe and we prove the existence of free models in universal Horn classes. The possibility given by fuzzy logic of defining the term structure associated to a theory using a similarity instead of the crisp equality leads us to a notion of free structure restricted to the class of reduced models of that theory (*reduced structures* are those whose Leibniz congruence is the identity).

Theorem 1. *Let Φ be a consistent set of formulas with $\|\Phi\|_{\mathbf{T}^\Phi, e^\Phi}^{\mathbf{F}_{\text{MTL}}(\bar{0})} = 1$. Then, $\langle \mathbf{F}_{\text{MTL}}(\bar{0}), \mathbf{T}^\Phi \rangle$ is a free structure in the class of the reduced models of Φ , i.e., for every reduced structure $\langle \mathbf{A}, \mathbf{M} \rangle$ and every evaluation v such that $\|\Phi\|_{\mathbf{M}, v}^{\mathbf{A}} = 1$, there is a unique homomorphism (f, g) from $\langle \mathbf{F}_{\text{MTL}}(\bar{0}), \mathbf{T}^\Phi \rangle$ such that for every $x \in \text{Var}$, $g(\bar{x}) = v(x)$.*

Theorem 2. *Let Φ be a consistent set of formulas. For every Horn clause φ , if $\Phi \vdash \varphi$, then $\|\varphi\|_{\mathbf{T}^{\Phi}, \epsilon^{\Phi}}^{\mathbf{F}_{\text{MTL}}(\bar{\emptyset})} = 1$.*

Herbrand structures are also introduced as structures associated to sets of equality-free sentences. We characterize the free Herbrand structure in the class of models of an equality-free universal Horn theory. Finally, we study a generalization of the notion of Herbrand structure, *fully named models*, and we prove that two concepts of minimality for these models are equivalent. A structure $\langle \mathbf{B}, \mathbf{N} \rangle$ is a *fully named model* if for any element n of the domain N , there exists a ground term t such that $\|t\|_{\mathbf{N}}^{\mathbf{B}} = n$.

Minimal structures have a relevant role in classical model theory and logic programming. Admitting minimal term structures make reasonable the concepts of *closed-word assumption* for databases and *negation as failure* for logic programming. These structures allow also a procedural interpretation for logic programs (for a reference see [14]). The minimality of a structure with respect to other structures of a certain class can be defined in different ways. On the one hand, a structure can be minimal from an algebraic point of view, in the sense of *free*, that is, if there is a unique homomorphism from this structure to any other structure in the class. On the other hand, a structure can be minimal from a model-theoretic point of view, in the sense of *atomic genericity*, if this structure is model exactly of those atomic sentences such that all the structures in the class are models of. The minimality of an atomically generic structure is revealed by the fact that it picks up the minimal positive information (that is, the information contained in atomic sentences).

Theorem 3. *Let \mathbf{K} be the class of all models of a consistent set of equality-free sentences which are (*w*-)Horn clauses. The intersection of the family of all *H*-structures in \mathbf{K} is the free model in \mathbf{K} .*

Definition 1. *Let \mathbf{K} be a class of structures. Given $\langle \mathbf{B}, \mathbf{N} \rangle \in \mathbf{K}$, we say that $\langle \mathbf{B}, \mathbf{N} \rangle$ is *A-generic* in \mathbf{K} if for every atomic sentence φ we have that*

$$\|\varphi\|_{\mathbf{N}}^{\mathbf{B}} = 1 \text{ if and only if for every structure } \langle \mathbf{A}, \mathbf{M} \rangle \in \mathbf{K}, \|\varphi\|_{\mathbf{M}}^{\mathbf{A}} = 1.$$

Theorem 4. *Let \mathbf{K} be a class of reduced structures and $\langle \mathbf{F}_{\mathbf{K}}(\bar{\emptyset}), \mathbf{N} \rangle \in \mathbf{K}$ be a fully named model. Then,*

$$\langle \mathbf{F}_{\mathbf{K}}(\bar{\emptyset}), \mathbf{N} \rangle \in \mathbf{K} \text{ is free in } \mathbf{K} \text{ if and only if } \langle \mathbf{F}_{\mathbf{K}}(\bar{\emptyset}), \mathbf{N} \rangle \text{ is A-generic in } \mathbf{K}.$$

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Interpreting Sequent Calculi as Client–Server Games*

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Resource consciousness is routinely cited as a main motivation for considering various kinds of substructural logics (see, e.g., [7]). But usually the reference to resources is kept informal and metaphoric, like in Girard’s well-known example of being able to buy a pack of Camels and/or a pack of Marlboro [5] with a single dollar, illustrating linear implication as well as the ambiguity of conjunction between the ‘multiplicative’ and ‘additive’ reading. The invitation to distinguish, e.g., between a ‘causal’, action-oriented interpretation of implication and a more traditional understanding of implication as a timeless, abstract relation between propositions is certainly inspiring and motivating. However, the specific shape and properties of proof systems for usual substructural logics owe more to a deep analysis of Gentzen’s sequent calculus [4] and, to some extent, also to Lambek’s calculus [6] than to action-oriented models of handling scarce resources of a specific kind. Various semantics, in particular so-called game semantics for (fragments of) linear logics [1, 2] offer additional leverage points for a logical analysis of resource consciousness. But these semantics hardly support a straightforward reading of sequent derivations as actions plans devised by resource conscious agents. Moreover, the inherent level of abstraction often does not match the appeal of Girard’s very concrete and simple picture of action-oriented inference.

Motivated by the above diagnosis, we introduce a two-person game based on the idea that a proof is an *action-plan*, i.e. a *strategy* for one of the players (the ‘client’) to establish particular *structured information*, given certain information provided the other player (the ‘server’). We will show that the rules of the game *directly* match the logical rules of a particular version of the sequent calculus for intuitionistic logic. More importantly in our context, the interpretation of game states as (single conclusion) sequents opens up a fairly wide space of variations of the initial game that leads to game based interpretations of various fragments of intuitionistic linear logic, but also of substructural logics based on variants of Lambek’s calculus [6].

To emphasize the indicated shift of perspective relative to traditional interpretations of formulas as sentences or propositions or types we introduce the notion of an *information package* (henceforth *ip*, for short). An ip is either *atomic* or else built up from given ips F_1, F_2, \dots, F_n ($n \geq 2$) using the following constructors:

- $\text{any_of}(F_1, \dots, F_n)$,
- $\text{some_of}(F_1, \dots, F_n)$,
- $(F_1 \text{ given } F_2)$.

Among the atomic ips is the *elementary inconsistent information* \perp .

In our $C/S(I)$ -game, a *client* C maintains that the information packaged as H can be obtained from the information represented by the ips G_1, \dots, G_n , provided by a *server* S , via stepwise reduction of complex ips into simpler ones. At any state of the game, the *bunch of information provided by* S is a (possibly empty) multiset of ips. The ip H which C currently

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claims to be obtainable from that information is called **C**'s *current ip*. The corresponding state is denoted by

$$G_1, \dots, G_n \triangleright H.$$

The game proceeds in *rounds* that are always initiated by **C** and, in general, solicit some action from **S**. There are two different types of requests that **C** may submit to **S**: (1) UNPACK a non-atomic ip provided by you (i.e. the server), and (2) CHECK my (i.e. the clients) current ip.

In a request of type UNPACK **C** points to an ip G in the bunch of information provided by **S** and the game proceeds as follows:

- (U_{any}^*) If $G = \text{any_of}(F_1, \dots, F_n)$ then **C** chooses an $i \in \{1, \dots, n\}$ and **S** has to add F_i to the bunch of provided information, accordingly.
- (U_{some}^*) If $G = \text{some_of}(F_1, \dots, F_n)$ then **S** chooses an $i \in \{1, \dots, n\}$ and adds F_i to the bunch of provided information, accordingly.
- (U_{given}^*) If $G = (F_1 \text{ given } F_2)$ then **S** chooses whether to add F_1 to the bunch of provided information or whether to force **C** to replace her current ip by F_2 .
- (U_{\perp}^+) If $G = \perp$ then the game ends and **C** wins.

If the request is of type CHECK then the game proceeds according to the form of **C**'s current ip H .

- (C_{any}) If $H = \text{any_of}(F_1, \dots, F_n)$ then **S** chooses an $i \in \{1, \dots, n\}$ and **C** has to replace the current ip by F_i , accordingly.
- (C_{some}) If $H = \text{some_of}(F_1, \dots, F_n)$ then **C** chooses an $i \in \{1, \dots, n\}$ and replaces the current ip by F_i , accordingly.
- (C_{given}) If $H = (F_1 \text{ given } F_2)$ then F_2 is added to the bunch of provided information and **C**'s current ip is replaced by F_1 .
- (C_{atom}^+) If H is atomic then the game ends and **C** wins if an occurrence of H is among the bunch of information provided by **S**.

At the beginning of each round of the game **C** is free to choose whether she wants to continue with a request of type UNPACK (if possible) or of type CHECK; moreover in the first case **C** can freely choose any occurrence of a non-atomic ip or an occurrence of \perp in the bunch of information provided by **S**. Formally, each initial state $G_1, \dots, G_n \triangleright H$ induces an extensive two-players win/lose (zero sum) game of perfect information in the usual game theoretic sense. The corresponding game tree is finitely branching, but may be infinite since **C** may request to unpack the same ip repeatedly. We will look at strategies only at the level of states resulting from fully completed rounds. A *winning strategy* τ for **C** can therefore be identified with a finite, downward growing, rooted tree of game states, where all leaves are winning states for **C** according to either rule U_{\perp}^+ or rule C_{atom}^+ . The root of τ is the initial state of the relevant instance of the **C/S(I)**-game in question. When, at a state S , the strategy τ tells **C** to continue the game with a round of type U_{some}^* , U_{given}^* , or C_{any} , then τ branches at S into two or more successor states according to the possible choices available to **S** as specified by the rules. On the other hand, no branching occurs at states where τ tells **C** to continue according to any other rule, since those rules do not involve a choice of **S**.

The reader presumably has no difficulties in recognizing ips as ordinary propositional formulas in disguise: restricting attention to binary versions of the operators, we obtain ordinary

formulas by writing $(F_1 \wedge F_2)$ for $\text{any_of}(F_1, F_2)$, $(F_1 \vee F_2)$ for $\text{some_of}(F_1, F_2)$, and $(F_2 \rightarrow F_1)$ for $(F_1 \text{ given } F_2)$. Negation is defined by $\neg F = (F \rightarrow \perp)$. Whenever it is appropriate to explicitly distinguish ips from corresponding formulas, we will overline ips.

In order to show that the $\mathbf{C/S(I)}$ -game is adequate for intuitionistic logic I we employ a specific variant \mathbf{LIk} of Gentzen’s sequent calculus \mathbf{LI} . \mathbf{LIk} arises from \mathbf{LI} by getting rid of exchange by using multisets, eliminating contraction by building into the logical rules, and eliminating weakening by generalizing the initial sequents (axioms) correspondingly (cf. [8]).

Theorem 1. *The client \mathbf{C} has a winning strategy for the $\mathbf{C/S(I)}$ -game starting in the state $\overline{G}_1, \dots, \overline{G}_n \triangleright \overline{H}$ iff H is an intuitionistic consequence of G_1, \dots, G_n . More specifically, there is an isomorphism between cut-free \mathbf{LIk} -derivations and $\mathbf{C/S(I)}$ -game winning strategies for \mathbf{C} .*

Instead of referring to \mathbf{LIk} one may introduce ‘bookkeeping rules’ into the game. In particular, weakening corresponds to the rule *Dismiss* that allows \mathbf{C} to eliminate an ip from the bunch of information provided by \mathbf{S} , while contraction corresponds to the rule *Copy*, enabling \mathbf{C} to duplicate a given ip. With these bookkeeping rules in place, one may modify the rules U_{any}^* , U_{some}^* , and U_{given}^* by dismissing the selected ip G after unpacking, instead of adding the unpacked components to the bunch of provided information. Similarly rules U_{\perp}^+ and C_{atom}^+ are modified to match the axioms of \mathbf{LI} , instead of those of \mathbf{LIk} .

The cut-rule plays a different role. Its eliminability corresponds to a ‘sanity check’ for client–server games, expressing that strategies can be combined as stated in the following corollary of the cut elimination theorem for $\mathbf{LI/LIk}$.

Corollary 1. *Suppose \mathbf{C} has a winning strategy τ for the $\mathbf{C/S(I)}$ -game starting in the state $\overline{F}_1, \dots, \overline{F}_n \triangleright \overline{H}$ and another winning strategy τ' for the $\mathbf{C/S(I)}$ -game starting in the state $\overline{H}, \overline{F}'_1, \dots, \overline{F}'_m \triangleright \overline{G}$, then τ and τ' can be combined into a winning strategy for the game starting in the state $\overline{F}_1, \dots, \overline{F}_n, \overline{F}'_1, \dots, \overline{F}'_m \triangleright \overline{G}$.*

Probably the most important step in converting the $\mathbf{C/S(I)}$ -game into a ‘resource conscious’ game, is based on the following observation regarding rules that entail a choice by \mathbf{S} and thus require \mathbf{C} to be prepared to act in more than just one possible successor state to the current state. The above rules allow \mathbf{C} to use *all* the information provided by \mathbf{S} in *each* of the possible successor states. If, instead, we require \mathbf{C} to declare which ips she intends to use for which of those options—taking care that she is using each occurrence of an ip exactly once—then we arrive at rules that match multiplicative instead of additive connectives. We illustrate this principle by the \mathbf{CHECK} -rule for the new constructor each_of , corresponding to multiplicative conjunction in intuitionistic linear logic \mathbf{ILL} .

(C_{each}) If $H = \text{each_of}(F_1, F_2)$ then \mathbf{C} has to split the bunch (multiset) Π of information provided by \mathbf{S} into $\Pi_1 \uplus \Pi_2 = \Pi$ and then let \mathbf{S} choose whether to continue the game in state $\Pi_1 \triangleright F_1$ or in state $\Pi_2 \triangleright F_2$.

Analogously one may define a multiplicative version U_{given}^m of the \mathbf{UNPACK} -rule for given , matching linear implication. To obtain a game for full \mathbf{ILL} we replace *Copy* and *Dismiss* by rules that allow \mathbf{C} to do one of the following with any occurrence ip of the form $\text{arbitrary_many}(F)$ in the bunch of formulas provided by \mathbf{S} :

- dismiss $\text{arbitrary_many}(F)$
- add another copy of $\text{arbitrary_many}(F)$
- replace $\text{arbitrary_many}(F)$ by F

Clearly the constructor `arbitrary_many` matches the ‘exponential’ $!$ of linear logic.

To obtain a variant of the $\mathbf{C/S}(I)$ -game that interprets *Full Lambek Calculus* \mathbf{FL} [6] one has to identify the bunch of information provided by \mathbf{S} with a *list* (instead of a multiset) of ips and replaces the rule C_{given} by two variants that specify whether F_2 of $(F_1 \text{ given } F_2)$ is added at the beginning or at the end of the list. In this manner we actually obtain a whole family of games characterizing substructural logics corresponding to different types of residuated lattices (see [3]). This in turn provides the basis for interpreting lattice elements as contents of information packages.

We finally emphasize that all mentioned versions of the basic client-server game can be lifted to the first-order level by treating quantifiers as operators for packaging schematic information in a natural and expected manner.

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Evaluation Driven Proof-Search in Natural Deduction Calculi for Intuitionistic Propositional Logic

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Starting from the seminal papers on *uniform proof systems* [9] and on *focusing* in linear logic [1], a considerable work has been done on the proof-theoretical characterization of goal-oriented proof-search that has led to the development of the proof-theory of *focused proof-systems* [8, 10]. This research is mainly devoted to sequent calculi and almost neglects the problem of proof-search in natural deduction calculi. As discussed in [14], this is probably motivated by the fact that natural deduction calculi lack the “deep symmetries” of sequent calculi which can be immediately exploited to control and reduce the search space.

We address the problem of proof-search in the natural deduction calculus for Intuitionistic Propositional Logic (IPL). We aim to improve the proof-search procedure based on the *intercalation calculus* [13, 14], where introduction rules are applied upwards and elimination rules downwards. To this purpose, we introduce the (sequent-style) natural deduction calculus \mathbf{Nbu}_θ , a variant of the usual natural deduction calculus for IPL where rule application is controlled by a labeling discipline and by some side-conditions exploiting *evaluation relations*. We claim that \mathbf{Nbu}_θ can be used as a base for a goal-oriented proof-search procedure.

The proof-search strategy of intercalation calculus

We represent natural deduction derivations in sequent style so that at each node $\Gamma \vdash H$ the open assumptions on which the derivation of H depends are put in evidence in the *context* Γ (a multiset). The strategy to build a natural deduction derivation of $\Gamma \vdash H$ based on the intercalation calculus presented in [14] consists of applying introduction rules (I-rules) reasoning bottom-up (from the conclusion to the premises) and elimination rules (E-rules) top-down (from the premises to the conclusion), in order to “close the gap” between Γ and H . Derivations built in this way are *normal* according to the standard definition [15]. This approach can be formalized using the calculus \mathbf{NJ} of Fig. 1 acting on two kinds of judgments, we denote by $\Gamma \vdash H \uparrow$ and $\Gamma \vdash H \downarrow$ (see [2, 3, 11] for similar calculi for IPL and intuitionistic linear logic). A derivation of \mathbf{NJ} with root sequent $\Gamma \vdash H \uparrow$ can be interpreted as a normal derivation of $\Gamma \vdash H$ having an I-rule, the rule $\perp E$ or $\vee E$ as root rule. An \mathbf{NJ} -derivation with root sequent $\Gamma \vdash H \downarrow$ represents a normal derivation of $\Gamma \vdash H$ having the rule Id, $\wedge E_k$ or $\rightarrow E$ as root rule. The rule \Downarrow (*coercion*), not present in the usual natural deduction calculus, is a sort of structural rule which “coerces” deductions in normal form. Actually, derivations of \mathbf{NJ} are in *long normal form* [15] since, differently from [3, 11], the coercion rule \Downarrow is restricted to *prime formulas* (namely, to propositional variables or \perp). We also deviate from [3, 11] since rule $\perp E$ is restricted to prime formulas and $\vee E$ is restricted to prime formulas and disjunctions. However, one can easily prove that these limitations do not affect the completeness of the calculus.

Using \mathbf{NJ} , the strategy of [14] to search for a derivation of $\Gamma \vdash H$ (that is, a normal derivation of H from assumptions Γ) can be sketched as follows. We start from the sequent $\Gamma \vdash H \uparrow$ and we \uparrow -expand it by applying bottom-up the I-rules; meanwhile, for every $H \in \Gamma$,

$$\begin{array}{c}
k \in \{0, 1\}, p \in \mathcal{V}, F \in \mathcal{V} \cup \{\perp\}, D \in \mathcal{V} \cup \{\perp\} \text{ or } D = D_0 \vee D_1 \\
\\
\frac{}{A, \Gamma \vdash A \downarrow} \text{Id} \quad \frac{\Gamma \vdash p \downarrow}{\Gamma \vdash p \uparrow} \Downarrow \quad \frac{\Gamma \vdash \perp \downarrow}{\Gamma \vdash F \uparrow} \perp E \\
\\
\frac{\Gamma \vdash A \uparrow \quad \Gamma \vdash B \uparrow}{\Gamma \vdash A \wedge B \uparrow} \wedge I \quad \frac{\Gamma \vdash A_0 \wedge A_1 \downarrow}{\Gamma \vdash A_k \downarrow} \wedge E_k \\
\\
\frac{\Gamma \vdash A_k \uparrow}{\Gamma \vdash A_0 \vee A_1 \uparrow} \vee I_k \quad \frac{\Gamma \vdash A \vee B \downarrow \quad A, \Gamma \vdash D \uparrow \quad B, \Gamma \vdash D \uparrow}{\Gamma \vdash D \uparrow} \vee E \\
\\
\frac{A, \Gamma \vdash B \uparrow}{\Gamma \vdash A \rightarrow B \uparrow} \rightarrow I \quad \frac{\Gamma \vdash A \rightarrow B \downarrow \quad \Gamma \vdash A \uparrow}{\Gamma \vdash B \downarrow} \rightarrow E
\end{array}$$

Figure 1: The natural deduction calculus **NJ**.

$$\begin{array}{c}
\frac{}{A, \Gamma \vdash A \downarrow} \text{Id} \quad \frac{\Gamma \vdash p \downarrow}{\Gamma \vdash p \uparrow^l} \Downarrow \quad \frac{\Gamma \vdash \perp \downarrow}{\Gamma \vdash F \uparrow^l} \perp E \\
\\
\frac{\Gamma \vdash A \uparrow^l \quad \Gamma \vdash B \uparrow^l}{\Gamma \vdash A \wedge B \uparrow^l} \wedge I \quad \frac{\Gamma \vdash A_0 \wedge A_1 \downarrow}{\Gamma \vdash A_k \downarrow} \wedge E_k \\
\\
\frac{\Gamma \vdash A_k \uparrow^b}{\Gamma \vdash A_0 \vee A_1 \uparrow^l} \vee I_k \quad \frac{\Gamma \vdash A \vee B \downarrow \quad A, \Gamma \vdash D \uparrow^u \quad B, \Gamma \vdash D \uparrow^u}{\Gamma \vdash D \uparrow^u} \vee E \quad \begin{array}{l} \Gamma \not\vdash_\theta A \\ \Gamma \not\vdash_\theta B \end{array} \\
\\
\frac{\Gamma \vdash A \rightarrow B \downarrow \quad \Gamma \vdash A \uparrow^b}{\Gamma \vdash B \downarrow} \rightarrow E \\
\\
\frac{\Gamma \vdash B \uparrow^l}{\Gamma \vdash A \rightarrow B \uparrow^l} \rightarrow I_1 \quad \Gamma \models_\theta A \quad \frac{A, \Gamma \vdash B \uparrow^u}{\Gamma \vdash A \rightarrow B \uparrow^l} \rightarrow I_2 \quad \Gamma \not\vdash_\theta A
\end{array}$$

Figure 2: The natural deduction calculus **Nbu**_θ ($l \in \{b, u\}$).

we \downarrow -expand the axiom sequent $\Gamma \vdash H \downarrow$ by applying downwards the rules $\wedge E_k$ and $\rightarrow E$. At each step, we get “open proof-trees” (henceforth, we call them *trees*); to successfully build a derivation in normal form, we have to glue such trees using the rules \Downarrow , $\perp E$ or $\vee E$. In general, the two expansion steps must be interleaved, hence the search space is huge. For instance, when in \downarrow -expansion we apply the rule $\rightarrow E$ to a tree with root $\Gamma \vdash A \rightarrow B \downarrow$, we get $\Gamma \vdash B \downarrow$ as new root and $\Gamma \vdash A \uparrow$ as new leaf. Thus, to turn the tree into a derivation, we must enter a new \uparrow -expansion phase to get a derivation of $\Gamma \vdash A \uparrow$. During \uparrow -expansion, contexts can increase. For instance, suppose to apply the rule $\rightarrow I$ to \uparrow -expand the sequent $\Gamma \vdash A \rightarrow B \uparrow$. Then, the context Γ is enlarged with the addition of A , hence we have to run a new \downarrow -expansion step from the axiom sequent $A, \Gamma \vdash A \downarrow$. A similar case concerns the application of $\vee E$: if the sequent to \uparrow -expand is $\Gamma \vdash H \uparrow$ and a derivation of $\Gamma \vdash A \vee B \downarrow$ has been obtained in previous steps, we continue the \uparrow -expansion with $A, \Gamma \vdash H \uparrow$ and $B, \Gamma \vdash H \uparrow$. Accordingly, we have to take into account the new contexts A, Γ and B, Γ for \downarrow -expansion. As pointed out in [14], the application of rule $\vee E$ is one of the main cause of inefficiency in proof-search for natural deduction.

Limiting the application of context extending rules

Our starting point is whether it is possible to restrict the search space by limiting the application of context extending rules. Taking advantages of ideas from [5, 6], we introduce the calculus \mathbf{Nbu}_θ of Fig.2, which allows us to limit the applications of $\rightarrow I$ and $\vee E$ during \uparrow -expansion. The rules of \mathbf{Nbu}_θ act on sequents where the arrow \uparrow is decorated by one of the labels b (*blocked*) and u (*unblocked*). In defining \mathbf{Nbu}_θ , we exploit an *evaluation relation* \models_θ , namely a decidable relation \models_θ between finite multisets of formulas and formulas with the following properties:

- ($\theta 1$) $A \in \Gamma$ implies $\Gamma \models_\theta A$;
- ($\theta 2$) $\Gamma \models_\theta A$ and $\mathbf{NJ} \vdash_d A, \Gamma \vdash H \uparrow^l$ imply $\mathbf{NJ} \vdash_d \Gamma \vdash H \uparrow^l$ ($l \in \{b, u\}$), where $\mathbf{NJ} \vdash_d \sigma$ means that there exists an \mathbf{NJ} -derivation of σ having depth at most d .

Intuitively, $\Gamma \models_\theta A$ means that the information conveyed by A is already available in Γ ; in proof-search, we avoid applying a rule which adds A to a context Γ if $\Gamma \models_\theta A$. The minimum evaluation relation is the membership relation ($\Gamma \models_\theta A$ iff $A \in \Gamma$). Another example is the *cover* relation [4, 12]: Γ *covers* E iff $E \in \Gamma$ or E is of the kind $E \wedge E$ or $E \vee A$ (A any formula) or $A \vee E$ or $A \rightarrow E$. One can easily check that *cover* is an evaluation relation (to prove ($\theta 2$)), one has to use the standard depth-preserving inversion principles of \mathbf{NJ} .

The rule for implication introduction is split into two versions to distinguish between applications which retain ($\rightarrow I_1$) and change ($\rightarrow I_2$) the context; A is added to Γ only in the case that $\Gamma \not\models_\theta A$ (rule $\rightarrow I_2$). The label b is mainly used to block a bottom-up application of rule $\vee E$: in an \uparrow -expansion of $\Gamma \vdash H \uparrow^b$ with H non-prime, we are forced to apply an introduction rule to eliminate the main connective of H (applications of $\vee E$ are forbidden). Hence \uparrow^b -arrows introduce a sort of *focus* on the right formula, according with [8]. If $\Gamma \vdash A \vee B \downarrow$ has been proved, we can \uparrow -expand $\Gamma \vdash D \uparrow^u$ by applying $\vee E$ with major formula $A \vee B$ provided that $\Gamma \not\models_\theta A$ and $\Gamma \not\models_\theta B$. Thus, the application of context extending rules $\vee E$ and $\rightarrow I_2$ is only allowed if the context Γ is enriched by relevant information. Note that the right premise of $\rightarrow E$ is an \uparrow^b -sequent: after having \downarrow -expanded a tree with root $\Gamma \vdash A \rightarrow B \downarrow$, the sequent to \uparrow -expand is $\Gamma \vdash A \uparrow^b$.

The calculus \mathbf{Nbu}_θ is equivalent to \mathbf{NJ} , namely: $\Gamma \vdash H \uparrow$ is provable in \mathbf{NJ} iff $\Gamma \vdash H \uparrow^u$ is provable in \mathbf{Nbu}_θ . This can be shown by exhibiting a simulation between the two calculi. While \mathbf{Nbu}_θ -derivations have a direct translation into \mathbf{NJ} (one has basically to erase the labels), the converse translation requires some non-trivial permutations steps involving applications of $\vee E$ and $\rightarrow I$.

Proof-search in \mathbf{Nbu}_θ

In \mathbf{Nbu}_θ , proof-search starts from an \uparrow^u -sequent. In \uparrow -expansions, the transition from \uparrow^u -sequents to \uparrow^b -sequents is marked by an application of rule $\vee I$. In expanding an \uparrow^b -sequent, the label turns to u only when $\rightarrow I_2$ is applied. As a result of the restrictions on rule applications, the search space for \mathbf{Nbu}_θ is smaller than the one for \mathbf{NJ} . More interestingly, for \mathbf{Nbu}_θ we can define a goal-oriented proof-search procedure, where all rules are applied bottom-up. The main point is to avoid the alternation between \uparrow -expansion and \downarrow -expansions phases. To simulate \downarrow -expansion in a goal-oriented style, we take advantage of the following property: if $\Gamma \vdash H \downarrow$ is provable in \mathbf{Nbu}_θ , then H is a *relevant positive subformula* of the context Γ , namely: $H \in \Gamma$ or H is of the kind $A \wedge H$ (A any formula) or $H \wedge A$ or $A \rightarrow H$.

In proof-search some caution must be taken to avoid loops. In [14], termination is guaranteed by loop-checking: whenever a sequent occurs twice in a branch of the derivation under

construction, the search is cut. In general, implementation of loop-checking is computationally expensive. To control termination, we associate with every sequent a *history set* [7] of prime formulas; such a set stores some of the prime formulas occurring in the right-hand side of the sequents in the branch under construction. In the proof-search procedure history sets are handled according with the following criteria: rules \Downarrow and $\perp E$ are only applied to $\sigma = \Gamma \vdash F \uparrow^l$ ($l \in \{b, u\}$) if F is not in the history set associated with σ . When \Downarrow and $\perp E$ are applied to $\Gamma \vdash F \uparrow^l$, the formula F is added to the related history set. An history set is emptied whenever the backward application of a rule extends the context (this involves rules $\vee E$ and $\rightarrow I_2$). This yields a complete and terminating goal-oriented proof-search procedure of \mathbf{Nbu}_θ . An implementation is available at <http://www.dista.uninsubria.it/~ferram/>.

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Hyperstates on strongly perfect MTL-algebras with cancellative radical

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The *radical* of an MTL-algebra \mathbf{A} ($Rad(\mathbf{A})$) is the intersection of its maximal filters, while the *co-radical* of \mathbf{A} is defined as $coRad(\mathbf{A}) = \{a \in A \mid \neg a \in Rad(\mathbf{A})\}$ (see [2]). Following [11], an MTL-algebra \mathbf{A} is said to be *perfect* if $A = Rad(\mathbf{A}) \cup coRad(\mathbf{A})$. In this contribution we will focus on a (proper sub)class of MTL-algebras that we call *strongly perfect* (SBP_0 -algebras for short). These are perfect MTL-algebras further satisfying

$$coRad(\mathbf{A}) = \neg Rad(\mathbf{A}) = \{\neg a \mid a \in Rad(\mathbf{A})\}.$$

It is worth to notice that every directly indecomposable product algebra and every perfect MV-algebra are SBP_0 -algebras. SBP_0 -algebras generate a variety that we will henceforth denote \mathbb{SBP}_0 .

In [8], the category \mathcal{T}_{CH} of *cancellative hoop-triplets* $(\mathbf{B}, \mathbf{C}, \vee_e)$ ¹ has been proved to be equivalent to the algebraic category \mathbb{P} of product algebras. It is possible to show that \mathcal{T}_{CH} is also equivalent to the category \mathbb{DLMV} generated by perfect MV-algebras [6]. Moreover, the results showing that \mathcal{T}_{CH} is equivalent to \mathbb{DLMV} and \mathbb{P} , can be proved in a uniform way by employing the notion of *dl-admissible operator* δ on a cancellative hoop [2] in order to define, for every triple $(\mathbf{B}, \mathbf{C}, \vee_e)$, the algebra $\mathbf{B} \otimes_e^\delta \mathbf{C}$ which generalises the construction used in [8] and whose elements are pairs $(b, c) \in B \times C$. In particular, if we choose $\delta_L : x \in C \mapsto 1 \in C$ as admissible operator, the structure $\mathbf{B} \otimes_e^{\delta_L} \mathbf{C}$ is a product algebra while, if we choose the identity map $\delta_D : x \in C \mapsto x \in C$, $\mathbf{B} \otimes_e^{\delta_D} \mathbf{C}$ is an algebra in \mathbb{DLMV} . In general (see [?]), for any cancellative hoop-triple $(\mathbf{B}, \mathbf{C}, \vee_e)$ and for any dl-admissible operator δ , $\mathbf{B} \otimes_e^\delta \mathbf{C} \in \mathbb{SBP}_0$.

States of MV-algebras have been introduced by Mundici in [10] as normalized and additive functions mapping every MV-algebra, in the real unit interval $[0, 1]$. Each state of an MV-algebra \mathbf{A} maps its infinitesimals (i.e., the elements of the co-radical) in 0. This fact is particularly evident when \mathbf{A} is perfect. In fact, every perfect MV-algebra has only one state: the function s mapping $Rad(\mathbf{A})$ in 1 and $coRad(\mathbf{A})$ in 0. In order to overcome this limitation, several alternative notions of state have been defined and, in particular, *local* states were introduced in [5]. In this contribution we are going to introduce a notion of hyperstate that, if from one side extends the previous attempts, from the other, exploiting the representation of SBP_0 -algebras in terms of triplets, can be defined on any structure of that kind. Indeed, the construction we will present in the following section is grounded on the fact that Boolean algebras and cancellative hoops (the basic components of a cancellative hoop-triple) already possess a well-established notion of *state*: probability functions on Boolean algebras and *states* of ℓ -groups (see [7]). Moreover, cancellative hoops are nothing but negative cones of ℓ -groups.

¹These are triplets in which \mathbf{B} is a Boolean algebra, \mathbf{C} is a cancellative hoop and $\vee_e : B \times C \rightarrow C$ is a suitably defined map intuitively representing the natural join between the elements of \mathbf{B} and those of \mathbf{C} .

Thus, our construction will define hyperstates for any structure of the form $\mathbf{B} \otimes_e^\delta \mathbf{C}$ and, in particular, for product algebras and any algebra in \mathbb{DLMV} . It is worth to mention that a similar, but slightly less general, approach to hyperstates on DLMV-algebras can be found in [3].

Hyperstates of DLMV-algebras

The following is our inspiring example: consider a perfect MV-algebra \mathbf{A} (i.e. any directly indecomposable DLMV-algebra) with its associated triple $(\mathbf{2}, \text{Rad}(\mathbf{A}), \vee)$. Recall that since \mathbf{A} is perfect, $A = \text{Rad}(\mathbf{A}) \cup \text{coRad}(\mathbf{A})$. $\text{Rad}(\mathbf{A})$ is (the domain of) the greatest cancellative hoop contained in \mathbf{A} , in symbols $\mathcal{H}(\mathbf{A})$. Thus, it coincides with the negative cone of an abelian ℓ -group $\mathbf{G}_{\mathcal{H}(\mathbf{A})} = (G, *_G, {}^{-1}_G, 1_G)$ where $G = (\mathcal{H}(\mathbf{A}) \times \mathcal{H}(\mathbf{A})) / \sim$ (see [6] for details). In particular, every element a of the perfect MV-algebra \mathbf{A} is mapped into $\mathbf{G}_{\mathcal{H}(\mathbf{A})}$ as follows:

$$a \mapsto \begin{cases} (1, a) & \text{if } a \in \text{Rad}(\mathbf{A}) \\ (\neg a, 1) & \text{if } a \in \text{coRad}(\mathbf{A}) \end{cases}$$

Following [7], by a *state* of $\mathbf{G}_{\mathcal{H}(\mathbf{A})}$, we mean a group homomorphism $h : \mathbf{G}_{\mathcal{H}(\mathbf{A})} \rightarrow \mathbf{R}$ (being \mathbf{R} the additive group of reals), such that $h(1_G) = 0$ and h is positive, that is, if $x \geq 1_G$, then $h(x) \geq 0$. Thus, for every state h of $\mathbf{G}_{\mathcal{H}(\mathbf{A})}$ we define a map $\sigma : \mathbf{A} \rightarrow \Gamma(\mathbf{Z} \times_{lex} \mathbf{R}, (1, 0))^2$ as follows: for every $a \in A$,

$$\sigma(a) = \begin{cases} (1, h([1, a])) & \text{if } a \in \text{Rad}(\mathbf{A}) \\ (0, h([\neg a, 1])) & \text{if } a \in \text{coRad}(\mathbf{A}) \end{cases} \quad (1)$$

Notice that, since in every perfect MV-algebra $\text{coRad}(\mathbf{A}) = \neg \text{Rad}(\mathbf{A})$ and $\text{Rad}(\mathbf{A}) = \neg \text{coRad}(\mathbf{A})$, in the equation (1) above, for $a \in \text{coRad}(\mathbf{A})$, $h([\neg a, 1]) = -h([1, \neg a])$. Hence, (1) can be equivalently rewritten as:

$$\sigma(a) = \begin{cases} (1, h([1, a])) & \text{if } a \in \text{Rad}(\mathbf{A}) \\ (0, -h([1, \neg a])) & \text{if } a \in \text{coRad}(\mathbf{A}). \end{cases}$$

Therefore, h can be restricted to the negative cone $\mathbf{G}_{\mathcal{H}(\mathbf{A})}^-$ of $\mathbf{G}_{\mathcal{H}(\mathbf{A})}$ without loss of generality. This remark justifies the following definition.

Definition 1. Let $\mathbf{C} = (C, \cdot, \rightarrow, \wedge, 1)$ be a cancellative hoop. By a *state* of \mathbf{C} we mean a map $h : C \rightarrow \mathbb{R}^-$ such that:

- (1) $h(1) = 0$,
- (2) $h(a \cdot b) = h(a) + h(b)$,

Notice that $h : C \rightarrow \mathbb{R}^-$ is a state of \mathbf{C} iff h is the restriction to $\mathbf{G}_{\mathbf{C}}^-$ of a state of $\mathbf{G}_{\mathbf{C}}$. Therefore, for every cancellative hoop \mathbf{C} , we will henceforth denote by h either a state of \mathbf{C} or a state of $\mathbf{G}_{\mathbf{C}}$ without danger of confusion.

Now, let \mathbf{A} be any algebra in \mathbb{DLMV} (not necessarily perfect), let $\mathcal{B}(\mathbf{A})$ be its Boolean skeleton and $(\mathcal{B}(\mathbf{A}), \mathcal{H}(\mathbf{A}), \vee)$ be its associated triple. Let us consider a pair (p, h) , where $p : \mathcal{B}(\mathbf{A}) \rightarrow [0, 1]$ is a probability measure, $h : \mathcal{H}(\mathbf{A}) \rightarrow \mathbf{R}$ is a state (in the sense of Definition 1) and define $\hat{\sigma} : \mathcal{B}(\mathbf{A}) \otimes_e^{\delta_D} \mathcal{H}(\mathbf{A}) \rightarrow \Gamma(\mathbf{R} \times_{lex} \mathbf{R}, (1, 0))$ as follows: for any pair (b, c)

$$\hat{\sigma}(b, c) = (p(b), h([b \vee_e \delta_D(c), \neg b \vee_e c])). \quad (2)$$

²Whenever (\mathbf{G}_1, u) is a totally ordered ℓ -group with strong unit and \mathbf{G}_2 is an ℓ -group, the *lexicographic* product $\mathbf{G}_1 \times_{lex} \mathbf{G}_2$ is an ℓ -group with strong unit $(u, 0)$. Thus, $\Gamma(\mathbf{G}_1 \times_{lex} \mathbf{G}_2, (u, 0))$ is the MV-algebra obtained from $\mathbf{G}_1 \times_{lex} \mathbf{G}_2$ by applying Mundici's functor Γ [9]

It is not difficult to show that, if \mathbf{A} is a perfect MV-algebra, then $\hat{\sigma}$ defined as in (2) coincides with the map σ defined in (1).

Notice that $\Gamma(\mathbf{R} \times_{lex} \mathbf{R}, (1, 0))$ is isomorphic to a suitable MV-subalgebra of a ultraproduct ${}^*[0, 1]$ of the standard MV-algebra (see [4] for details). In particular, for any nontrivial infinitesimal $\varepsilon \in {}^*[0, 1]$, $\Gamma(\mathbf{R} \times_{lex} \mathbf{R}, (1, 0)) \cong \langle [0, 1] \cup \{\varepsilon\} \rangle_{{}^*[0, 1]}$ (the MV-subalgebra of ${}^*[0, 1]$ generated by $[0, 1] \cup \{\varepsilon\}$). (For the sake of a lighter notation, we will write $\mathcal{L}(\mathbb{R})$ instead of $\langle [0, 1] \cup \{\varepsilon\} \rangle_{{}^*[0, 1]}$). As a consequence, each element (r_1, r_2) of $\Gamma(\mathbf{R} \times_{lex} \mathbf{R}, (1, 0))$ can be uniquely represented in $\mathcal{L}(\mathbb{R})$ as $r_1 + \varepsilon r_2$. Thus, also recalling that δ_D is the identity map on $\mathcal{H}(\mathbf{A})$, we can write

$$\hat{\sigma}(b, c) = p(b) + \varepsilon h([b \vee_e c, \neg b \vee_e c]).$$

Such construction can be clearly extended to any algebra of the form $\mathbf{B} \otimes_e^\delta \mathbf{C}$ where \mathbf{B} is a Boolean algebra, \mathbf{C} is a cancellative hoop, and δ is any dl-admissible operator, putting $\hat{\sigma}(b, c) = p(b) + \varepsilon h([b \vee_e \delta(c), \neg b \vee_e c])$.

Hyperstates on SBP₀-algebras with cancellative radical

We are now going to define hyperstates on those SBP₀-algebra whose radical is cancellative. For their axiomatization we will use to abbreviate $\neg(\neg a \odot \neg b)$ as $a \oplus b$. For every $n \in \mathbb{N}$ and every element $x \in \mathcal{L}(\mathbb{R})$, we will write $n.a$ for $a \oplus \dots \oplus a$ (n -times).

Definition 2. For any SBP₀-algebra \mathbf{A} with cancellative radical, we define a *hyperstate* of \mathbf{A} as a map $s : A \rightarrow \mathcal{L}(\mathbb{R})$ such that

1. $s(1) = 1$,
2. $s(a \oplus b) + s(a \odot b) = s(a) + s(b)$,
3. If $a \vee \neg a = 1$, then there is $n \in \mathbb{N}$, such that $n.s(a) = 1$.

Proposition 3. *The following properties hold for hyperstates of SBP₀-algebras with cancellative radical:*

- (i) $s(\neg x) = 1 - s(x)$, and hence $s(0) = 0$,
- (ii) if $a \leq b$, then $s(a) \leq s(b)$,
- (iii) if $a \odot b = 0$, $s(a \oplus b) = s(a) + s(b)$,
- (iv) if $a \oplus b = 1$, $s(a \odot b) = s(a) \odot s(b)$,
- (v) $s(a \wedge b) + s(a \vee b) = s(a) + s(b)$,
- (vi) *The restriction p of s to $\mathcal{B}(\mathbf{A})$ is a $[0, 1]$ -valued and finitely additive probability measure.*

Our main result shows that the representation of SBP₀-algebras with a cancellative radical in terms of cancellative hoop-triplets, has a counterpart for their hyperstates. Indeed, the following theorem shows that every hyperstate of a SBP₀-algebra with cancellative radical \mathbf{A} coincides, up to identifying \mathbf{A} with $\mathcal{B}(\mathbf{A}) \otimes_e^\delta \mathcal{H}(\mathbf{A})$, with the operator $\hat{\sigma}$ we discussed in the previous section. Thus, a state of \mathbf{A} is represented by a probability measure on its Boolean skeleton and a state on its largest cancellative subhoop.

Theorem 4. *For every SBP₀-algebra \mathbf{A} with cancellative radical and for every hyperstate s of \mathbf{A} , there is a probability measure p on $\mathcal{B}(\mathbf{A})$ and a state h of $\mathcal{H}(\mathbf{A})$ such that, for every $a \in A$,*

$$s(a) = p(b_a) + \varepsilon h([b_a \vee_e \delta(c_a), \neg b_a \vee_e c_a]).$$

Hyperstates on product algebras

As we already noticed, product algebras constitute a relevant subvariety of SBP₀-algebras with cancellative radical. Thus, Definition 2 above provides, in particular, a notion of hyperstate for product algebras.

We finally notice that, if \mathbf{A} is a directly indecomposable product algebra and s is a hyperstate of \mathbf{A} , the image $s(A)$ of \mathbf{A} under s is the domain of a product subalgebra of $\mathcal{L}(\mathbb{R})$. Indeed, since $\text{coRad}(\mathbf{A}) = \{0\}$, Theorem 4 and Proposition 3 show that $s(A)$ is a product subalgebra of $\mathbf{2} \otimes_e^{\delta_L} \mathbb{R}^-$.

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Almost structural completeness and structural completeness of nilpotent minimum logics

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1 Preliminaries

The *nilpotent minimum logic*, NML for short, was firstly introduced by Esteva and Godo in [3] in order to formalize the logic of the nilpotent minimum t-norms. Fodor in [4] defined the nilpotent minimum t-norms as examples of some involutive left continuous t-norms which are not continuous. NML is the logic obtained from the monoidal t-norm logic defined in [3], by adding the involutive condition $\neg\neg\varphi \rightarrow \varphi$ and the nilpotent minimum condition $(\psi * \varphi \rightarrow \perp) \vee (\psi \wedge \varphi \rightarrow \psi * \varphi)$.

A logic is *structurally complete* if each of its proper extensions has new theorems. Admissible rules of a logic are those rules under which the set of theorems are closed. If L is a logic, an L -unifier of a formula φ is a substitution σ such that $\vdash_L \sigma\varphi$. A *single-conclusion rule* is an expression of the form Γ/φ where φ is a formula and Γ is a finite set of formulas. As usual Γ/φ is derivable in L iff $\Gamma \vdash_L \varphi$. The rule Γ/φ is *admissible* in L iff every common L -unifier of Γ is also an L -unifier of φ . Γ/φ is *passive L -admissible* iff Γ has no common L -unifier. So equivalently, a logic is *structurally complete* iff every admissible rule is a derivable rule (See for instance [9]). We say that a logic is *almost structurally complete* iff every admissible rule is either derivable or passive. Our goal is to prove that NML is not structurally complete but almost structurally complete.

It is well known that NML is algebraizable and the class \mathbf{NM} of all nilpotent minimum algebras is its equivalent algebraic quasivariety semantics. A *nilpotent minimum algebra*, *NM-algebra* for short, is a bounded integral commutative residuated lattice $\mathbf{A} = \langle A, *, \rightarrow, \wedge, \vee, \bar{0}, \bar{1} \rangle$ satisfying the conditions (L) $(x \rightarrow y) \vee (y \rightarrow x) \approx \bar{1}$ of prelinearity, (I) $\neg\neg x \approx x$ of involution and (WNM) $(x * y \rightarrow 0) \vee (x \wedge y \rightarrow x * y) \approx \bar{1}$ of weak nilpotent minimum axiom, where \neg is the associated negation (i.e. $\neg x =_{def} x \rightarrow 0$).

We say that a NM-algebra is a NM-chain, provided that it is totally ordered. The canonical standard NM-chain is $[0, 1] = \langle [0, 1], *, \rightarrow, \wedge, \vee, 0, 1 \rangle$ where \wedge and \vee are the meet and join with the usual order and for every $a, b \in [0, 1]$,

$$a * b = \begin{cases} \bar{0}, & \text{if } b \leq 1 - a; \\ \min\{a, b\}, & \text{otherwise.} \end{cases} \quad a \rightarrow b = \begin{cases} \bar{1}, & \text{if } a \leq b; \\ \max\{1 - a, b\}, & \text{otherwise.} \end{cases}$$

Generalizing the behaviour of $[0, 1]$, it is easy to see that given a totally ordered set A with upper bound $\bar{1}$ and lower bound $\bar{0}$ equipped with an involutive negation \neg dually order preserving, if we define \wedge, \vee as meet and join and for every $a, b \in A$,

$$a * b = \begin{cases} \bar{0}, & \text{if } b \leq \neg a; \\ a \wedge b, & \text{otherwise.} \end{cases} \quad a \rightarrow b = \begin{cases} \bar{1}, & \text{if } a \leq b; \\ \neg a \vee b, & \text{otherwise.} \end{cases} ,$$

then $\mathbf{A} = \langle A, *, \rightarrow, \wedge, \vee, \bar{0}, \bar{1} \rangle$ is a NM-chain. Moreover every NM-chain is of this form. Therefore up to isomorphism for each finite $n \in \mathbb{N}$, there is only one NM-chain \mathbf{A}_n with exactly n elements.

For every $n > 1$ we define the canonical finite NM-chains as follows $\mathbf{A}_{2n+1} = \langle [-n, n] \cap \mathbb{Z}, *, \rightarrow, \wedge, \vee, -n, n \rangle$ and $\mathbf{A}_{2n} = \langle A_{2n+1} \setminus \{0\}, *, \rightarrow, \wedge, \vee, -n, n \rangle$. Notice that \mathbf{A}_1 is the trivial algebra, \mathbf{A}_2 the 2-element Boolean algebra and \mathbf{A}_3 the 3-element MV-algebra.

Given an NM-algebra \mathbf{A} , an element $a \in A$ is a *negation fixpoint* (or just *fixpoint*, for short) if, and only if, $\neg a = a$. In [6], Höhle proves that there exists at most one fixpoint. It is easy to see that if \mathbf{C} is a NM-chain and $c \in C$ is a fixpoint then $C \setminus \{c\}$ is the universe of a subalgebra of \mathbf{C} which we denote by \mathbf{C}^- . Notice that $\mathbf{A}_{2n} = \mathbf{A}_{2n+1}^-$.

Since the class of all bounded integral commutative residuated lattices is equational and the three conditions of (L), (I) and (WNM) are identities, then \mathbf{NML} is a variety. All proper subvarieties of \mathbf{NML} , and therefore all proper axiomatic extensions of NML, were described in [5] and they can be characterized by the the fixpoint and the number of elements of their finite chains. In fact every axiomatic extension of NML is strong finite complete with respect the class of finite chains in the associated variety [8].

2 Main Results

Proposition 2.1. *NML is not structurally complete.*

Proof. Structural completeness fails because the rule $\neg p \leftrightarrow p/\perp$ is NML-admissible but not derivable in NML. \square

In [2], Dzik and Wronski proved the structural completeness of the Gödel logic by proving that any finite Gödel chain is embeddable into the free Gödel algebra. Structural completeness of the positive fragment of Gödel logic was similarly proved in [1]. We use the same method to prove the structural completeness of \mathbf{NM}^- , the axiomatic extension of NML by the axiom $\nabla(\varphi) \leftrightarrow \Delta(\varphi)$ where $\nabla(x) = \neg(\neg x^2)^2$ and $\Delta(x) = (\neg(\neg x)^2)^2$.

Proposition 2.2. *For every $n \in \mathbb{N}$, $n > 1$, \mathbf{A}_{2n} is embeddable into $Free_{\mathbf{NM}^-}(\omega)$.*

Proof. Let $p_1, \dots, p_{n-1} \in X$ be distinct variables, we define

$$\begin{aligned} \varphi_1 &= p_1 \vee \neg p_1 \\ \varphi_i &= (((p_i \vee \neg p_i) \rightarrow \varphi_{i-1}) \rightarrow (p_i \vee \neg p_i)) \rightarrow (p_i \vee \neg p_i) \quad i = 2 \div n-1 \\ \varphi_n &= \top \end{aligned}$$

then $f : A_{2n} \rightarrow Free_{\mathbf{NM}^-}(\omega)$ defined by $f(i) = \begin{cases} \overline{\varphi_i}, & \text{if } i > 0; \\ \neg \varphi_i, & \text{if } i < 0. \end{cases}$ gives the embedding. \square

We recall that \mathbf{NM}^- is finite strong complete with respect $\{\mathbf{A}_{2n} : n > 1\}$ and every proper non trivial subvariety of \mathbf{NM}^- is $\mathbf{NM}2n$, the variety generated by \mathbf{A}_{2n} , for some $n > 0$. Then, since $Free_{\mathbf{NM}2n}(n-1) \cong Free_{\mathbf{NM}^-}(n-1) \subseteq Free_{\mathbf{NM}^-}(\omega)$, we have that

Theorem 2.3. *\mathbf{NM}^- is hereditarily structurally complete.*

In order to prove the almost structural completeness of NML we need the following previous result of embeddability into the free \mathbf{NM} -algebra.

Proposition 2.4. *For every $n \in \mathbb{N}$, $n > 1$, $\mathbf{A}_2 \times \mathbf{A}_{2n+1}$ is embeddable into $Free_{\mathbf{NM}}(\omega)$*

Proof. Let $p_0, \dots, p_{n-1} \in X$ be distinct variables, and define $\varphi_0, \dots, \varphi_n$ as in the proof of previous proposition but starting with $\varphi_0 = p_0 \vee \neg p_0$,

then $h : A_2 \times A_{2n+1} \rightarrow Free_{NML}(\omega)$ defined by

$$\begin{array}{ll} h((1, i)) & = \overline{\nabla(\varphi_0) \vee \varphi_i} & h((-1, i)) & = \overline{\neg \nabla(\varphi_0) \wedge \varphi_i} \\ h((1, 0)) & = \overline{\nabla(\varphi_0) \vee \varphi_0} & h((-1, 0)) & = \overline{\neg \nabla(\varphi_0) \wedge \varphi_0} \\ h((1, -i)) & = \overline{\nabla(\varphi_0) \vee \neg \varphi_i} & h((-1, -i)) & = \overline{\neg \nabla(\varphi_0) \wedge \neg \varphi_i} \end{array}$$

for $i = 1 \div n$, gives the desired embedding. \square

Using a characterization of almost structurally complete quasivarieties given in [7, Theorem 4.10] by Metcalfe and Röthlisberger we obtain our last result:

Theorem 2.5. *NML is almost structurally complete and all their axiomatic extensions are almost structurally complete.*

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A new view of effects in a Hilbert space

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Abstract

Any physical theory determines a class of *event-state systems* $\langle E, S \rangle$, where E contains the events that may occur relative to a given system, while S contains the states that such a physical system, described by the theory, may assume. In the case of quantum mechanics, and in particular within the framework of the traditional Hilbert space model [3, p. 52 ff.], it is customary to identify E with the set $\Pi(\mathbf{H})$ of all projection operators of a Hilbert space \mathbf{H} (which is in bijective correspondence with the set of all closed subspaces thereof), and S with the set $\mathcal{S}(\mathbf{H})$ of density operators of \mathbf{H} . It is well-known that $\Pi(\mathbf{H})$ can be made into the universe of an *orthomodular lattice* if we endow it with intersection, as well as with the operations so defined, for all $P, Q \in \Pi(\mathbf{H})$:

- $P' = \mathbb{I} - P$, where \mathbb{I} is the identity operator;
- $P \vee Q$ is the projection onto the closed subspace generated by $P \cup Q$.

Although this model has triggered a conspicuous amount of research into orthomodular lattices (see e.g. [9, 1]), the relevance of such an approach for an algebraic treatment of quantum mechanics has been called into question, as a result of the discovery that the class of orthomodular lattices based on lattices of projection operators does not generate the whole variety of orthomodular lattices [7], whence there are equational properties of event-state systems which are not correctly captured by the proposed mathematical abstraction.

More recently, a different formal counterpart of the set of quantum events has been suggested within the so-called *unsharp approach* to quantum theory [3, Ch. 4 ff.]. Once $\Pi(\mathbf{H})$ is fixed, Gleason's theorem guarantees that $\mathcal{S}(\mathbf{H})$ corresponds to an optimal notion of state: any probability measure defined on $\Pi(\mathbf{H})$ is determined by a member of $\mathcal{S}(\mathbf{H})$, provided the dimension of \mathbf{H} is 3 or higher [6]. On the contrary, $\Pi(\mathbf{H})$ does not represent the largest set of operators that are assigned a probability value. There are, in fact, bounded linear operators E of \mathbf{H} that fail to be projections, yet satisfy the Born rule:

$$\forall \rho \in \mathcal{S}(\mathbf{H}), \operatorname{tr}(\rho E) \in [0, 1].$$

As a consequence, it makes sense to “liberalise” the notion of quantum event so as to include all *effects* of \mathbf{H} — where an effect of \mathbf{H} is precisely a (not necessarily idempotent) bounded linear operator satisfying the Born rule. The set $\mathcal{E}(\mathbf{H})$ of effects of a Hilbert space can be partially ordered by letting, for all $E, F \in \mathcal{E}(\mathbf{H})$,

$$E \leq F \text{ iff } \forall \rho \in \mathcal{S}(\mathbf{H}), \operatorname{tr}(\rho E) \leq \operatorname{tr}(\rho F).$$

Structures having underlying posets whose properties are abstracted away from partially ordered sets of the form $\langle \mathcal{E}(\mathbf{H}), \leq \rangle$ — including *effect algebras* [4], *quantum MV algebras* [5], *Brouwer-Zadeh posets* [2] — have been variously investigated over the years. In all these cases,

the main drawback is that the partial ordering \leq defined above is not, in general, a lattice ordering, whereby we cannot help ourselves to the powerful tools of lattice theory in their algebraic study.

There is, however, a different but just as natural way to define a partial ordering on $\mathcal{E}(\mathbf{H})$. Recall that a (bounded) *spectral family* [10, Ch. 7] on a separable Hilbert space \mathbf{H} is a map $M : \mathbb{R} \rightarrow \Pi(\mathbf{H})$ that satisfies the following conditions:

- $\forall \lambda, \mu \in \mathbb{R}$: if $\lambda \leq \mu$, then $M(\lambda) \leq M(\mu)$ (*monotonicity*);
- $\forall \lambda \in \mathbb{R}$: $M(\lambda) = \bigwedge_{\mu > \lambda} M(\mu)$ (*right-continuity*);
- $\exists \lambda, \mu \in \mathbb{R}$ ($\lambda \leq \mu$) such that $\forall \eta \in \mathbb{R}$: $M(\eta) = \begin{cases} \mathbb{O}, & \text{if } \eta < \lambda; \\ \mathbb{I}, & \text{if } \eta \geq \mu. \end{cases}$

If A is a bounded self-adjoint operator of \mathbf{H} , then there exists a unique spectral family M^A such that $A = \int_{-\infty}^{\infty} \lambda dM^A(\lambda)$, where the integral is meant in the sense of norm-converging Riemann-Stieltjes sums [12, Ch. 1]. Moreover, every spectral family $M : \mathbb{R} \rightarrow \Pi(\mathbf{H})$ determines a *unique* bounded self-adjoint operator on \mathbf{H} according to the previous equality. In particular, if A is a projection operator, then

$$M^A(\lambda) = \begin{cases} \mathbb{O}, & \text{if } \lambda < 0; \\ \mathbb{I} - A, & \text{if } 0 \leq \lambda < 1; \\ \mathbb{I}, & \text{if } \lambda \geq 1. \end{cases}$$

Now define, for $E, F \in \mathcal{E}(\mathbf{H})$,

$$E \leq_s F \text{ iff } \forall \lambda \in \mathbb{R} : M^F(\lambda) \leq M^E(\lambda).$$

Olson [11] and de Groote [8] have essentially shown that $\mathcal{E}(\mathbf{H})$ is the universe of a lattice under the meet and join operations \wedge_s and \vee_s induced by \leq_s , where $E \wedge_s F$ and $E \vee_s F$ are the unique effects associated, respectively, to the spectral families $M_1 \wedge_s M_2$ and $M_1 \vee_s M_2$ such that

- $\forall \lambda \in \mathbb{R} : (M_1 \wedge_s M_2)(\lambda) := \bigwedge_{\mu > \lambda} (M_1(\lambda) \vee M_2(\lambda));$
- $\forall \lambda \in \mathbb{R} : (M_1 \vee_s M_2)(\lambda) := M_1(\lambda) \wedge M_2(\lambda).$

Moreover, the ordering \leq_s coincides with \leq when restricted to the set of projection operators.

In this paper we intend to elucidate the structure of the bounded lattices of the form $\langle \mathcal{E}(\mathbf{H}), \wedge_s, \vee_s, \mathbb{O}, \mathbb{I} \rangle$ endowed with the further unary operations $E' = \mathbb{I} - E$ and $E^\sim = P_{\ker(E)}$ (the projection onto the kernel of E ; observe that $E' = E^\sim$ whenever E is a projection). To this end, we introduce an abstract counterpart of this notion that bears to orthomodular lattices the same relationship that these lattices of effects bear to lattices of projection operators. These algebras will be given, for reasons that will become clear in what follows, the name of *PBZ*-lattices* (paraorthomodular Brouwer-Zadeh lattices with the star condition). In light of the above, PBZ*-lattices can be seen as generalisations of orthomodular lattices where the usual orthomodular complement splits into two distinct operations, playing different roles in the more general context of lattices of effects.

The main goal of this paper, besides observing that the above-mentioned lattices of effects make instances of PBZ*-lattices, is to scratch the surface of the structure theory of this class

of algebras by resorting to the toolbox of universal algebra. The starting point is to observe that the condition to the effect that for all $a, b \in L$, if $a \leq b$ and $a' \wedge b = 0$, then $a = b$, called *paraorthomodularity*, which is equivalent to orthomodularity in the context of ortholattices, is a strictly weaker condition in the more general setting of bounded involution lattices. Indeed, $\langle \mathcal{E}(\mathbf{H}), \wedge_s, \vee_s, ', \mathbb{O}, \mathbb{I} \rangle$ is, in general, a non-orthomodular but paraorthomodular Kleene lattice. We show that the class of paraorthomodular Kleene lattices is a proper quasivariety which generates the variety of Kleene lattices, and investigate a deductive system associated to this quasivariety. Successively, we expand the language of paraorthomodular Kleene lattices by an intuitionistic-like complement \sim , with an eye to modelling the behaviour of the operation $E\sim$ mentioned above. As a result of this move, paraorthomodular Kleene lattices are turned into instances of *Brouwer-Zadeh lattices* (BZ-lattices). A BZ-lattice \mathbf{L} is said to be \diamond -orthomodular in case for all $a, b \in L$,

$$(\diamond a \rightarrow \diamond b) \wedge \diamond a \leq \diamond b,$$

where \rightarrow is the Sasaki hook and $\diamond a = a\sim\sim$. We prove that all paraorthomodular BZ-lattices are \diamond -orthomodular, but not the other way around. Then, we consider paraorthomodular BZ-lattices \mathbf{L} satisfying the following *star condition* for all $a \in L$:

$$(*) \quad (a \wedge a')\sim \leq a\sim \vee a'\sim.$$

These *PBZ*-lattices* are remarkable under two respects: first, three distinct notions of “sharpness” that come apart in general BZ-lattices turn out to coincide with one another in the present context — and every PBZ*-lattice has an orthomodular sublattice of sharp elements in this sense. Second, under (*), the quasiequational paraorthomodularity condition is equivalent to \diamond -orthomodularity, whence PBZ*-lattices form a variety. The main non-orthomodular examples of PBZ*-lattices are the lattices of effects

$$\langle \mathcal{E}(\mathbf{H}), \wedge_s, \vee_s, ', \sim, \mathbb{O}, \mathbb{I} \rangle,$$

whose sharp elements are exactly the projections. More precisely, every PBZ*-lattice of effects has an orthomodular subalgebra of projection operators. We show that every bounded lattice can be embedded as a sublattice into a PBZ*-lattice, whereby the variety of PBZ*-lattices satisfies no nontrivial lattice equation. Finally, we begin the investigation of the lattice of PBZ* varieties. Upon defining an *antiorthomodular lattice* as a PBZ*-lattice whose unique sharp elements are 0 and 1, we axiomatise the variety generated by antiorthomodular lattices and show that the single atom in the lattice of PBZ* varieties, corresponding to Boolean algebras, has a unique non-orthomodular cover, generated by a certain antiorthomodular lattice over the 3-element Kleene chain.

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Poset Product and BL-chains

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Basic Fuzzy Logic (BL for short) was introduced by Hájek in [Há98b] to formalize fuzzy logics in which the conjunction is interpreted by a continuous t-norm on the real segment $[0, 1]$ and the implication by its corresponding adjoint. BL-algebras are the algebraic counterpart of BL. These algebras form the variety \mathcal{BL} , which has many well known subvarieties, including the variety \mathcal{MV} of MV-algebras (the algebraic semantics for Łukasiewicz's logic), the variety of Product algebras and the variety of Gödel algebras.

One of the most important properties of \mathcal{BL} is that it is generated by its totally ordered members, called BL-chains. This reason explains why the first attempts to investigate \mathcal{BL} focused on the structure of BL-chains. For instance, in [CEGT00] the authors characterized saturated BL-chains as ordinal sums of MV-chains, product chains and Gödel chains, resembling the famous Mostert-Schild decomposition for t-norms. Although this characterization cannot be extended to every BL-chain, it gives information on the structure of BL-chains in general, since each BL-chain can be isomorphically embedded in a saturated chain.

Clearly, to have a representation theorem in terms of simpler or better known structures helps to understand BL-algebras.

\mathcal{BL} can also be seen as the subvariety of bounded basic hoops \mathcal{BH} , i.e. bounded hoops that are isomorphic to subdirect products of totally ordered hoops. Thus, the general algebraic theory of hoops applies to BL-algebras as well. As Aglianò and Montagna noted in [AM03], the fundamental structures in the study of BL-algebras are Wajsberg hoops, whose structure are simpler than BL. They proved that each BL-chain can be uniquely decomposed into an ordinal sum of totally ordered Wajsberg hoops. It is worth to say that the definition for ordinal sum in [AM03] differs from the one given in [CEGT00].

However, none of the decompositions mentioned until here can be generalized to non totally ordered BL-algebras.

The poset product is a construction introduced by Jipsen and Montagna in [JM09]. In a sense, the poset product generalizes the notions of ordinal sum and direct product. Given a poset $P = \langle P, \leq \rangle$ and a collection $\{\mathbf{L}_p : p \in P\}$ of commutative residuated lattices sharing the same neutral element 1 and the same minimum element 0, the poset product $\bigotimes_{p \in P} \mathbf{L}_p$ is the lattice \mathbf{L} defined as follows:

1. The domain of \mathbf{L} is the set of all x belonging to $\prod_{p \in P} L_p$ such that for all $i \in P$, if $x_i \neq 1$, then $x_j = 0$ provided that $j > i$.
2. The monoid operation and the lattice operations are defined pointwise.
3. The residual is

$$(x \rightarrow_{\mathbf{L}} y)_i = \begin{cases} x_i \rightarrow_{\mathbf{L}_i} y_i & \text{if } x_j \leq y_j \text{ for all } j < i; \\ 0 & \text{otherwise.} \end{cases}$$

Based on the results of [JM06], [JM09] and [JM10], one can find in [BM11, Theorem 3.5.4] a proof that every BL-algebra is a subdirect poset product of MV-chains and product chains indexed by a poset P which is a forest. Therefore each BL-algebra is a subalgebra of a poset

product of MV-chains and product chains. Unfortunately, this embedding is not surjective in general, even when dealing with chains.

Our work is framed in the study of BL-algebras that admit a representation as poset product. In this communication we will present a necessary and sufficient condition to establish when an ordinal sum of BL-chains coincides with the poset product of the same collection. From the study of BL-algebras that are not representable in this sense will arise the significant role that the index poset plays in the poset product construction. For instance, we will see that those BL-chains that are isomorphic to a poset product of MV-chains and product chains form a proper subset of the set of saturated BL-chains.

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Structural Multi-type Sequent Calculus for Modal Intuitionistic Dependence Logic

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In this paper, we introduce a multi-type sequent calculus for modal intuitionistic dependence logic that is sound, complete and enjoys Belnap-style cut-elimination and subformula property.

Dependence logic, introduced by Väänänen [12], is a logical formalism that captures the ubiquitous notion of *dependence* in social and natural sciences. The modal version of the logic, *modal dependence logic*, defined in [13], extends the usual modal logic by adding a new type of atoms $\text{=}(p_1, \dots, p_n, q)$, called *dependence atoms*, to express dependencies between propositions, and by lifting the usual single-possible world semantics to the so-called *team semantics*, introduced by Hodges [7, 8]. Formulas of modal dependence logic are evaluated on sets of possible worlds of Kripke models, called *teams*. Intuitively, a dependence atom $\text{=}(p_1, \dots, p_n, q)$ is true if within a team the truth value of the proposition q is *functionally determined* by the truth values of the propositions p_1, \dots, p_n .

Although the research around modal dependence logic and its variants concerning their model theoretic properties, expressive power, computational complexity and other topics has been active in recent years (see e.g., [11, 6, 10] among many other research articles), no previous proposal for sequent calculi for modal dependence logics exists. In this paper, we take a first step in this direction by providing a multi-type sequent calculus for modal dependence logic with intuitionistic connectives. This variant of modal dependence logic is known in the literature as *modal intuitionistic dependence logic (MID)* and it was axiomatized recently in [15]. The logic **MID** is a type of intermediate modal logic that is not closed under uniform substitution. To tackle the hurdle of the non-schematicity of the Hilbert-style presentation of **MID** we design the calculus for **MID** in the style of a generalization of Belnap's display calculi, the so-called *multi-type sequent calculi* in the sense of [3]. Our calculus for **MID** is a natural extension of the multi-type calculus defined recently in [4] for *inquisitive logic* [1], which is essentially the propositional fragment of **MID**.

1 Multi-type modal intuitionistic dependence logic

For the purpose of the multi-type calculus to be introduced in this paper we define the language of modal intuitionistic dependence logic (**MID**) as a multi-type language, whose formulas are given in two types, the Flat type and the General type, defined inductively as follows:

$$\begin{aligned} \text{Flat } \exists \alpha &::= p \mid 0 \mid \alpha \sqcap \alpha \mid \alpha \rightarrow \alpha \mid \diamond \alpha \mid \Box \alpha \\ \text{General } \exists A &::= \downarrow \alpha \mid \text{=}(\downarrow p_1, \dots, \downarrow p_n, \downarrow q) \mid A \wedge A \mid A \vee A \mid A \rightarrow A \mid \diamond A \mid \Box A \end{aligned}$$

We write $\sim \alpha$ for $\alpha \rightarrow 0$ and $\neg A$ for $A \rightarrow \downarrow 0$. The above-defined multi-type modal intuitionistic dependence logic is essentially equivalent to the standard (single-type) modal intuitionistic dependence logic known in the literature. The reader is referred to [4] for further discussion on the multi-type environment for dependence logics.

Formulas of Flat type are the usual modal formulas that are evaluated on single worlds w of the usual Kripke models $\mathfrak{M} = (W, R, V)$, and the satisfaction relation $\mathfrak{M}, w \models \alpha$ for flat formulas α is defined as in the usual model logic. Formulas of General type are evaluated on *teams*. A team is a set $T \subseteq W$ of possible worlds of a Kripke model $\mathfrak{M} = (W, R, V)$. The satisfaction relation $\mathfrak{M}, T \models A$ for general formulas A is defined inductively as follows:

- $\mathfrak{M}, T \models \downarrow \alpha$ iff $\mathfrak{M}, w \models \alpha$ for all $w \in T$
- $\mathfrak{M}, T \models =(\downarrow p_1, \dots, \downarrow p_n, q)$ iff for any $w, u \in T$, $\mathfrak{M}, w \models \downarrow p_i \Leftrightarrow \mathfrak{M}, u \models \downarrow p_i$ for all $1 \leq i \leq n$ implies $\mathfrak{M}, w \models q \Leftrightarrow \mathfrak{M}, u \models q$
- $\mathfrak{M}, T \models A \wedge B$ iff $\mathfrak{M}, T \models A$ and $\mathfrak{M}, T \models B$
- $\mathfrak{M}, T \models A \vee B$ iff $\mathfrak{M}, T \models A$ or $\mathfrak{M}, T \models B$
- $\mathfrak{M}, T \models A \rightarrow B$ iff for all $T' \subseteq T$, $\mathfrak{M}, T' \models A$ implies $\mathfrak{M}, T' \models B$
- $\mathfrak{M}, T \models \Box A$ iff $\mathfrak{M}, R(T) \models A$, where $R(T) = \{w \in W \mid \exists v \in T \text{ s.t. } vRw\}$
- $\mathfrak{M}, T \models \Diamond A$ iff there exists $T' \subseteq W$ such that $T' \subseteq R(T)$, $R(w) \cap T' \neq \emptyset$ for every $w \in T$, and $\mathfrak{M}, T' \models A$

Formulas of General type satisfy the *downward closure property*, that is, $\mathfrak{M}, T \models A$ and $T' \subseteq T$ imply $\mathfrak{M}, T' \models A$, and they also admit the *disjunction property*, that is, $\models A \vee B$ implies $\models A$ or $\models B$.

The single-type intuitionistic modal dependence logic, as axiomatized in [15], can be viewed as a logic obtained by adding to Fischer Servi's intuitionistic modal logic [2] the axiom $(\neg p \rightarrow q \vee r) \rightarrow (\neg p \rightarrow q) \vee (\neg p \rightarrow r)$ of Kreisel-Putnam logic [9], the double negation law $\neg\neg p \rightarrow p$ only for propositional variables and other axioms characterizing the team semantics, and, as such, the logic is *not* closed under uniform substitution. The reader is referred to [14] for further discussion. Below we present the multi-type presentation of the Hilbert system given in [15] for our multi-type **MID**.

- Axioms:**
1. all axioms of the normal modal logic **K** for flat formulas:
 - (1) all axioms of **CPC**
 - (2) $\Box(\alpha \rightarrow \beta) \rightarrow (\Box\alpha \rightarrow \Box\beta)$
 - (3) $\Diamond\alpha \leftrightarrow \sim\Box\sim\alpha$
 2. $\neg\neg\downarrow\alpha \rightarrow \downarrow\alpha$
 3. $(\downarrow\alpha \rightarrow A \vee B) \rightarrow (\downarrow\alpha \rightarrow A) \vee (\downarrow\alpha \rightarrow B)$
 4. $=(\downarrow p_1, \dots, \downarrow p_k, \downarrow q) \leftrightarrow ((\downarrow p_1 \vee \neg\downarrow p_1) \wedge \dots \wedge (\downarrow p_k \vee \neg\downarrow p_k) \rightarrow (\downarrow q \vee \neg\downarrow q))$
 5. all axioms of Fischer Servi's intuitionistic modal logic for general formulas:
 - (1) all axioms of **IPC**
 - (2) $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$
 - (3) $\Diamond(A \rightarrow B) \rightarrow (\Box A \rightarrow \Diamond B)$
 - (4) $\neg\Diamond\downarrow 0$
 - (5) $\Diamond(A \vee B) \rightarrow (\Diamond A \vee \Diamond B)$
 - (6) $(\Diamond A \rightarrow \Box B) \rightarrow \Box(A \rightarrow B)$
 6. $\Box(A \vee B) \rightarrow (\Box A \vee \Box B)$
 7. $\neg\Diamond\neg\downarrow\alpha \rightarrow \Box\downarrow\alpha$

- Rules:**
1. Modus Ponens: $A, A \rightarrow B / B$
 2. Necessitation: $A / \Box B$

In view of axiom 4 dependence atoms can actually be eliminated. In the calculus of **MID** to be given, we will regard dependence atoms $=(\downarrow p_1, \dots, \downarrow p_k, \downarrow q)$ only as abbreviations for their defining formulas. The propositional fragment of **MID** without dependence atoms turns out to be exactly the *inquisitive logic* [1]. A complete and cut-free multi-type calculus for inquisitive logic has recently been introduced in [4]. In the next section we will extend this calculus of inquisitive logic to a calculus of **MID**.

2 Structural Multi-type Sequent Calculus

Our multi-type modal intuitionistic dependence logic can be viewed as an extension of the multi-type inquisitive logic defined in [4] with dependence atoms, box and diamond modalities (for both types). The new flat-type and the general-type modalities are normal, so they have a natural adjoint. All the rules governing their behaviour are standard and they verify all the conditions of a proper display calculus given in [3]. Therefore, adding these (sound) rules to the structural multi-type calculus for inquisitive logic preserves cut-elimination. To prove the completeness of our calculus it suffices to derive all the axioms of the Hilbert system of **MID**. All the axioms of Fischer Servi's intuitionistic modal logic can be

derived in the standard way. In particular, the derivations of the peculiar Fisher Servi's axioms 5.3 and 5.6 makes use of the rule Gen adj and the rules FS (on the right of the turnstile) for structural modalities of General type. The axioms 6 and 7 characterizing the team semantics can be derived using the rules dis and, respectively, the rules swap_L, FS_R for modalities of Flat type, Flat adj plus CG_R, d dis and d adj (see [4] for the last three rules).

Let us now define the language and the interpretation of the structural connectives of our calculus.

- Structural and operational languages of type Flat and General:

Flat	General
$\alpha ::= p \mid 0 \mid \alpha \sqcap \alpha \mid \alpha \rightarrow \alpha \mid \diamond \alpha \mid \Box \alpha$	$A ::= \downarrow \alpha \mid A \wedge A \mid A \vee A \mid A \rightarrow A \mid \diamond A \mid \Box A$
$\Gamma ::= \alpha \mid \Phi \mid \Gamma, \Gamma \mid \Gamma \sqsupset \Gamma \mid FX \mid \odot \Gamma \mid \bullet \Gamma$	$X ::= A \mid \Downarrow \Gamma \mid F^* \Gamma \mid X; X \mid X > X \mid \circ X \mid \bullet X$

- Interpretation of structural Flat connectives as their operational (i.e. logical) counterparts:¹

Structural symbols	Φ	,	\sqsupset	\odot	\bullet
Operational symbols	(1) 0	\sqcap (\sqcup)	(\mapsto) \rightarrow	\diamond \Box	(\blacklozenge) (\blacksquare)

- Interpretation of structural General connectives as their operational counterparts:

Structural symbols	;	$>$	\circ	\bullet
Operational symbols	\wedge \vee	(\triangleright) \rightarrow	\diamond \Box	(\blacklozenge) (\blacksquare)

- Interpretation of multi-type connectives:

Structural symbols	F^*	F	\Downarrow
Operational symbols	(f^*)	(f) (f)	\downarrow \downarrow

Our calculus contains all the rules in [4] that involve propositional connectives, together with the additional rules governing the behaviour of the modalities listed below.

- Structural rules common to both types: Let $\otimes \in \{\odot, \bullet\}$ and $*$ $\in \{\circ, \bullet\}$.

Flat adj $\frac{\odot \Gamma \vdash \Delta}{\Gamma \vdash \bullet \Delta}$	$\frac{\Gamma \vdash \odot \Delta}{\bullet \Gamma \vdash \Delta}$ Flat adj	Gen adj $\frac{\circ X \vdash Y}{X \vdash \bullet Y}$	$\frac{X \vdash \circ Y}{\bullet X \vdash Y}$ Gen adj
nec $\frac{\Phi \vdash \Gamma}{\otimes \Phi \vdash \Gamma}$	$\frac{\Gamma \vdash \Phi}{\Gamma \vdash \otimes \Phi}$ nec	nec $\frac{I \vdash X}{*I \vdash X}$	$\frac{X \vdash I}{X \vdash *I}$ nec
FS $\frac{\otimes \Gamma \sqsupset \otimes \Delta \vdash \Sigma}{\otimes (\Gamma \sqsupset \Delta) \vdash \Sigma}$	$\frac{Y \vdash \otimes X \sqsupset \otimes Z}{Y \vdash \otimes (X \sqsupset Z)}$ FS	FS $\frac{*Y > *Z \vdash X}{*(Y > Z) \vdash X}$	$\frac{Y \vdash *X > *Z}{Y \vdash *(X > Z)}$ FS
mon $\frac{\otimes \Gamma, \otimes \Delta \vdash \Sigma}{\otimes (\Gamma, \Delta) \vdash \Sigma}$	$\frac{\Gamma \vdash \otimes \Delta, \otimes \Sigma}{\Gamma \vdash \otimes (\Delta, \Sigma)}$ mon	mon $\frac{*X; *Y \vdash Z}{*(X; Y) \vdash Z}$	$\frac{Z \vdash *Y; *X}{Z \vdash *(Y; X)}$ mon

¹We follow the notational conventions introduced in [5]: Each structural connective in the upper row of the synoptic tables is interpreted as the logical connective(s) in the two slots below it in the lower row. In particular, each of its occurrences in antecedent (resp. succedent) position is interpreted as the logical connective in the left-hand (resp. right-hand) slot.

- Structural rules specific to the Gen type:

$$\frac{X \vdash \bullet(Y;Z)}{X \vdash \bullet Y; \bullet Z} \text{ dis}$$

- Structural rules governing the interaction between \Downarrow and the new modalities:

$$\text{swap} \frac{* \Downarrow \Gamma \vdash X}{\Downarrow \otimes \Gamma \vdash X} \quad \frac{X \vdash * \Downarrow \Gamma}{X \vdash \Downarrow \otimes \Gamma} \text{ swap}$$

- Introduction rules common to both types:

$$\frac{\odot \alpha \vdash \Gamma}{\diamond \alpha \vdash \Gamma} \quad \frac{\Gamma \vdash \alpha}{\odot \Gamma \vdash \diamond \alpha} \quad \frac{\circ A \vdash X}{\diamond A \vdash X} \quad \frac{X \vdash A}{\circ X \vdash \diamond A}$$

$$\frac{\alpha \vdash \Gamma}{\square \alpha \vdash \odot \Gamma} \quad \frac{\Gamma \vdash \odot \alpha}{\Gamma \vdash \square \alpha} \quad \frac{A \vdash X}{\square A \vdash \circ X} \quad \frac{X \vdash \circ A}{X \vdash \square A}$$

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Finitely Protoalgebraic and Finitely Weakly Algebraizable Logics

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The operational approach in abstract algebraic logic was initiated by Blok and Pigozzi in [1]. In this context, an operator is a function that assigns a binary relation of the formula algebra to any theory of a logic \mathcal{L} . The aim is to classify logics according to the properties that the operator has on the lattice of \mathcal{L} -theories, denoted $\text{Th}\mathcal{L}$, for a particular logic \mathcal{L} . The most important classification of logics is the one into the Leibniz hierarchy according to behaviour of the Leibniz operator Ω . The Leibniz operator assigns to any \mathcal{L} -theory T the Leibniz congruence ΩT , which is an indiscernibility relation relative to the theory T defined as follows:

$$\langle \alpha, \beta \rangle \in \Omega T \text{ if for all formulas } \varphi \text{ and all variables } x \text{ we have } \varphi(x/\alpha) \in T \Leftrightarrow \varphi(x/\beta) \in T.$$

In this talk we present new characterizations for two classes of logics via the Leibniz operator.

Protoalgebraic logics were introduced by Blok and Pigozzi in [1]. A logic \mathcal{L} is protoalgebraic if Ω is monotone on the lattice of all \mathcal{L} -theories. Protoalgebraic logics can also be characterized as the logics having a parameterized equivalence. Given a set $\Delta(x, y, \bar{z})$ of formulas with main variables x and y and parameters \bar{z} , we define for any formulas φ and ψ the set

$$\Delta(\langle \varphi, \psi \rangle) = \{\delta(\varphi, \psi, \bar{\chi}) : \delta(x, y, \bar{z}) \in \Delta(x, y, \bar{z}), \bar{\chi} \in \text{Fm}\}.$$

A set $\Delta(x, y, \bar{z})$ is a parameterized equivalence for a logic \mathcal{L} , if the following three conditions hold:

- (i) $\vdash_{\mathcal{L}} \Delta(\langle x, x \rangle)$;
- (ii) $x, \Delta(\langle x, y \rangle) \vdash_{\mathcal{L}} y$;
- (iii) $\Delta(\langle x_1, y_1 \rangle), \dots, \Delta(\langle x_n, y_n \rangle) \vdash_{\mathcal{L}} \Delta(\langle \lambda x_1 \dots x_n, \lambda y_1 \dots y_n \rangle)$ for any n -ary connective λ .

A logic \mathcal{L} is equivalential if it has a parameter-free equivalence and finitely equivalential if it has a finite parameter-free equivalence. Analogously, we say that a logic is finitely protoalgebraic if it has a finite parameterized equivalence. It is natural to ask whether we can give alternative characterizations for the class of finitely protoalgebraic logics.

Following Czelakowski and Jansana [3], we say that a protoalgebraic logic \mathcal{L} is weakly algebraizable if Ω is injective on the lattice of all \mathcal{L} -theories. Analogously, a finitely protoalgebraic logic \mathcal{L} is finitely weakly algebraizable if Ω is injective on $\text{Th}\mathcal{L}$.

Given a set X of variables, we say that an \mathcal{L} -theory T is X -invariant if T is closed under all substitutions σ such that $\sigma x = x$ for all $x \in X$. We denote the set of all X -invariant \mathcal{L} -theories by $\text{Th}_{\text{inv}}^X \mathcal{L}$. It is easy to see that $\text{Th}_{\text{inv}}^X \mathcal{L}$ is a complete sublattice of the lattice of all \mathcal{L} -theories for any set X of variables. We are here particularly interested in the lattice of $\{x, y\}$ -invariant theories. We denote this lattice here by $\text{Th}_{\text{inv}}^{xy} \mathcal{L}$.

As a first step in obtaining the results, we note that in order to show that a logic \mathcal{L} is protoalgebraic it is enough to consider only the $\{x, y\}$ -invariant \mathcal{L} -theories, as the following lemma shows.

Lemma 1. *Let \mathcal{L} be a logic. Then the following are equivalent:*

- (i) Ω is monotone on $\text{Th}\mathcal{L}$.
- (ii) Ω is monotone on $\text{Th}_{\text{inv}}^{xy}\mathcal{L}$.

The characterization we obtain for finitely protoalgebraic logics is analogous to the well-known characterization for finitely equivalential logics [2, 5]: a logic \mathcal{L} is finitely equivalential if and only if Ω is continuous on $\text{Th}\mathcal{L}$. Recall that we say that Ω is continuous on $\text{Th}\mathcal{L}$ if for any directed family $\{T_i : i \in I\}$ of \mathcal{L} -theories such that $\bigcup_I T_i$ is an \mathcal{L} -theory we have that

$$\bigcup_I \Omega T_i = \Omega \bigcup_I T_i.$$

This notion of continuity generalizes straightforwardly to any complete sublattice of $\text{Th}\mathcal{L}$. Now we can state our characterization for finitely protoalgebraic logics.

Theorem 1. *Let \mathcal{L} be a logic. Then the following are equivalent:*

- (i) \mathcal{L} is finitely protoalgebraic.
- (ii) Ω is continuous on $\text{Th}_{\text{inv}}^{xy}\mathcal{L}$.

Note that, unlike other known characterizations for the classes in the Leibniz hierarchy, this characterization does not lift to the lattice of \mathcal{L} -filters of an arbitrary algebra.

Now we know that a logic is finitely weakly algebraizable if Ω is continuous on $\text{Th}_{\text{inv}}^{xy}\mathcal{L}$ and injective on $\text{Th}\mathcal{L}$. We can however obtain also a cleaner characterization, as the following lemma shows.

Lemma 2. *Let \mathcal{L} be a protoalgebraic logic. Then the following are equivalent:*

- (i) Ω is injective on $\text{Th}\mathcal{L}$.
- (ii) Ω is injective on $\text{Th}_{\text{inv}}^{xy}\mathcal{L}$.

And thus we obtain the following corollary.

Corollary 1. *Let \mathcal{L} be a logic. Then the following are equivalent:*

- (i) \mathcal{L} is finitely weakly algebraizable.
- (ii) Ω is continuous and injective on $\text{Th}_{\text{inv}}^{xy}\mathcal{L}$.

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The strong version of a sentential logic

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In many settings, we find pairs of logics (taken as consequence relations) strongly related one to the other. For example, in modal logic we have for a given class of frames the local consequence and the global consequence relations determined by it, being the latter an extension of the former and both with the same consequences from the empty set. A similar situation is encountered for any logic with a relational semantics; it determines a local as well as a global consequence relation. Also in the area of many-valued logics, we find for every variety of integral residuated lattices the usual assertional logic and the logic preserving degrees of truth.

In [5] we developed in the setting of protoalgebraic logics¹ a theory to account for the phenomena of the pairs of logics just mentioned. Given a protoalgebraic logic, we managed to define in [5] what we called its strong version. According to the theory, it turns that the global consequence relation of a class of frames is the strong version of its local consequence relation. But we can not encompass under the theory, for example, the relation between the logic of degrees of truth of the variety of MV-algebras and its extension the usual infinite-valued Łukasiewicz logic because the first one is not protoalgebraic [4].

Recently we extended the theory by defining the notion of the strong version of an arbitrary logic. When applied to a protoalgebraic logic we obtain the results already obtained in [5]. We will present the definition of the strong version, the main results about it and will apply it to several examples of non protoalgebraic logics.

One of the main ingredients of the theory we develop is the concept of Leibniz filter introduced in [2]. It is a generalization of the concept with the same name defined in [5], where it was used to obtain the concept of the strong version of a protoalgebraic logic.

Let \mathcal{S} be a logic, i.e. an algebra of formulas \mathbf{Fm} of a given propositional language $\mathcal{L}_{\mathcal{S}}$ together with a single conclusion consequence relation $\vdash_{\mathcal{S}} \subseteq \mathcal{P}(\mathbf{Fm}) \times \mathbf{Fm}$ that is substitution invariant, i.e. for every homomorphism $\sigma : \mathbf{Fm} \rightarrow \mathbf{Fm}$ if $\Gamma \vdash_{\mathcal{S}} \varphi$, then $\sigma[\Gamma] \vdash_{\mathcal{S}} \sigma(\varphi)$. Let \mathbf{A} be an $\mathcal{L}_{\mathcal{S}}$ -algebra. A set $F \subseteq A$ is an \mathcal{S} -filter of \mathbf{A} if it is closed under the interpretation on \mathbf{A} of the rules of \mathcal{S} . This means that if $\Gamma \vdash_{\mathcal{S}} \varphi$ and $h : \mathbf{Fm}_{\mathcal{L}} \rightarrow \mathbf{A}$ is a homomorphism such that $h[\Gamma] \subseteq F$, then $h(\varphi) \in F$. We denote the lattice of \mathcal{S} -filters of \mathbf{A} by $\mathcal{F}i_{\mathcal{S}}\mathbf{A}$. A congruence θ of \mathbf{A} is compatible with a set $X \subseteq A$ if X is a union of equivalence classes of θ . Given $X \subseteq A$, the set of congruences of \mathbf{A} compatible with X has a largest element which is called the *Leibniz congruence* of X and is denoted by $\Omega^{\mathbf{A}}(X)$. An \mathcal{S} -filter F of \mathbf{A} is a *Leibniz \mathcal{S} -filter* of \mathbf{A} if it is the least element of the set $\{G \in \mathcal{F}i_{\mathcal{S}}\mathbf{A} : \Omega^{\mathbf{A}}(F) \subseteq \Omega^{\mathbf{A}}(G)\}$. It exists because for every family $\{X_i : i \in I\}$ of subsets of A , $\bigcap_{i \in I} \Omega^{\mathbf{A}}(X_i) \subseteq \Omega^{\mathbf{A}}(\bigcap_{i \in I} X_i)$. We denote the set of the Leibniz \mathcal{S} -filters of \mathbf{A} by $\mathcal{F}i_{\mathcal{S}}^*\mathbf{A}$.

The *strong version* of a logic \mathcal{S} , denoted by \mathcal{S}^+ , can be defined in several equivalent ways. One is as the logic given by the class of the matrices $\langle \mathbf{A}, F \rangle$ where F is a Leibniz \mathcal{S} -filter of \mathbf{A} . Another, as the logic given by the class of the matrices $\langle \mathbf{A}, F \rangle$ where F is the least \mathcal{S} -filter of \mathbf{A} (i.e. $F = \bigcap \mathcal{F}i_{\mathcal{S}}\mathbf{A}$), namely the logic whose consequence relation is defined by

$$\Gamma \vdash_{\mathcal{S}^+} \varphi \text{ iff } \forall \mathbf{A} \forall h \in \text{Hom}(\mathbf{Fm}_{\mathcal{L}}, \mathbf{A})(h[\Gamma] \subseteq \bigcap \mathcal{F}i_{\mathcal{S}}\mathbf{A} \Rightarrow h(\varphi) \in \bigcap \mathcal{F}i_{\mathcal{S}}\mathbf{A})$$

¹Protoalgebraic logics are the logics with a set of formulas $\Delta(x, y)$, in two variables, that satisfies modus ponens (from x and $\Delta(x, y)$ follows y) and for which $\Delta(x, x)$ is a set of theorems.

for every $\Gamma \subseteq Fm$ and every $\varphi \in Fm$. It follows from the definition that the strong version of \mathcal{S} is an extension of \mathcal{S} with the same theorems.

In the talk, we will present the main results of the theory of the strong version of an arbitrary logic and will apply them to finding the strong versions of the following logics: positive modal logic, Belnap-Dunn four-valued logic, the logic preserving degree of truth of the variety of commutative and integral residuated lattices, as well as of some of their extensions like Łukasiewicz's infinite valued logic preserving degrees of truth.

The main issues that we will address for these logics are the characterization of their Leibniz filters, ways of defining them, whether on every algebra the Leibniz filters of the logic are exactly the logical filters of the strong version, and whether the logic and its strong version have the same algebraic counterpart.

When \mathcal{S} does not have theorems, its strong version is the quasi-inconsistent logic (the logic with only two theories, the empty set and the set of all formulas). This fact shows that the theory of the strong version of a logic is useful only for logics with theorems. Nevertheless, if \mathcal{S} does not have theorems but we are interested in an extension \mathcal{S}' with theorems of \mathcal{S} we can always ask the following question. Is \mathcal{S}' the strong version of the least extension of \mathcal{S} with the theorems of \mathcal{S}' ? We will address this issue for the algebraizable logic $\mathcal{F}\mathcal{L}_e$ associated with the variety of commutative residuated lattices as well as for some of its extensions, like some systems related to relevance logic.

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Syntax Meets Semantics to Prove Interpolation*

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The blending of syntactic and semantic methods through correspondence theory has provided both new generic ways of creating analytic cut-free calculi and a novel modular constructive method for proving interpolation properties in modal logics. We show that the two methods can be combined to the mutual benefit of both methods by demonstrating that the Craig and Lyndon interpolation properties (IPs) can be proved based on a wide variety of sequent-like calculi, including hypersequents, nested sequents, and labelled sequents. We also provide sufficient criteria on the Kripke-frame characterization of a modal logic that guarantee the IPs. In particular, we show that classes of frames definable by quantifier-free Horn formulas correspond to logics with the IPs. Our criteria capture the modal cube between K and $S5$ and the infinite family of transitive Geach logics.

The Craig Interpolation Property (CIP) is one of the fundamental properties desired of a logic. It states that, for any theorem $A \rightarrow B$ of the logic, there must exist an *interpolant* C in the common language of A and B such that both $A \rightarrow C$ and $C \rightarrow B$ are theorems of the logic. The interpolation properties have numerous, well-established connections to both mathematics (e.g., to algebra via amalgamation) and computer science. Here we consider modal logics based on classical propositional logic and understand *common language* to mean common propositional atoms. The Lyndon Interpolation Property (LIP) strengthens the CIP by requiring that not only propositional atoms in C but even their polarities be common to A and B . Since the LIP implies the CIP, we will often write LIP to mean both CIP and LIP.

One of the standard methods of proving both CIP and LIP, or IPs for short, is constructing an interpolant by induction on a derivation of (a representation of) $A \rightarrow B$ in an analytic sequent calculus. Apart from its constructiveness, the method is also modular: if the sequent system is strengthened by an extra rule, only this additional rule needs to be checked to extend the IPs to the resulting stronger logic.

Until recently, a major weakness of the method was the limited expressivity of analytic sequent calculi. In particular, it was unclear how to extend the syntactic sequent formulations of the two implications in the IPs to more expressive sequent-like calculi.

The semantic view of these calculi, inspired in part by tableaux, provides an alternative solution. This semantic view is quite apparent in labelled sequents, where each formula is assigned a label and labels are mapped to worlds in a Kripke model. The same paradigm can also be applied to hypersequent, nested sequents, and similar calculi if the maps are applied to sequent components rather than to labels. According to this approach, a sequent of any of these types, whether one- or two-sided, can be viewed as one disjunction or, equivalently, as an implication from a conjunction to a disjunction, with a twist that each formula is to be evaluated at the world of a Kripke model, to which its label/component is mapped. The set of allowable maps, called *good maps* is chosen individually for each logic/formalism. This view provides a simple way of defining the two implications in the IPs semantically in a way that is suitable for an inductive proof on the depth of a derivation in the proof system in question. We call this semantic definition the *Componentwise Interpolation Property* (CWIP).

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The method was first applied to nested sequents [FK15] (in collaboration with Melvin Fitting), then adapted to hypersequents [Kuz16a]. These results were unified and generalized to a wide range of internal sequent-like formalisms [Kuz16b]. In this presentation, we will use the notation inspired by labelled sequents, both for results on labelled sequents proper and for less expressive internal calculi. The method works surprisingly well and uniformly for either, and notation can be adapted to a particular formalism if need be.

The following are sufficient criteria for the reduction of LIP to CWIP. These criteria are nothing more than stronger forms of completeness statements. Let $\mathsf{L} \vdash C$ denote derivability in a logic L , $\mathsf{SL} \vdash \Gamma \Rightarrow \Delta$ denote derivability of a sequent $\Gamma \Rightarrow \Delta$ in a corresponding sequent-like calculus SL , and $\vDash_{\mathcal{C}_L}$ denote the local logical consequence in the class of Kripke models \mathcal{C}_L .

Requirement I. $\mathsf{L} \vdash A \rightarrow B$ implies $\mathsf{SL} \vdash w : A \Rightarrow w : B$ for all labels w .

Requirement II. For each sequent containing only a single label/component w , each model $\mathcal{M} \in \mathcal{C}_L$, and each world $w \in \mathcal{M}$, any map $\llbracket \cdot \rrbracket$ with $\llbracket w \rrbracket = w$ is good.

We also explain what is expected from a semantic representation of the formalism.

Requirement III. If $\mathsf{SL} \vdash \Gamma \Rightarrow \Delta$, then for each model $\mathcal{M} \in \mathcal{C}_L$ and for each good map, either $\mathcal{M}, \llbracket w \rrbracket \not\vDash A$ for some $w : A \in \Gamma$ or $\mathcal{M}, \llbracket o \rrbracket \vDash B$ for some $o : B \in \Delta$.

A great advantage of labelled sequents is the existence of general methods of generating sequent rules from first-order frame conditions for Kripke-complete logics. We harness this strength by outlining sufficient criteria on the frame conditions to guarantee the CIP and LIP. Moreover, we describe an algorithm for constructing an interpolant of a formula $A \rightarrow B$ from a given derivation of $w : A \Rightarrow w : B$ in the labelled calculus for the logic. A full paper detailing these results is available as a technical report [Kuz16c].

We work with sequent-like objects, from now on *multisequents*, that are two-sided and *symmetric*, meaning that formulas must be in negation normal form (NNF) and that rules do not move formulas between the antecedent and consequent. Both conditions are for ease of presentation only. However, so far the method only works for multi-conclusion classical multisequent formalisms.

We now present the sequent-based Componentwise Interpolation Property (CWIP). In order to apply the semantic view of generalized sequents to interpolants, it is necessary to make componentwise interpolants more complex than mere formulas.

Definition 1 (Uniformula). A *uniformula* $w : A$ is obtained from a multisequent \mathcal{G} by replacing all sequent components in \mathcal{G} with such multisets of formulas that the union of these multisets contains exactly one formula, formula A for the component/label w .

Definition 2 (Multiformula). Each uniformula $w : C$ is a *multiformula*. If \mathcal{U}_1 and \mathcal{U}_2 are multiformulas, then $\mathcal{U}_1 \otimes \mathcal{U}_2$ and $\mathcal{U}_1 \oplus \mathcal{U}_2$ are also *multiformulas*. In other words, a multiformula is a conjunctive-disjunctive combination of formulas to be evaluated at different worlds of a Kripke model.

Let $\llbracket \cdot \rrbracket$ be an interpretation into a model \mathcal{M} . A uniformula $w : C$ is *forced* by this interpretation, written $\mathcal{M} \vDash \llbracket w : C \rrbracket$, iff $\mathcal{M}, \llbracket w \rrbracket \vDash C$. $\mathcal{M} \vDash \llbracket \mathcal{U}_1 \otimes \mathcal{U}_2 \rrbracket$ ($\mathcal{M} \vDash \llbracket \mathcal{U}_1 \oplus \mathcal{U}_2 \rrbracket$) iff $\mathcal{M} \vDash \llbracket \mathcal{U}_i \rrbracket$ for some (each) $i = 1, 2$.

Definition 3. Let \mathcal{M} be a Kripke model and $\llbracket \cdot \rrbracket$ be a good \mathcal{M} -map on a labelled sequent $\Gamma \Rightarrow \Delta$. We write $\mathcal{M} \vDash \llbracket \text{Ant}(\Gamma) \rrbracket$ if $\mathcal{M}, \llbracket w \rrbracket \vDash A$ for each $w : A \in \Gamma$. We write $\mathcal{M} \vDash \llbracket \text{Cons}(\Delta) \rrbracket$ if $\mathcal{M}, \llbracket o \rrbracket \vDash B$ for some $o : B \in \Delta$.

In other words, forcing an antecedent means forcing all antecedent formulas (each in its respective world) and forcing a consequent means forcing at least one consequent formula (in its respective world), which is a standard sequent interpretation.

Definition 4 (Componentwise (Lyndon) Interpolation Property, CWIP). A multiformula $\bar{\cup}$ is a (*componentwise*) *interpolant* of a labelled sequent $\Gamma \Rightarrow \Delta$, written $\Gamma \stackrel{\bar{\cup}}{\Rightarrow} \Delta$, if the following conditions hold:

- each component/label w occurring in $\bar{\cup}$ must occur either in Γ or in Δ ;
- each positive propositional atom P occurring in $\bar{\cup}$ must occur both in Γ and Δ ;
- each negative propositional atom \bar{P} occurring in $\bar{\cup}$ must occur both in Γ and Δ ;
- for each model $\mathcal{M} \in \mathcal{C}_{\mathbf{L}}$ and each good \mathcal{M} -map $\llbracket \cdot \rrbracket$ on $\Gamma \Rightarrow \Delta$, both implications are true:

$$\mathcal{M} \models \llbracket \text{Ant}(\Gamma) \rrbracket \quad \text{implies} \quad \mathcal{M} \models \llbracket \bar{\cup} \rrbracket, \quad (1)$$

$$\mathcal{M} \models \llbracket \bar{\cup} \rrbracket \quad \text{implies} \quad \mathcal{M} \models \llbracket \text{Cons}(\Delta) \rrbracket. \quad (2)$$

A labelled calculus \mathbf{SL} has the CWIP iff every \mathbf{SL} -derivable labelled sequent has an interpolant.

Theorem 5 (Reduction of LIP to CWIP). *Let a logic \mathbf{L} have multisequent proof system \mathbf{SL} and be complete with respect to the class of Kripke models $\mathcal{C}_{\mathbf{L}}$. If requirements I-III are fulfilled and \mathbf{SL} has the CWIP, then \mathbf{L} has the LIP. Moreover, a Lyndon interpolant can be retrieved from the constructed componentwise interpolant.*

Corollary 6. *Let a logic \mathbf{L} have a labelled proof system \mathbf{SL} generated based on a class of Kripke models $\mathcal{C}_{\mathbf{L}}$ by the general method from [NvP11]. If \mathbf{SL} enjoys the CWIP, then \mathbf{L} enjoys the LIP.*

The most general results obtained using this reduction are for the more expressive labelled sequents:

Theorem 7. *Modal logics enjoy the CWIP, LIP, and CIP if they are complete w.r.t. any class $\mathcal{C}_{\mathbf{L}}$ of frames described by Horn clauses of the forms*

$$\begin{aligned} w_1 R u_1 \wedge \dots \wedge w_m R u_m &\rightarrow v R z \\ w_1 R u_1 \wedge \dots \wedge w_m R u_m &\rightarrow \perp \\ w_1 R u_1 \wedge \dots \wedge w_m R u_m &\rightarrow v = z \end{aligned}$$

and by properties of the form

$$\bigwedge_{i=1}^m w_i R o_i \rightarrow \exists y_1 \dots \exists y_k (x R y_1 \wedge y_1 R y_2 \wedge \dots \wedge y_{k-1} R y_k)$$

Further, we define properties of labelled-sequent rules that can be viewed as weak versions of transitivity and connectedness and show that in their presence, the list of frame conditions generating interpolable rules above can be further extended.

In the following corollary we collect all the interpolation results obtainable by our methods so far:

Corollary 8. *Modal logics enjoy both CIP and LIP if they are complete w.r.t. the class of Kripke models defined by any combination of the following properties:*

- *reflexivity, transitivity, symmetry, seriality, and Euclideaness;*
- *shift reflexivity, shift transitivity, shift symmetry, shift seriality, and shift Euclideaness;*
- *any property obtained by replacing the “shift” condition above by an arbitrary conjunction of relational atoms;*
- *functionality;*
- *(1, n)-transitivity;*
- *strictly irreflexive, strictly reflexive, or unspecified discreteness of the frame;*
- *hijk-convergence with $h, i, j, k \geq 1$ (in presence of transitivity);*
- *hijk-convergence with $i, k \geq 1$ and $h, j \geq 2$ (in presence of shift transitivity);*
- *density (in presence of transitivity and Euclideaness);*
- *(n, m)-transitivity for $m < n$ (in presence of transitivity and Euclideaness).*

In particular, the list of logics with CIP/LIP proved using labelled sequents includes all 15 logics of the so-called modal cube, K4.2, S4.2, Triv, Verum, and K4_{1,n}, as well as the infinite family of non-degenerate Geach logics over K4 and almost the full family of Geach logics over K5 (due to the shift transitivity of the latter).

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Some observations regarding cut-free hypersequent calculi for intermediate logics

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Abstract

We address the question of which intermediate logics admit cut-free structural hypersequent calculi by considering a semantically defined subclass of the so-called $(0, \wedge, \vee, 1)$ -stable intermediate logics [3]. This class encompasses many well-known intermediate logics for which good calculi are already known. We show that our class coincides with the class of intermediate logics axiomatisable (over **IPC**) by \mathcal{P}_3 -formulas of the substructural hierarchy of [11]. This yields a purely semantic criterion for determining which intermediate logics admit cut-free structural hypersequent calculi.

Constructing well-behaved proof calculi for intermediate logics can be notoriously difficult. For example by [11, Cor. 7.2] no proper intermediate logic can be captured by extending Gentzen's single-succedent¹ sequent calculus **LJ** with so-called structural rules. However, by moving to the framework of hypersequent calculi [14, 1] it is possible to construct cut-free hypersequent calculi for many well-known intermediate logics see e.g. [2, 9, 8, 13]. Moreover, in [11, 12] a systematic approach to the problem of constructing well-behaved proof calculi is developed and a class of formulas, called \mathcal{P}_3 , is defined for which corresponding cut-free structural single-succedent hypersequent calculi may be obtained in a uniform manner. However, negative results demarcating the class of intermediate logics admitting cut-free structural hypersequent calculi is still to some extent lacking. It can be shown [11, Cor. 7.3] that any structural hypersequent rule is either derivable in **HSM**, where **HSM** is a hypersequent calculus for the strongest proper intermediate logic **Sm**, or derives the formula $\phi \vee \neg \phi^n$ in **HFL**_{ew} for some $n \in \omega$.² Unfortunately, this condition is not particularly informative when it comes to intermediate logics. Similarly, it has been established that if an extension L of **FL**_e, i.e. Full Lambek Calculus with exchange, admits a cut-free structural hypersequent calculus then the corresponding variety $\mathbb{V}(L)$ is closed under so-called hyper-MacNeille completions [10, Thm. 6.8]. However, this is a condition which can be quite difficult to verify. Our—admittedly very modest—contribution consists in singling out a purely semantic criterion determining when an intermediate logic can be axiomatised by \mathcal{P}_3 -formulas and thus be captured by a cut-free structural single-succedent hypersequent calculus extending **LJ**. This is done by considering a subclass of the so-called $(0, \wedge, \vee, 1)$ -stable logics studied in [3, 4, 6]. More precisely, we consider intermediate logics determined by classes of Heyting algebras which are well-behaved with respect to so-called $(0, \wedge, 1)$ -embeddings, viz. injective functions preserving the $(0, \wedge, 1)$ -reduct of the Heyting algebra signature.

We first recall the definitions of some of the main concepts mentioned above.

¹That is, at most one formula is allowed on the right-hand side of the sequent arrow.

²Here ϕ^n is defined by the following recursion $\phi^1 = \phi$ and $\phi^{n+1} = \phi \cdot \phi^n$.

Definition 1 ([11, Def. 3.1], see also [13]). The *substructural hierarchy* is defined by the following grammar based on the propositional language of **LJ**.³

$$\begin{aligned}\mathcal{P}_{n+1} &::= \perp \mid \top \mid \mathcal{N}_n \mid \mathcal{P}_{n+1} \wedge \mathcal{P}_{n+1} \mid \mathcal{P}_{n+1} \vee \mathcal{P}_{n+1}, \\ \mathcal{N}_{n+1} &::= \perp \mid \top \mid \mathcal{P}_n \mid \mathcal{N}_{n+1} \wedge \mathcal{N}_{n+1} \mid \mathcal{P}_{n+1} \rightarrow \mathcal{N}_{n+1},\end{aligned}$$

where $\mathcal{P}_0 = \mathcal{N}_0$ is a set of propositional variables.

Definition 2 (cf. [11, 10]). A *structural hypersequent rule* is a rule of the form:

$$\frac{H \mid \Upsilon_1 \Rightarrow \Psi_1 \quad \cdots \quad H \mid \Upsilon_m \Rightarrow \Psi_m}{H \mid \Upsilon_{m+1} \Rightarrow \Psi_{m+1} \mid \cdots \mid \Upsilon_n \Rightarrow \Psi_n} (r)$$

where for each $i \in \{1, \dots, n\}$, Υ_i is a (possibly empty) sequence of meta-variables for formulas and for sequences of formulas and Ψ_i either a meta-variable for stoups, a meta-variable for formulas or empty.

The main virtue of structural hypersequent rules is that every structural hypersequent rule (r) may (effectively) be transformed into a structural hypersequent rule (r') which preserves cut-elimination when added to the hypersequent version **HLJ** of **LJ** [11, Thm. 7.1, Cor. 8.6].

The following theorem explains the relationship between structural hypersequent rules and the substructural hierarchy.

Theorem 3 ([11], see also [13, Thm. 1]). *Any formula $\phi \in \mathcal{P}_3$ may (effectively) be transformed into a finite set of structural hypersequent rules which are sound and complete for the logic $\text{IPC} + \phi$. Furthermore, cut-elimination is preserved by adding these rules to **HLJ**.*

By a *structural hypersequent calculus* we shall here understand any extension of **HLJ** by structural hypersequent rules. We should like to emphasise that results obtained below very much depend on this particular definition of structural hypersequent calculus.

Definition 4. A class of Heyting algebras \mathcal{K} is called $(0, \wedge, 1)$ -stable if whenever $\mathfrak{B} \in \mathcal{K}$ and $h: \mathfrak{A} \hookrightarrow \mathfrak{B}$ is an $(0, \wedge, 1)$ -embedding then $\mathfrak{A} \in \mathcal{K}$.

Definition 5. An intermediate logic L is said to be $(0, \wedge, 1)$ -stable if whenever $\mathfrak{B} \in \mathbb{V}(L)$ is subdirectly irreducible and $h: \mathfrak{A} \hookrightarrow \mathfrak{B}$ is an $(0, \wedge, 1)$ -embedding of Heyting algebras then $\mathfrak{A} \in \mathbb{V}(L)$.

Definition 6. Let \mathfrak{A} be a finite Heyting algebra and for each $a \in A$ let p_a be a propositional letter. The $(0, \wedge, 1)$ -stable rule $\xi(\mathfrak{A})$ associated with \mathfrak{A} is defined to be the multi-conclusion rule Γ/Δ , where

$$\Gamma := \{p_0 \leftrightarrow \perp\} \cup \{p_1 \leftrightarrow \top\} \cup \{p_{a \wedge b} \leftrightarrow p_a \wedge p_b : a, b \in A\} \quad \text{and} \quad \Delta := \{p_a \rightarrow p_b : a, b \in A, a \not\leq b\}.$$

Finally, the $(0, \wedge, 1)$ -stable formula $\delta(\mathfrak{A})$ associated with \mathfrak{A} is defined as the formula $\bigwedge \Gamma \rightarrow \bigvee \Delta$.

Analogous to the case of $(0, \wedge, \vee, 1)$ -stable logics [4, Thm. 5.7] the $(0, \wedge, 1)$ -stable logics can be characterised by the following equivalent conditions.

Theorem 7. *Let L be an intermediate logic. Then the following are equivalent:*

1. *The logic L is $(0, \wedge, 1)$ -stable;*

³Note that, as originally defined in [11], the hierarchy is based on the language of Full Lambek Calculus.

2. The logic L is axiomatisable by $(0, \wedge, 1)$ -stable formulas;
3. The logic L is axiomatisable by $(0, \wedge, 1)$ -stable multi-conclusion rules;
4. The variety $\mathbb{V}(L)$ is generated by a $(0, \wedge, 1)$ -stable universal class;
5. The variety $\mathbb{V}(L)$ is generated by a $(0, \wedge, 1)$ -stable class.

Using the above characterisation of $(0, \wedge, 1)$ -stable logics one may prove that these are precisely the logics which can be captured by a structural hypersequent calculus in the sense defined above.

Theorem 8. *Let L be an intermediate logic. Then L admits a (cut-free) structural hypersequent calculus if and only if L is $(0, \wedge, 1)$ -stable. Moreover, in case L is a finitely axiomatisable intermediate logic, the corresponding calculus will be determined by a finite set of structural hypersequent rules.*

Using an argument similar to the one establishing [12, Prop. 7.5] one may show that

Proposition 9. *Every $(0, \wedge, 1)$ -stable logic is axiomatisable by \mathcal{P}_3 -formulas.*

In [11] it is shown that any intermediate logic axiomatised by \mathcal{P}_3 -formulas admits a structural hypersequent calculus. Consequently, we obtain as an immediate consequence of Theorem 8 that every intermediate logic axiomatised by \mathcal{P}_3 -formulas is $(0, \wedge, 1)$ -stable. Thus, we have the following theorem.

Theorem 10. *Let L be an intermediate logic. Then the following are equivalent:*

1. The logic L is axiomatised by \mathcal{P}_3 -formulas;
2. The logic L is $(0, \wedge, 1)$ -stable.

The novelty of Theorem 10 is that it provides a purely semantic criterion enabling us to determine whether or not an intermediate logic L can be axiomatised by \mathcal{P}_3 -formulas and thereby whether or not L admits a cut-free structural hypersequent calculus.

Example 11. Given Theorem 10 one may conclude that the following intermediate logics

$$\mathbf{LC}, \mathbf{LC}_n, \mathbf{KC}, \mathbf{BTW}_n, \mathbf{BW}_n, \mathbf{BC}_n, \quad (n \geq 2)$$

are all $(0, \wedge, 1)$ -stable, as these can be axiomatised by formulas which are ostensibly \mathcal{P}_3 -formulas. Of course, that these logics are $(0, \wedge, 1)$ -stable may also be established by purely algebraic arguments or alternatively by appealing to the duality theory for distributive meet-semilattice as developed in [5].

Remark 12. Note that all of the above logics were already known to be $(0, \wedge, \vee, 1)$ -stable cf. [3, Thm. 7.3]. However, it can be shown that not all $(0, \wedge, \vee, 1)$ -stable logics are $(0, \wedge, 1)$ -stable.

For $n \geq 2$, none of the intermediate logics \mathbf{BD}_n are $(0, \wedge, \vee, 1)$ -stable cf. [3, Thm. 7.4(2)], and so in particular not $(0, \wedge, 1)$ -stable. Consequently, as a corollary of Theorem 8 we have.

Corollary 13. *The logics \mathbf{BD}_n do not admit (cut-free) structural single-succedent hypersequent calculi for $n \geq 2$.*

In particular there exist decidable and finitely axiomatisable intermediate logics not having any (cut-free) structural single-succedent hypersequent calculus.

Remark 14. We emphasise that Theorem 8 only refers to the existence or non-existence of certain very particular hypersequent calculi, viz., calculi obtained by extending \mathbf{HLJ} with structural hypersequent rules. For instance, the logic \mathbf{BD}_2 admits a logical, i.e. non-structural, hypersequent calculus [13], and in general, for $n \geq 2$, the logic \mathbf{BD}_n can be captured using so-called path-hypertableaux calculi [7] or path-hypersequent calculi [15].

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NPc-algebras and Gödel hoops

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This is a joint work with S. Aguzzoli, M. Busaniche and B. Gerla.

The algebraic models of paraconsistent Nelson logic were introduced by Odinstov [6, 7] under the name of $\mathbf{N4}$ -lattices. These algebras form a variety, and can be represented by twist-structures of generalized Heyting algebras (also known as implicative lattices or Brouwerian algebras).

The expansions of $\mathbf{N4}$ -lattices by a constant e fulfilling certain equations are called expanded $\mathbf{N4}$ -lattices or $\mathbf{eN4}$ -lattices. It follows that the variety of $\mathbf{eN4}$ -lattices is the algebraic semantics (in the sense of [3]) of the expansion of paraconsistent Nelson logic by a constant fulfilling the appropriate axioms. S. P. Odinstov observed that this constant may be considered as a counterpart of Belnap's truth value 'both' [2] (see also [8, page 308]).

In [5] the authors define NPc-lattices as non-integral commutative residuated lattices with involution satisfying certain equations and a quasiequation, and then prove NPc-lattices and $\mathbf{eN4}$ -lattices are termwise equivalent, thus NPc-lattices form a variety. Then paraconsistent Nelson logic can be studied within the framework of substructural logics.

In this work we are particularly interested in the representation of NPc-lattices by twist-product obtained from their negative cones, which turn to be Brouwerian algebras. More precisely:

Theorem 1. (see [10, Corollary 3.6]) *Let $\mathbf{L} = (L, \vee, \wedge, \Rightarrow, e)$ be a Brouwerian algebra. Then*

$$\mathbf{K}(\mathbf{L}) = (L \times L, \sqcup, \sqcap, *, \rightarrow, (e, e))$$

with the operations $\sqcup, \sqcap, *, \rightarrow$ given by

$$(a, b) \sqcup (c, d) = (a \vee c, b \wedge d) \tag{1}$$

$$(a, b) \sqcap (c, d) = (a \wedge c, b \vee d) \tag{2}$$

$$(a, b) * (c, d) = (a \wedge c, (a \Rightarrow d) \wedge (c \Rightarrow b)) \tag{3}$$

$$(a, b) \rightarrow (c, d) = ((a \Rightarrow c) \wedge (d \Rightarrow b), a \wedge d) \tag{4}$$

is an NPc-lattice. Moreover, the correspondence

$$(a, e) \mapsto a$$

defines an isomorphism from the negative cone of $\mathbf{K}(\mathbf{L})$ onto \mathbf{L} .

The NPc-lattice $\mathbf{K}(\mathbf{L})$ is called the *full twist-product* obtained from \mathbf{L} , and every subalgebra \mathbf{A} of $\mathbf{K}(\mathbf{L})$ containing the set $\{(a, e) : a \in L\}$ is called a *twist-product* obtained from \mathbf{L} .

NPc-lattices are always isomorphic to a twist-product obtained from their negative cones. This result is a corollary of the more general theorem [4, Theorem 3.7].

Theorem 2. *For every NPC-lattice \mathbf{B} , the application $\phi_{\mathbf{B}} : \mathbf{B} \rightarrow \mathbf{K}(\mathbf{B}^-)$ given by*

$$x \mapsto (x \wedge e, \sim x \wedge e)$$

is an injective morphism.

The first aim of this work is to prove a categorical equivalence between the category of NPC-lattices and residuated lattice morphisms and a category whose objects are pairs of Brouwerian algebras and some filters that we call regular. The point is to reformulate the characterization of N4-lattices given by Odintsov ([9]) in terms of residuated lattices. Some similar ideas were presented in [4], but those ideas strongly rely on the fact that the twist product is obtained from a bounded integral residuated lattice, which is not the present case.

In details, if \mathbf{L} is a Brouwerian algebra, an element $x \in L$ is *dense* if it is of the form $x = w \vee (w \Rightarrow z)$. The set of dense elements is a (lattice) filter, and filters containing this set are called *regular*. Consider the category \mathcal{NPC} of NPC-lattices together with NPC-lattice morphisms, and the category \mathcal{BF} that has as objects pairs of the form (\mathbf{L}, ∇) where \mathbf{L} is a Brouwerian algebra and $\nabla \subset L$ is a regular filter, and as arrows $f : (\mathbf{L}, \nabla) \rightarrow (\mathbf{L}', \nabla')$ such that $f : \mathbf{L} \rightarrow \mathbf{L}'$ is a Brouwerian morphism that satisfies $f(\nabla) \subset \nabla'$.

Theorem 3. *The functor $F : \mathcal{BF} \rightarrow \mathcal{NPC}$ that acts on objects as*

$$F((\mathbf{L}, \nabla)) = \mathbf{Tw}(\mathbf{L}, \nabla) = \{(a, b) \in L \times L : a \vee b \in \nabla\}$$

and on arrows, for $f : (\mathbf{L}, \nabla) \rightarrow (\mathbf{L}', \nabla')$ obtaining $F(f) : \mathbf{Tw}(\mathbf{L}, \nabla) \rightarrow \mathbf{Tw}(\mathbf{L}', \nabla')$ given by

$$F(f)(x, y) = (f(x), f(y)),$$

gives an equivalence of categories.

Observe that the restriction of the functor F to the category \mathcal{GBF} of pairs consisting of Gödel hoops (Brouwerian algebras satisfying the prelinearity equation $(x \Rightarrow y) \vee (y \Rightarrow x) = e$) and regular filters, also gives an equivalence of categories between \mathcal{GBF} and the category \mathcal{GNPC} of Gödel NPC-lattices, the subcategory of \mathcal{NPC} that has as objects NPC-lattices satisfying the equation

$$(((x \wedge e) \rightarrow y) \vee ((y \wedge e) \rightarrow x)) \wedge e = e.$$

This connection between Gödel NPC-lattices and Gödel hoops suggests a duality for \mathcal{GNPC} . In [1] the authors prove the dual of the category \mathcal{GH}_{fin} of finite Gödel hoops is the category \mathcal{T}_{fin} of finite trees and open maps. Using this result we construct a dual category for \mathcal{GNPC} , which we call $\mathcal{T}_{t,fin}$, and has as objects pairs of finite trees and certain subtrees.

These relations between categories allows us to obtain a representation of the free algebras in \mathcal{GNPC} . In first place, we can show that $\text{Free}_{\mathcal{GNPC}}(n)$, the free Gödel NPC-lattice of n generators, is isomorphic to a twist-product obtained from $\text{Free}_{\mathcal{GH}}(2n)$, the free Gödel hoop of $2n$ generators. For the case $n = 1$ we find the regular filter $\nabla_{\text{Free},2}$ such that

$$\text{Free}_{\mathcal{GNPC}}(1) \cong Tw(\text{Free}_{\mathcal{GH}}(2), \nabla_{\text{Free},2})$$

Now, for $n > 1$, as $\text{Free}_{\mathcal{GNPC}}(n)$ is the coproduct of n copies of $\text{Free}_{\mathcal{GNPC}}(1)$, in the dual category we obtain the product of n copies of the dual of $\text{Free}_{\mathcal{GNPC}}(1)$. Therefore, describing the product in this category is sufficient to characterize the free Gödel NPC-lattice with n generators.

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Order-Based and Continuous Modal Logics

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Many-valued modal logics are easy enough to define. We simply generalize the Kripke frame semantics of classical modal logic to allow a many-valued semantics at each world based on an algebra with a complete lattice reduct, where the accessibility relation may also take values in this algebra (see [3] for a careful account of this approach). Such logics can be designed to model modal notions such as necessity, belief, and spatio-temporal relations in the presence of uncertainty, possibility, or vagueness, and applications have included modelling fuzzy belief [11], spatial reasoning with vague predicates [17], many-valued tense logics [7], and fuzzy similarity measures [12]. Many-valued multi-modal logics also provide a basis, as in the classical case, for defining fuzzy description logics (see, e.g., [13, 1, 2]). More generally, many-valued modal logics provide a first foray, again following the classical approach, into investigating useful and computationally feasible fragments of corresponding first-order logics.

Although it is easy to define a many-valued modal logic, studying the formal properties of such a logic can be quite challenging. In particular, the relational semantics may give no clue as to how to obtain a reasonable axiom system or algebraic semantics for the logic. Computational properties such as the decidability or complexity class of the validity or satisfiability problems for the logic may also be difficult to determine. Quite general ways of tackling these issues are to restrict attention to finite-valued logics (see, e.g., [9, 10, 3, 14]), where the resemblance to classical modal logic is more apparent, or to witnessed semantics, where suprema and infima of values at a set of worlds are witnessed by values at particular worlds (see, e.g., [13, 1, 2]). Alternatively, extra constants may be introduced into the language which simplify the task of axiomatizing the logic (see, e.g. [18]). Such approaches are all well and good, but do not do full justice to intuitions, usually accepted at the propositional and even first-order level, that infinite-valued logics provide an appropriate level of generality for considering arbitrary numbers of truth values, that requiring witnessed models is a rather artificial condition, and that adding extra constants unduly constrains the algebraic semantics of the logic.

The aim of my talk will be to describe recent joint work on many-valued modal logics based on an infinite algebra. Such logics fall very loosely into two core families:

- *Order-based modal logics* extend the semantics of infinite-valued Gödel logics and are defined based over a complete chain of real numbers with additional operations depending only on the given order. Axiomatizations, algebraic semantics, and limited decidability and complexity results for (fragments) of these logics based on the standard Gödel algebra have been provided in [5, 16, 6]. In joint work with X. Caicedo, R. Rodríguez, and J. Rogger [4], decidability and PSPACE-complexity results have been obtained for a very general family of order-based modal logics, using a new semantics that, unlike the standard semantics, admits the finite model property. Decidability and co-NP complexity results have also been obtained for one-variable fragments of first-order order-based logics, answering positively the open decidability problem for the one-variable fragment of first-order Gödel logic.

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- *Continuous modal logics* may be understood as real-valued modal logics with propositional connectives interpreted by continuous operations such as those of Łukasiewicz infinite-valued logic. Although finite-valued Łukasiewicz modal logics have been axiomatized in [14], the axiom system provided there for the infinite-valued Łukasiewicz modal logic includes a rule with infinitely many premises. As a first step to addressing this problem, a finitary axiomatization has been provided in joint work with D. Diaconescu and L. Schnüriger [8] for a simple continuous modal logic with propositional connectives interpreted as the usual group operations over the real numbers.

Let us remark finally that the expressivity of many-valued (in particular, order-based and continuous) modal logics is addressed in a joint paper with M. Marti [15], which provides characterizations of logics whose image-finite models admit the Hennessy-Milner property.

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Adjunctions as translations between relative equational consequences

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1 Outline of the talk

In this talk we provide a logical and algebraic characterization of adjunctions between generalized quasi-varieties. Our approach to this problem is inspired by the work of McKenzie on category equivalences [8]. Roughly speaking, McKenzie showed that if two prevarieties \mathbf{K} and \mathbf{K}' are categorically equivalent, then we can transform \mathbf{K} into \mathbf{K}' by applying two kinds of deformations to \mathbf{K} .

The first of these deformations is the *matrix power* construction. The matrix power with exponent $n \in \omega$ of an algebra \mathbf{A} is a new algebra $\mathbf{A}^{[n]}$ with universe A^n and whose basic m -ary operations are all n -sequences of $(m \times n)$ -ary term functions of \mathbf{A} , which are applied component-wise. The other basic deformation is defined as follows. Suppose that $\sigma(x)$ is a unary term. Then, given an algebra \mathbf{A} , we let $\mathbf{A}(\sigma)$ be the algebra whose universe is the range of the term-function $\sigma^{\mathbf{A}}: A \rightarrow A$ and whose m -ary operations are the restrictions to $\sigma[A]$ of the term functions of \mathbf{A} of the form $\sigma t(x_1, \dots, x_m)$, where t is an m -ary term of \mathbf{A} . McKenzie showed that the prevarieties categorically equivalent to \mathbf{K} are exactly the ones obtained deforming \mathbf{K} by means of the matrix power and $\sigma(x)$ constructions, where σ is a unary *idempotent and invertible* term. This algebraic approach to the study of category equivalence has been reformulated in categorical terms for example in [9, 10] and has an antecedent in [5].

Building on McKenzie's work [8] and on the theory of *locally presentable* categories [1], we show that every right adjoint functor between generalized quasi-varieties (which are particular kinds of prevarieties) can be decomposed into a combination of two deformations that generalize the ones devised in the special case of category equivalence. These deformations are matrix powers with (possibly) *infinite* exponent and the following generalization of the $\sigma(x)$ construction. Given an algebra \mathbf{A} , we say that a set of equations θ in a single variable is *compatible* with a sublanguage \mathcal{L} of the language of \mathbf{A} if the set of solutions of θ in \mathbf{A} is closed under the restriction of the operations in \mathcal{L} . In this case, we let $\mathbf{A}(\theta, \mathcal{L})$ be the algebra obtained by equipping the set of solutions of θ in \mathbf{A} with the restriction of the operations in \mathcal{L} .

This characterization of right adjoint functors is achieved by developing a correspondence between the concept of adjunction and a new notion of *translation* between equational consequences relative to classes of algebras. More precisely, we introduce a notion of translation that satisfies the following condition: Given two generalized quasi-varieties \mathbf{K} and \mathbf{K}' , every translation of the equational consequence relative to \mathbf{K} into the one relative to \mathbf{K}' corresponds to a right adjoint functor from \mathbf{K}' to \mathbf{K} and vice-versa. In a slogan, translations between relative equational consequences are the *duals* of right adjoint functors. Examples of this correspondence between right adjoint functors and translations abound in the literature, e.g., Gödel's translation [6] of intuitionistic logic into the modal system $\mathcal{S}4$ corresponds to the functor that extracts the Heyting algebra of open elements from an interior algebra, and Kolmogorov's translation [7] of classical logic into intuitionistic logic corresponds to the functor that extracts the Boolean algebra of regular elements out of a Heyting algebra.

If time allows, we will discuss some computational aspects of the decomposition of right adjoint functors [2] and some applications to the theory of algebraizable logics [3] and to natural dualities [4].

2 Detailed content

Our aim is to show that every right adjoint functor between generalized quasi-varieties can be decomposed into a combination of two basic kinds of deformations. The first one is the following generalization of the usual matrix power construction. Let $\kappa > 0$ be a cardinal and \mathbf{X} a class of similar algebras. Then $\mathcal{L}_{\mathbf{X}}^{\kappa}$ is the algebraic language whose n -ary operations (for every $n \in \omega$) are all κ -sequences $\langle t_i : i < \kappa \rangle$ of terms t_i of the language of \mathbf{X} built up with variables among

$$\{x_m^j : 1 \leq m \leq n \text{ and } j < \kappa\}.$$

Observe that each t_i has a finite number of variables, possibly none, of each sequence $\vec{x}_m := \langle x_m^j : j < \kappa \rangle$ with $1 \leq m \leq n$. We write $t_i = t_i(\vec{x}_1, \dots, \vec{x}_n)$ to denote this fact.

Consider an algebra $\mathbf{A} \in \mathbf{X}$ and a cardinal $\kappa > 0$. We let $\mathbf{A}^{[\kappa]}$ be the algebra of type $\mathcal{L}_{\mathbf{X}}^{\kappa}$ with universe A^{κ} where a n -ary operation $\langle t_i : i < \kappa \rangle$ is interpreted as

$$\langle t_i : i < \kappa \rangle(a_1, \dots, a_n) = \langle t_i^{\mathbf{A}}(a_1/\vec{x}_1, \dots, a_n/\vec{x}_n) : i < \kappa \rangle$$

for every $a_1, \dots, a_n \in A^{\kappa}$. If \mathbf{X} is a class of similar algebras, we set

$$\mathbf{X}^{[\kappa]} := \mathbb{I}\{\mathbf{A}^{[\kappa]} : \mathbf{A} \in \mathbf{X}\}$$

and call it the κ -th *matrix power* of \mathbf{X} . Now, let $[\kappa]$ be the functor defined as follows:

$$\begin{aligned} \mathbf{A} &\longmapsto \mathbf{A}^{[\kappa]} \\ f : \mathbf{A} \rightarrow \mathbf{B} &\longmapsto f^{[\kappa]} : \mathbf{A}^{[\kappa]} \rightarrow \mathbf{B}^{[\kappa]} \end{aligned}$$

where $f^{[\kappa]} \langle a_i : i < \kappa \rangle := \langle f(a_i) : i < \kappa \rangle$, for every $\mathbf{A}, \mathbf{B} \in \mathbf{X}$ and every homomorphism f .

Theorem 2.1. *Let \mathbf{X} be a generalized quasi-variety and $\kappa > 0$ a cardinal. If \mathbf{Y} is a generalized quasi-variety such that $\mathbf{X}^{[\kappa]} \subseteq \mathbf{Y}$, then $[\kappa] : \mathbf{X} \rightarrow \mathbf{Y}$ is a right adjoint functor.*

It is worth to remark that if κ is finite and \mathbf{X} is a prevariety, then $\mathbf{X}^{[\kappa]}$ is a prevariety categorically equivalent to \mathbf{X} [8]. However this needs not to be the case when κ is infinite.

The second basic deformation of classes of algebras that we take into account is the following. Let \mathbf{X} be a class of similar algebras and $\mathcal{L} \subseteq \mathcal{L}_{\mathbf{X}}$, where $\mathcal{L}_{\mathbf{X}}$ is the language of \mathbf{X} . A set of equations θ of \mathbf{X} in a single variable is *compatible* with \mathcal{L} in \mathbf{X} if for every n -ary operation $\varphi \in \mathcal{L}$ we have that:

$$\theta(x_1) \cup \dots \cup \theta(x_n) \models_{\mathbf{X}} \theta(\varphi(x_1, \dots, x_n))$$

where $\models_{\mathbf{X}}$ is the equational consequence relative to \mathbf{X} .

In other words θ is compatible with \mathcal{L} in \mathbf{X} when the solution sets of θ in \mathbf{C} are closed under the interpretation of the operations and constants in \mathcal{L} . In this case, for every $\mathbf{A} \in \mathbf{X}$ we let $\mathbf{A}(\theta, \mathcal{L})$ be the algebra of type \mathcal{L} whose universe is

$$A(\theta, \mathcal{L}) := \{a \in A : \mathbf{A} \models \theta(a)\}$$

equipped with the restriction of the operations in \mathcal{L} . Moreover, given a homomorphism $f: \mathbf{A} \rightarrow \mathbf{B}$ in \mathbf{X} , we denote its restriction to $\mathbf{A}(\theta, \mathcal{L})$ by

$$\theta_{\mathcal{L}}(f): \mathbf{A}(\theta, \mathcal{L}) \rightarrow \mathbf{B}(\theta, \mathcal{L}).$$

Now, consider the following class of algebras:

$$\mathbf{X}(\theta, \mathcal{L}) := \mathbb{I}\{\mathbf{A}(\theta, \mathcal{L}) : \mathbf{A} \in \mathbf{X}\}.$$

Let $\theta_{\mathcal{L}}: \mathbf{X} \rightarrow \mathbf{X}(\theta, \mathcal{L})$ be the functor defined by the following rule:

$$\begin{aligned} \mathbf{A} &\mapsto \mathbf{A}(\theta, \mathcal{L}) \\ f: \mathbf{A} \rightarrow \mathbf{B} &\mapsto \theta_{\mathcal{L}}(f): \mathbf{A}(\theta, \mathcal{L}) \rightarrow \mathbf{B}(\theta, \mathcal{L}). \end{aligned}$$

Theorem 2.2. *Let \mathbf{X} be a generalized quasi-variety and θ a set of equations of \mathbf{X} in a single variable compatible with $\mathcal{L} \subseteq \mathcal{L}_{\mathbf{X}}$. If \mathbf{Y} is a generalized quasi-variety such that $\mathbf{X}(\theta, \mathcal{L}) \subseteq \mathbf{Y}$, then $\theta_{\mathcal{L}}: \mathbf{X} \rightarrow \mathbf{Y}$ is a right adjoint functor.*

Our main result shows that every right adjoint functor between generalized quasi-varieties can be obtained as a combination of the two deformations defined above. More precisely, we have the following:

Theorem 2.3.

1. *Every non-trivial right adjoint between generalized quasi-varieties is naturally isomorphic to a functor of the form $\theta_{\mathcal{L}} \circ [\kappa]$.*
2. *Every functor of the form $\theta_{\mathcal{L}} \circ [\kappa]$ between generalized quasi-varieties is a right adjoint.*

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About some Categories of Translations between Logics

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1 Introduction

In this work we introduce the concept of abstract contextual translation, based on the concept of contextual translation. So we study a new category determined by the abstract contextual translations and its relations with the categories **Tr**, **TrCon** and **TrCx**.

We start with the concepts of Tarski's consequence operator and Tarski's logic.

Definition 1.1. If X is a non-empty set, then a *consequence operator* on X is a function $\mathbf{C} : \wp(X) \rightarrow \wp(X)$ such that for every $A, B \subseteq X$:

- (i) $A \subseteq \mathbf{C}(A)$,
- (ii) if $A \subseteq B$, then $\mathbf{C}(A) \subseteq \mathbf{C}(B)$,
- (iii) $\mathbf{C}(\mathbf{C}(A)) \subseteq \mathbf{C}(A)$.

Definition 1.2. A *logic* \mathbf{L} is a pair (L, \mathbf{C}) such that L is a set, the *domain* of \mathbf{L} , and \mathbf{C} is a consequence operator on L .

Now, the definition of *translation* as in [3].

Definition 1.3. A *translation* from the logic $\mathbf{L}_1 = (L_1, \mathbf{C}_1)$ into the logic $\mathbf{L}_2 = (L_2, \mathbf{C}_2)$ is a function $t : L_1 \rightarrow L_2$ such that, for every $A \cup \{x\} \subseteq L_1$, if $x \in \mathbf{C}_1(A)$, then $t(x) \in \mathbf{C}_2(t(A))$, with $t(A) = \{t(y) : y \in A\}$.

In according this definition, the central attribution of a translation is to preserve the derivability.

In [3], it was introduced the category **Tr**, whose objects are logics and morphisms are translations between these logics. Of course, the composition of translations is associative and the identity function is a translation too. It was proved that this category is bicomplete.

Conservative translations between logics, characterize a special class of translations between logics, as [5].

Definition 1.4. *Conservative translation* from the logic $\mathbf{L}_1 = (L_1, \mathbf{C}_1)$ into the logic $\mathbf{L}_2 = (L_2, \mathbf{C}_2)$ is a function $t : L_1 \rightarrow L_2$ such that, for every $A \cup \{x\} \subseteq L_1$, it holds that $x \in \mathbf{C}_1(A)$ if, and only if, $t(x) \in \mathbf{C}_2(t(A))$.

\mathbf{Trcon} is a subcategory of \mathbf{Tr} , whose objects are the logics and morphisms are conservative translations between logics [5]. The category \mathbf{Trcon} has equalizer, coequalizer and coproduct.

Coniglio [2] proposed a specific notion of translation named meta-translation, because these functions should preserve some meta-properties between very restrict formal languages L and L' . He was interested in recovering a logic when fibring its fragments.

If $\Gamma_1 \vdash_{\mathcal{L}} \varphi_1, \dots, \Gamma_n \vdash_{\mathcal{L}} \varphi_n \Rightarrow \Gamma \vdash_{\mathcal{L}} \varphi$, then

$h(\Gamma_1) \vdash_{\mathcal{L}'} h(\varphi_1), \dots, h(\Gamma_n) \vdash_{\mathcal{L}'} h(\varphi_n) \Rightarrow h(\Gamma) \vdash_{\mathcal{L}'} h(\varphi)$.

In the paper [1] appeared a simplified version of meta-translations that was called *contextual translation*.

2 The Category \mathbf{TrCx}

The contextual translations are defined for logics with specific language. We introduce a variation of this notion, the *abstract contextual translation*, that determines another category whose morphisms are yet between the above logics.

Definition 2.1. Let $\mathbf{L}_1 = (L_1, \mathbf{C}_1)$ and $\mathbf{L}_2 = (L_2, \mathbf{C}_2)$ be two logics. An *abstract contextual translation* is a function $t : L_1 \rightarrow L_2$ such that, for every set $A_i \cup \{x_i\} \subseteq L_1$, with $i \in \{1, 2, \dots, n\}$, if $x_1 \in \mathbf{C}_1(A_1), x_2 \in \mathbf{C}_1(A_2), \dots, x_{n-1} \in \mathbf{C}_1(A_{n-1}) \Rightarrow x_n \in \mathbf{C}_1(A_n)$, then $t(x_1) \in \mathbf{C}_2(t(A_1)), t(x_2) \in \mathbf{C}_2(t(A_2)), \dots, t(x_{n-1}) \in \mathbf{C}_2(t(A_{n-1})) \Rightarrow t(x_n) \in \mathbf{C}_2(t(A_n))$.

It is immediate that each abstract contextual translation is a particular case of translation and that each conservative translation is a particular case of abstract contextual translation.

We characterize the category of abstract contextual translations, \mathbf{TrCx} , whose objects are logics and the morphisms are the abstract contextual translations between logics. This way, we can compare these concepts. We mentioned that \mathbf{TrCon} is a subcategory of \mathbf{Tr} , but it is not bicomplete. Moreover, \mathbf{TrCon} is a subcategory of \mathbf{TrCx} . Besides, we show that the category \mathbf{TrCx} has equalizer, coequalizer, product and coproduct, that is, \mathbf{TrCx} is bicomplete.

3 Final considerations

We couldn't completely compare the concept of contextual translation with translations and conservative translations because the objects in the cases are different. So, we proposed the definition of abstract contextual translation. This case is not better than the other ones, it is only different. But, so we could produce a comparison.

We saw that the category \mathbf{TrCx} is a bicomplete subcategory of \mathbf{Tr} and \mathbf{TrCon} is a subcategory of \mathbf{TrCx} .

Jeřábek [8] relates the ubiquity of conservative translation, namely the existence of conservative translation evolving logics of a large class of logics. With a particular algorithm it is possible to explain the result but not to obtain simple conservative translations with applicability.

As each conservative translation is an abstract contextual translation, then there so many such functions.

However, to explicit the functions must be relevant. Many interesting examples of conservative translation used to justify logical results have been described, as [9], [7] and [6].

We are yet interested in to understand more about these functions. Of course, we must improve definitions, change some of them, but a general view yet is in our desire. Besides, in context of natural languages, it is always possible to translate from one language to another.

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Neighborhood Semantics for Many Valued Modal Logics

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Our starting point is that the framework of classical logic is not enough to reason with vague concepts or with modal notions such as belief, uncertainty, knowledge, obligations, time, etc. Many-valued logical systems under the umbrella of mathematical fuzzy logic (in the sense of Hájek [8, 5]) appear as a suitable logical framework to formalize reasoning with vague or gradual predicates, while a variety of modal logics address the logical formalization to reason about different notions as the ones mentioned above. Therefore, if one is interested in a logical account of both vagueness and some sorts of modality, one is led to study systems of many-valued modal logic.

The basic idea of this presentation is to systematically introduce modal extensions of many-valued or fuzzy logics. These logics, under different forms and contexts, have appeared in the literature for different reasoning modeling purposes. For instance, in [7], Fitting introduces a modal logic on logics valued on finite Heyting algebras, and provides a satisfactory justification to study such modal systems to deal with opinions of experts with a dominance relationship among them. In [3, 4], the authors have proposed to extend Gödel fuzzy logic with modal operators. They provide a systematic study of this Gödel modal logic, which has been complemented in [2]. In [1], a detailed description of many-valued modal logics (with a necessity operator) over finite residuated lattices is proposed. In [9], a modal extension of Łukasiewicz logic is developed following an algebraic approach. Finally, in [13], a general approach to modal expansions of t-norm based logics is also introduced with the help of rational constants and possibly infinitary inference rules.

In most of these mentioned papers, many-valued modal logics are endowed with a Kripke-style semantics, generalizing the classical one, where propositions at each possible world, and possibly accessibility relations between worlds as well, are valued in a residuated lattice. The natural next step in this line of research is to axiomatize such semantics. However, this has turned apparently to be a considerable overall challenge because it is difficult to transfer some usual techniques from Boolean algebras to residuated lattices. For instance, the K axiom ($\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$) plays a central role in the construction of the canonical models in order to prove completeness in the classical case. However, except for either Gödel modal logic or many-valued modal logics defined from Kripke frames with crisp accessibility relations, the K axiom is not sound.

In order to overcome this difficulty, we propose to study an alternative semantics which is a generalization of the classical *neighborhood semantics*. This will be elaborated based on two preliminary workshop papers by the same authors [11, 12]. At this moment, it is worth mentioning some works from other authors which consider a generalization of neighborhood semantics in the same way we have done it. Namely, Kroupa and Teheux consider in [10] a neighborhood semantics for playable L_n -valued effectivity function. They want to characterize the notion of

coalitional effectivity within game form models. Also we must mention a very recent paper by Cintula et al. ([6]) where the authors explore a fuzzified version of the classical neighborhood semantics and prove a relationship between fuzzy Kripke and neighborhood semantics in a very precise way (much better than the one proposed in our previous work). In fact, the authors of this paper propose to attack the problem of characterizing the modal extensions of MTL logics under a neighborhood semantics with algebraic tools. According to their algebraic approach, they characterize a global MTL modal logic, leaving open the case of characterizing the local consequence relation.

In summary, in this presentation, we will mainly focus on the development of a theoretical and general framework. Considering our motivation, our main goal, at large, is a systematic presentation of the minimum many-valued modal logics and their extensions. In this sense, we will firstly present minimum many-valued modal logics with necessity and possibility operators, \Box, \Diamond , defined on top of logics of residuated lattices under a neighborhood semantics. In this context, we are going to introduce the concepts of filtrations and bounded models (see [2]) in this new fuzzy framework in order to study decidability.

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Dually pseudocomplemented Heyting algebras: discriminator varieties

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In this talk we will see that, for certain classes of algebras with a Heyting algebra reduct, a variety is semisimple if and only if it is a discriminator variety. It is well-known that congruences on a Heyting algebra are in one-to-one correspondence with filters on the underlying lattice. If an algebra \mathbf{A} has a Heyting algebra reduct, it is of natural interest to characterise which filters still correspond to congruences on \mathbf{A} . Such a characterisation was given by Hasimoto [6]. When the filters can be sufficiently described by a single unary term, many useful properties come to life. The traditional example arises from boolean algebras with operators.

Definition 1.1. Let \mathbf{A} be a bounded lattice and let f be a unary operation on A . We say that f is an *operator* if $f(x \wedge y) = x \wedge y$, and f is *normal* if $f1 = 1$. More generally, let f be an n -ary operation on A . For all $a \in A$ and all $k \leq n$, let $f_k(a)$ be the $(n - 1)$ -ary map given by

$$f_k(a)(x_1, \dots, x_{n-1}) = f(x_1, \dots, x_{k-1}, a, x_k, \dots, x_{n-1}).$$

As a special case, let $f^k a = f_k(a)(0, \dots, 0)$. We say that f is an *operator on A* provided that, for all $k \leq n$ and all $x_1, \dots, x_{n-1} \in A$, if $y, z \in A$, then,

$$f_k(y \wedge z)(x_1, \dots, x_{n-1}) = f_k(y)(x_1, \dots, x_{n-1}) \wedge f_k(z)(x_1, \dots, x_{n-1}),$$

and f is *normal* if, for all $k \leq n$ and all $x_1, \dots, x_{n-1} \in A$, we have $f_k(1)(x_1, \dots, x_{n-1}) = 1$.

The reader is warned that, conventionally, the definition of an operator and a normal map is dual to the definition given here (see, for example, Goldblatt [5]). However, when the algebra of interest is a Heyting algebra, it turns out that meet-preserving operations are more natural than join-preserving operations. Here, we say that a *boolean algebra with operators* (BAO for short) is an algebra $\mathbf{B} = \langle B; M, \vee, \wedge, \neg, 0, 1 \rangle$ such that $\langle B; \vee, \wedge, \neg, 0, 1 \rangle$ is a boolean algebra, and M is a set of normal operators on \mathbf{B} . If \mathbf{B} is of finite type, then congruences on \mathbf{B} are determined by filters closed under the map t , where t is defined by

$$tx = \bigwedge \{f^k x \mid f \in M \text{ and } k \leq \text{arity}(f)\}.$$

For more detail, see, for example, Jipsen [7]. Hasimoto gave a construction which generalises the term above to Heyting algebras equipped with an arbitrary set of arbitrarily many operations (note that Hasimoto uses the word “operator” for an arbitrary unary operation). The construction does not apply in all cases, and even when it does, it does not guarantee that the result is a term function on the algebra. Having said that, natural constraints exist which guarantee both that the construction applies, and produces a term function.

Definition 1.2. We will say that an algebra $\mathbf{A} = \langle A; M, \vee, \wedge, \rightarrow, 0, 1 \rangle$ is an *expanded Heyting algebra* (EHA for short) if $\langle A; \vee, \wedge, \rightarrow, 0, 1 \rangle$ is a Heyting algebra, and M is an arbitrary set of operations on A . Let f be an n -ary operation on A and let $x \leftrightarrow y = (x \rightarrow y) \wedge (y \rightarrow x)$. We

say that a filter $F \subseteq A$ is *normal with respect to f* if the following implication is satisfied for all $x_1, y_1, \dots, x_n, y_n \in A$:

$$\{x_i \leftrightarrow y_i \mid i \leq n\} \subseteq F \implies f(x_1, \dots, x_n) \leftrightarrow f(y_1, \dots, y_n) \in F.$$

We say that F is a *normal filter (on \mathbf{A})* if F is normal with respect to f for every operation f on \mathbf{A} . For all $x \in A$ we will write $\text{Fg}^{\mathbf{A}}(x)$ for the normal filter generated by x . Let $\text{Fil}(\mathbf{A})$ denote the set of normal filters on \mathbf{A} . It is easily verified that $\text{Fil}(\mathbf{A})$ forms a complete lattice, and so we will let $\mathbf{Fil}(\mathbf{A})$ denote the lattice of normal filters on \mathbf{A} . For all $F \in \text{Fil}(\mathbf{A})$, let $\theta(F)$ be the equivalence relation defined by

$$\theta(F) = \{(x, y) \mid x \leftrightarrow y \in F\}.$$

Just as congruences on a Heyting algebra are determined by filters, congruences on an EHA are determined by normal filters.

Theorem 1.3 (Hasimoto [6]). *Let \mathbf{A} be an EHA. Then the map $\theta: \mathbf{Fil}(\mathbf{A}) \rightarrow \mathbf{Con}(\mathbf{A})$, defined by $F \mapsto \theta(F)$, is an isomorphism, with its inverse given by $\alpha \mapsto 1/\alpha$.*

In some special cases, we can simplify the description of normal filters.

Definition 1.4. Let \mathbf{A} be an EHA and let t be a unary term in the language of \mathbf{A} . We say that t is a *normal filter term (on \mathbf{A})* if,

1. $t^{\mathbf{A}}$ is order-preserving, and,
2. if F is a filter on \mathbf{A} , then F is a normal filter on \mathbf{A} if and only if F is closed under $t^{\mathbf{A}}$.

If \mathcal{K} is a class of EHAs with a common signature, and t is a term in the language of \mathcal{K} , we say that t is a *normal filter term on \mathcal{K}* if t is a normal filter term on \mathbf{A} for every $\mathbf{A} \in \mathcal{K}$. Note that, since the singleton set $\{1\}$ is a normal filter, a normal filter term t must always have $t^{\mathbf{A}}1 = 1$.

Henceforth, we will not take care to distinguish between terms and term functions. The map t for BAOs seen before is an example of a normal filter term. In general, if \mathbf{A} is an EHA and t is a normal filter term on \mathbf{A} , then the map $dx = x \wedge tx$ is also a normal filter term on \mathbf{A} with the further property that $dx \leq x$. We will then say that d is *descending*. An easy description of congruences via a descending normal filter term then allows a deeper investigation of congruence-related properties. In particular, we can characterise equationally definable principal congruences in a very straightforward manner.

Definition 1.5. A variety \mathcal{V} has *equationally definable principal congruences (EDPC)* if there exists a finite set of 4-ary terms $\{p_1, q_1, \dots, p_n, q_n\}$ such that, for all $\mathbf{A} \in \mathcal{V}$, and all $a, b, c, d \in A$, we have $(a, b) \in \text{Cg}^{\mathbf{A}}(c, d)$ if and only if $\mathbf{A} \models p_i(a, b, c, d) = q_i(a, b, c, d)$ for each $i \leq n$.

Theorem 1.6. *Let \mathcal{V} be a variety of EHAs and assume that d is a descending normal filter term on \mathcal{V} . Then \mathcal{V} has EDPC if and only if there exists $n \in \omega$ such that $\mathcal{V} \models d^{n+1}x = d^n x$.*

Varieties with EDPC were studied extensively by Blok, Köhler and Pigozzi [1–4, 8]. Closely connected to varieties with EDPC are discriminator varieties, and semisimple varieties.

Definition 1.7. A variety \mathcal{V} is *semisimple* if every subdirectly irreducible member of \mathcal{V} is simple. For all $\mathbf{A} \in \mathcal{V}$, a ternary term t in the language of \mathcal{V} is called a *discriminator term on \mathbf{A}* if the corresponding term function is the *discriminator function* on \mathbf{A} , i.e.,

$$t^{\mathbf{A}}(x, y, z) = \begin{cases} x & \text{if } x \neq y, \\ z & \text{if } x = y. \end{cases}$$

If there is a term t in the language of \mathcal{V} such that t is a discriminator term on every subdirectly irreducible member of \mathcal{V} , we say that \mathcal{V} is a *discriminator variety*.

It is well known, for instance, that every discriminator variety is semisimple (Werner [16]) and has EDPC (Blok and Pigozzi [2]). In the presence of congruence permutability, this characterises discriminator varieties. That is to say, a variety is a discriminator variety if and only if it is congruence permutable, semisimple and has EDPC (see Corollary 3.4 of Blok, Köhler and Pigozzi [1]). En route to our main result, we characterise discriminator varieties of *dually pseudocomplemented* EHAs with a normal filter term.

Definition 1.8. An algebra $\mathbf{A} = \langle A; M, \vee, \wedge, \rightarrow, 0, 1 \rangle$ is called a *dually pseudocomplemented EHA* if $\langle A; M, \vee, \wedge, \rightarrow, 0, 1 \rangle$ is an EHA and M contains a unary operation \sim which is a *dual pseudocomplement* operation on \mathbf{A} , i.e.,

$$x \vee y = 1 \iff y \geq \sim x.$$

A *dually pseudocomplemented Heyting algebra* is a dually pseudocomplemented EHA with $M = \{\sim\}$. Sankappanavar [13] proved directly that if \mathbf{A} is a dually pseudocomplemented Heyting algebra, then congruences are determined by filters closed under the operation $\neg\sim$. In our terminology, we say that $\neg\sim$ is a normal filter term on the class of dually pseudocomplemented Heyting algebras. If \mathbf{A} is a dually pseudocomplemented EHA, and t is a normal filter term on \mathbf{A} , then the map $dx = \neg\sim x \wedge tx$ is also a normal filter term on \mathbf{A} , with the additional property that $dx \leq \neg\sim x \leq x$. We will say then that d is *strongly descending*. Then, we can also characterise discriminator varieties in the presence of the dual pseudocomplement.

Theorem 1.9. *Let \mathcal{V} be a variety of dually pseudocomplemented EHAs and let d be a strongly descending normal filter term on \mathcal{V} . Then the following are equivalent:*

1. \mathcal{V} is a discriminator variety.
2. \mathcal{V} has EDPC and there exists $m \in \omega$ such that $\mathcal{V} \models x \leq d \sim d^m \neg x$.
3. There exists $n \in \omega$ such that $\mathcal{V} \models d^{n+1}x = d^n x$ and $\mathcal{V} \models d \sim d^n x = \sim d^n x$.

The proofs of Theorem 1.6 and Theorem 1.9 are not particularly deep. Our main result is that the conditions of Theorem 1.9 also characterise semisimple varieties. As mentioned, every discriminator variety is semisimple, but the converse direction is not as straightforward.

Theorem 1.10. *Let \mathcal{V} be a variety of dually pseudocomplemented EHAs and assume \mathcal{V} has a normal filter term. Then \mathcal{V} is a discriminator variety if and only if \mathcal{V} is semisimple.*

The argument is based on an argument by Kowalski and Kracht [11] proving the same characterisation for BAOs. The proof proceeds by assuming \mathcal{V} is semisimple, and then through a series of intermediate results, concludes that there exists $n \in \omega$ such that $\mathcal{V} \models d^{n+1}x = d^n x$. By Theorem 1.6, this implies that \mathcal{V} has EDPC. Courtesy of the underlying Heyting algebra, if \mathcal{V} is a variety of dually pseudocomplemented EHAs then \mathcal{V} is congruence permutable. So Blok, Köhler and Pigozzi's characterisation of discriminator varieties holds: \mathcal{V} is a discriminator variety if and only if \mathcal{V} is semisimple and has EDPC. It then follows that \mathcal{V} is a discriminator variety.

Kowalski and Kracht's result is a corollary of our main theorem. The present author also proved the same characterisation for double-Heyting algebras [15], which is also a corollary. Another example is the class of Heyting algebras with an involution (i.e., dual automorphism)

studied by Meskhi [12]. Let i be the involution operation. The dual pseudocomplement can be defined by $\sim x = i \neg i x$, and Meskhi proves that $tx := \neg i x$ is a normal filter term for these algebras. Since $t^2 x = \neg i \neg i x = \neg \sim x$, we have that $dx := t^2 x = \neg \sim x$ is a strongly descending normal filter term. Meskhi's investigation focused on the class of Heyting algebras with involution satisfying the identity $i \neg x = \neg \neg x$, denoted by HRI. One consequence of the identity is that $t^2 x = tx$, implying further that $d^2 x = dx$. It is also not hard to show that HRI satisfies $d \sim dx = \sim dx$. Meskhi's result that HRI is a discriminator variety is then a special case of Theorem 1.9 with $n = 1$.

By way of sufficiency conditions for normal filter terms, in this talk, we will also see some new cases for which the characterisation applies to. On the other hand, certain classes of residuated lattices have the same characterisation (Kowalski [9], Kowalski and Ferreirim [10], Takamura [14]), using a similar proof technique, but our main theorem does not apply to them. We believe that this is no coincidence, and further research will attempt to unite these results.

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Undecidability of some product modal logics

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Along the last years, the expansion of propositional fuzzy logics with modal operators and the development of the resulting systems has become a fertile field of study. One of the most interesting features of these logics is that they have a high expressive power while, in general, enjoying a lower level of complexity than predicate fuzzy logics. For this, studies on the complexity of the different problems that can be formulated over it gain a particular interest. In general, there are several ways to expand a particular fuzzy logic with the usual modal operators expressing necessity (\Box) and possibility (\Diamond) like notions, mainly due to two reasons. First, in fuzzy modal logic the modal operators \Box and \Diamond are not in general inter-definable (as it happens in the classical case) and, therefore, it makes sense to consider modal expansions of a given fuzzy logic either with \Box or with \Diamond or both with \Box and \Diamond . Second, the relational semantics for fuzzy modal logics can be defined in terms of crisp or fuzzy accessibility relation and this gives rise to different modal logics over the same fuzzy logic. Special attention has been devoted to modal expansion of the fuzzy logics associated with the basic continuous t-norms: Łukasiewicz modal logics [9, 8], Gödel modal logics [4, 5] and Product modal logics [11]. For further studies on modal expansions of t-norm based logics see [2, 10]. The study of decision problems in fuzzy modal logic has focused on Gödel-style logics [3] and fuzzy description logics [1, 6, 7]. In the current contribution we present some undecidability results of the logical consequence relation over different modal logics with both \Box and \Diamond operators expanding the usual Product logic.

Let us first formally define the logics of our study. The set of formulas Fm is build, as usual, from a countable set of (propositional) variables \mathcal{V} using the operations $\langle \{\bar{0}, \bar{1}\}/0, \{\neg, \Box, \Diamond\}/1, \{\rightarrow, \& \}/2 \rangle$, where \star/n denotes that operation \star is of arity n . A **(crisp) product Kripke model** \mathfrak{M} is a triple $\langle W, R, e \rangle$ where: W is a non-empty set (worlds), $R \subseteq W \times W$ is an accessibility relation in W^1 and $e : W \times \mathcal{V} \rightarrow [0, 1]$ is an evaluation of the propositional variables at each world. Evaluation e is uniquely extended to Fm by interpreting the propositional operations as their corresponding ones in the standard product algebra² and letting

$$e(u, \Box\varphi) = \inf\{e(v, \varphi) : Ruv\} \quad e(u, \Diamond\varphi) = \sup\{e(v, \varphi) : Ruv\}$$

Let \mathbb{K}_Π denote the class of all product Kripke models. For a class of product Kripke models \mathbb{C} , we say that a formula φ follows from a set of premises Γ in the **local modal logic** arising from \mathbb{C} , and write $\Gamma \vdash_{\mathbb{C}} \varphi$, whenever for each $\mathfrak{M} \in \mathbb{C}$ and each $u \in W$ (of \mathfrak{M}), if $e(u, \Gamma) \subseteq \{1\}$ then $e(u, \varphi) = 1$. We say that φ follows from Γ in the **global modal logic** arising from \mathbb{C} , and write $\Gamma \Vdash_{\mathbb{C}} \varphi$, whenever for each $\mathfrak{M} \in \mathbb{C}$, if for all $u \in W$ it holds that $e(u, \Gamma) \subseteq \{1\}$, then for all $u \in W$ $e(u, \varphi) = 1$. We will focus on the following modal logics based on the previous definitions. For $\Gamma, \varphi \subseteq Fm$ we write:

- $\Gamma \Vdash \varphi$ whenever $\Gamma \Vdash_{\mathbb{K}_\Pi} \varphi$,
- $\Gamma \Vdash^f \varphi$ whenever $\Gamma \Vdash_{\{\mathfrak{M} \in \mathbb{K}_\Pi : W \text{ is finite}\}} \varphi$,

¹We write Ruv instead of $\langle v, w \rangle \in R$

² $\&$ corresponds to the usual real product in $[0, 1]$, \rightarrow to its corresponding residuated operation and $\neg p$ equals $p \rightarrow 0$

- $\Gamma \vdash_4 \varphi$ whenever $\Gamma \vdash \{\mathfrak{M} \in \mathbb{K}_H : R \text{ is transitive}\} \varphi$,
- $\Gamma \vdash_4^f \varphi$ whenever $\Gamma \vdash \{\mathfrak{M} \in \mathbb{K}_H : R \text{ is transitive and } W \text{ is finite}\} \varphi$.

We prove that all four previous consequence relations are undecidable, by reducing the well-known Post correspondence problem to each one of them. Both for the global case and for the transitive local case, we approach the problem in general (without restricting to finite structures), but the same reduction can be shown to work for finite structures.

Lemma

1. The problem of determining, for arbitrary $\Gamma, \varphi \subseteq Fm$ whether $\Gamma \Vdash \varphi$ is undecidable.
2. The problem of determining, for arbitrary $\Gamma, \varphi \subseteq Fm$ whether $\Gamma \Vdash^f \varphi$ is undecidable.
3. The problem of determining, for arbitrary $\Gamma, \varphi \subseteq Fm$ whether $\Gamma \vdash_4 \varphi$ is undecidable.
4. The problem of determining, for arbitrary $\Gamma, \varphi \subseteq Fm$ whether $\Gamma \vdash_4^f \varphi$ is undecidable.

In what follows, we will sketch the proof of the previous statements.

Recall that an instance of the Post Correspondence Problem (PCP) is a list $\{\langle \mathbf{v}_1, \mathbf{w}_1 \rangle, \dots, \langle \mathbf{v}_m, \mathbf{w}_m \rangle\}$ of numbers in base $s \in \mathbb{N}^+$, and a solution for it is a finite list i_1, \dots, i_k with $i_j \in \{1, \dots, m\}$ such that $\mathbf{v}_{i_1} \dots \mathbf{v}_{i_k} = \mathbf{w}_{i_1} \dots \mathbf{w}_{i_k}$ ³. Given an instance of the PCP we can define two particularly interesting finite sets of formulas, $\Gamma_G \cup \{\varphi_G\}$ and $\Gamma_L \cup \{\varphi_L\}$ using only the variables $\mathcal{V}_P = \{x, y, z, v, w\}$. We denote by $\|\mathbf{u}\|$ to the length of number \mathbf{u} (in base s), and φ^k stands, as usual, for the formula $\varphi \& \varphi^{k-1}$, with $\varphi^0 = \top$.

Concerning the global deduction, consider the following set of formulas:

Formulas of Γ_G	Intended meaning
$(\neg \Box \bar{0}) \rightarrow (\Box p \leftrightarrow \Diamond p)$ for each $p \in \mathcal{V}$	[If a world has successors, p takes the same value in all of them, for all $p \in \mathcal{V}_P$]
$\neg \neg y, \neg \neg z, \neg \neg v$	$[y, z, v > 0]$
$\neg \Box \bar{0} \rightarrow ((y \leftrightarrow \Box y) \wedge (z \leftrightarrow \Box z))$	[If a world has successors then y has the same value at this world than in all the successors (when also the above formulas are true). Same for z .]
$\bigvee_{1 \leq i \leq m} (x \leftrightarrow z^i)$	$[x = z \text{ or } x = z^2 \text{ or } \dots \text{ or } x = z^m]$
$(x \leftrightarrow z^i) \rightarrow (v \leftrightarrow (\Box v)^{s^{\ \mathbf{v}_i\ }} \& y^{\mathbf{v}_i})$ for $1 \leq i \leq m$	[If $x = z^i$ then $v = (\Box v)^{s^{\ \mathbf{v}_i\ }} \& y^{\mathbf{v}_i}$]
$(x \leftrightarrow z^i) \rightarrow (w \leftrightarrow (\Box w)^{s^{\ \mathbf{w}_i\ }} \& y^{\mathbf{w}_i})$ for $1 \leq i \leq m$	[If $x = z^i$ then $w = (\Box w)^{s^{\ \mathbf{w}_i\ }} \& y^{\mathbf{w}_i}$]
$\varphi_G : (v \leftrightarrow w) \rightarrow z \vee y$	[If $v = w$, then either $z = 1$ or $y = 1$]

It is possible to check that if $\Gamma_G \not\Vdash \varphi_G$, then there is a product Kripke model \mathfrak{M} with the structure of Figure 1 such that:

1. \mathfrak{M} globally satisfies Γ_G , $e(u_k, \varphi_G) < 1$, and this implies that $e(u_k, v) = e(u_k, w)$,
2. y and z take the same value (α_y and α_z) in all worlds of the model, and $0 < \alpha_y, \alpha_z < 1$,
3. x equals z^j for some $1 \leq j \leq m$ at each world of the model,
4. For $1 \leq j \leq k$, $e(u_j, v) = \alpha_y^{\mathbf{v}_{i_1} \dots \mathbf{v}_{i_j}}$, where $i_n = r$ for $e(u_n, x) = e(u_n, z)^r$. The analogous thing happens for $e(u_j, w)$.

³The concatenation of numbers.



Figure 1: Frame for the Global logic proof

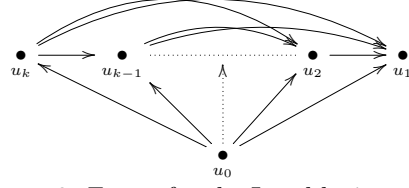


Figure 2: Frame for the Local logic proof

Using the previous results, and given that \mathfrak{M} is finite and that the converse direction follows easily we can prove that

$$P \text{ is satisfiable} \iff \Gamma_G \not\models \varphi_G \iff \Gamma_G \not\models_f \varphi_G.$$

Concerning the local case, we can instead consider a set of formulas that will lead to a similar result, but taking into account that we need to address the behavior at each world of the model from just one point (since the deduction is local). This is achieved partially due to the fact that we will assume transitivity.

Formulas of Γ_L	Intended meaning in the evaluation world
$\Box y \leftrightarrow \Diamond y, \Box z \leftrightarrow \Diamond z$	[The world has successors and in all of them y (and z) take the same value]
$\neg\neg\Box y, \neg\neg\Box z, \neg\neg\Box v$	[In all the successor worlds, $y, z, v > 0$]
$\Diamond\Box\bar{0},$ $\Box(\Box\bar{0} \& x) \leftrightarrow \Diamond(\Box\bar{0} \& x)$	[There is some successor with no related worlds, and in all these worlds, x takes the same value]
$\Box(\bigvee_{1 \leq i \leq m} (x \leftrightarrow z^i))$	[In each successor world $x = z$ or $x = z^2$ or \dots $x = z^m$]
$\Box((x \leftrightarrow z^i) \rightarrow (v \leftrightarrow (\Box v)^{s^{\ v_i\ }} \& y^{v_i}))$ for $1 \leq i \leq m$	[In each successor, if $x = z^i$ then $v = (\Box v)^{s^{\ v_i\ }} \& y^{v_i}$]
$\Box((x \leftrightarrow z^i) \rightarrow (w \leftrightarrow (\Box w)^{s^{\ w_i\ }} \& y^{w_i}))$ for $1 \leq i \leq m$	[In each successor, if $x = z^i$ then $w = (\Box w)^{s^{\ w_i\ }} \& y^{w_i}$]
$\Box(\Box(v \& w) \rightarrow (\Box v \& \Box w))$	[In each successor world $\Box(v \& w) \leq \Box v \& \Box w$ (and in fact, coincide).]
$\varphi_L : \Box((v \leftrightarrow w) \rightarrow (y \vee z))$	[In all successors, if $v = w$ then $y = 1$ or $z = 1$. Moreover, if $\Box\varphi_L$ is < 1 then $z, y < 1$ in all successors.]

The proof that this formulas suffice to prove a reduction of a PCP instance to a local deduction is more involved, since proving that the resulting Kripke model has the desired shape is more complicated, but the concepts are essentially very similar. It is again possible to check that if $\Gamma_L \not\models_4 \varphi_L$ then there is a transitive product Kripke model \mathfrak{M} with the structure of Figure 2 such that $e(u_0, \Gamma_L) = \{1\}$ and $e(u_0, \varphi_L) < 1$. This implies that $e(u_j, z), e(u_j, y) < 1$ for all $1 \leq j \leq k$, and so all the characteristics listed for the global case are preserved. In particular, $e(u_k, v) = e(u_k, w)$, which gives us a solution for the PCP instance as it happened before. Since again this model is finite, and the converse direction can be proven with some calculus, we get that

$$P \text{ is satisfiable} \iff \Gamma_L \not\models_4 \varphi_L \iff \Gamma_L \not\models_4^f \varphi_L.$$

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