Fuzzy Logic and Algebra
An overview

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Overview

1. Introduction
   - T-norms

2. BL logic

3. Three important systems
   - Göedel Logic
   - Product Logic
   - Łukasiewicz Logic

4. Advanced Topic
   - Fixed points
   - $\mu\mathcal{LP}$ logic
The mathematical core of Fuzzy Logic

- Fuzzy Logic has undoubtedly gained an important role in engineering and industry.
  This is due to its **flexibility** and **feasibility**.
Fuzzy Logic has undoubtedly gained an important role in engineering and industry. This is due to its **flexibility** and **feasibility**. But it lacks a solid **mathematical background**. The aim is to give strong mathematical/logical foundations. To this end we start back from the core of the logic.
Starting with the connectives

What kind of assumptions have to be made?

- We want to **generalize** classical logic, expanding its set of truth values.
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- The conjunction has to be: **commutative, associative** and **non decreasing** in both arguments.
Starting with the connectives

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- We want to generalize classical logic, expanding its set of truth values.
- The conjunction has to be: commutative, associative and non decreasing in both arguments.
- If we want a Logic the conjunction needs to be related with the implication.
T-norms and their residua

**Definition**

A *t-norm* $\ast$ is a function from $[0, 1]^2$ to $[0, 1]$ that is

- $1 \ast x = x$ and $x \ast 0 = 0$
- associative and commutative
- non-decreasing in both argument, i.e. $x_1 \leq x_2$ implies $x_1 \ast y \leq x_2 \ast y$ and $x_1 \leq x_2$ implies $y \ast x_1 \leq y \ast x_2$

Let $\ast$ be a continuous t-norm. The unique operation $x \Rightarrow y$ satisfying the following condition:

$$(x \ast z) \leq y \text{ if and only if } z \leq (x \Rightarrow y)$$

namely:

$$x \Rightarrow y = \max\{z \mid x \ast z \leq y\}$$

is called the **residuum** of $\ast$
T-norms and their residua

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Let \( \ast \) be a continuous t-norm. The unique operation \(x \Rightarrow y\) satisfying the following condition:

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namely: \(x \Rightarrow y = \max\{z \mid x \ast z \leq y\}\) is called the \textit{residuum} of \(\ast\)
Examples

- Łukasiewicz t-norm: $x \ast y = \max\{0, x + y - 1\}$; and its residuum $x \Rightarrow y = \min\{1, 1 - x + y\}$
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- Gödel t-norm: $x \ast y = \min\{x, y\}$; and its residuum $x \Rightarrow y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$

Remark. The above three functions form a complete system in the sense that every other t-norm is locally isomorphic to them.
Examples

- Łukasiewicz t-norm: \( x \ast y = \max\{0, x + y - 1\} \);
  and its residuum \( x \Rightarrow y = \min\{1, 1 - x + y\} \)

- Gödel t-norm: \( x \ast y = \min\{x, y\} \); and its residuum
  \[ x \Rightarrow y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases} \]

- Product t-norm: \( x \ast y = x \cdot y \); and its residuum
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If we fix a t-norm, we fix a **logic system**, letting the t-norm as the truth function of the conjunction.
Back to the logic

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**Definition**

The *propositional calculus* PC(*) has propositional variables $p_1, \ldots, p_n, \ldots$, connectives $\&$ and $\rightarrow$. Formulas are built as usual. Further connectives are defined:

- $\neg \varphi = \varphi \rightarrow 0$
- $\varphi \leftrightarrow \psi = (\varphi \rightarrow \psi) \& (\psi \rightarrow \varphi)$
- $\varphi \land \psi = \varphi \& (\varphi \rightarrow \psi)$
- $\varphi \lor \psi = (\varphi \rightarrow \psi) \rightarrow \psi \land \neg (\psi \rightarrow \varphi) \rightarrow \varphi$
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Evaluation

Definition

An evaluation \( e \) is a function from propositional variables to \([0, 1]\).
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- $e(0) = 0$
- $e(\varphi \& \psi) = e(\varphi) \ast e(\psi)$
- $e(\varphi \rightarrow \psi) = e(\varphi) \Rightarrow e(\psi)$
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An evaluation \( e \) is a function from propositional variables to \([0, 1]\) . It extends in a unique way to formulas according to the following constraints:

- \( e(0) = 0 \)
- \( e(\varphi \& \psi) = e(\varphi) * e(\psi) \)
- \( e(\varphi \rightarrow \psi) = e(\varphi) \Rightarrow e(\psi) \)

Definition
A formula \( \varphi \) of PC(\( \ast \)) is a 1-tautology iff for any evaluation one has \( e(\varphi) = 1 \)
The aim, then, is to find a **calculus** for this system.
BL logic

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Definition

The following are the axioms of BL Logic

- \((\phi \rightarrow \psi) \rightarrow ((\psi \rightarrow \theta) \rightarrow (\phi \rightarrow \theta))\)
- \((\phi \& \phi) \rightarrow \phi\)
- \((\phi \& \psi) \rightarrow (\psi \& \phi)\)
- \((\phi \& (\phi \rightarrow \psi)) \rightarrow (\psi \& (\psi \rightarrow \phi))\)
- \((\phi \rightarrow (\psi \rightarrow \theta)) \rightarrow ((\phi \& \psi) \rightarrow \theta)\)
- \(((\phi \rightarrow \psi) \rightarrow \theta) \rightarrow (((\psi \rightarrow \phi) \rightarrow \theta) \rightarrow \theta)\)
- \(0 \rightarrow \phi\)
Definition

A residuated lattice is a structure $\mathcal{A} = \langle A, *, \Rightarrow, \wedge, \vee, 0, 1 \rangle$
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- $\langle A, \wedge, \vee, 0, 1 \rangle$ is a lattice with greatest and least element being respectively 1 and 0
- $\langle A, \ast, 1 \rangle$ is a commutative monoid
- $\ast$ and $\Rightarrow$ form an adjoint pair, i.e. $z \leq (x \Rightarrow y)$ iff $x \ast z \leq y$
**BL algebras**

**Definition**

A **residuated lattice** is a structure $\mathcal{A} = \langle A, *, \Rightarrow, \land, \lor, 0, 1 \rangle$

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**Definition**

A residuated lattice $\mathcal{A} = \langle A, *, \Rightarrow, \land, \lor, 0, 1 \rangle$ is a **BL algebra** if it satisfies

- $x \land y = x \ast (x \Rightarrow y)$ (divisibility)
- $(x \Rightarrow y) \lor (y \Rightarrow x) = 1$ (pre-linearity)
Lindenbaum-Tarski algebra

**Definition**

Let $T$ be a theory over BL. For each formula $\varphi$ let $[\varphi]_T$ be the set of formula $\psi$ such that $T \vdash \psi \leftrightarrow \varphi$.

$[0]_T = \bot$, $[1]_T = \top$, $[\varphi \& \psi]_T = [\varphi]_T \cap [\psi]_T$, $[\varphi \rightarrow \psi]_T = [\neg \varphi]_T \cup [\psi]_T$, $[\varphi \vee \psi]_T = [\varphi]_T \cup [\psi]_T$.

This algebra will be denoted as $LT$. 

**Lemma**

$LT$ is a BL algebra.
Lindenbaum-Tarski algebra

Definition

Let $T$ be a theory over BL. For each formula $\varphi$ let $[\varphi]_T$ be the set of formula $\psi$ such that $T \vdash \psi \leftrightarrow \varphi$. Then define

- $0 = [0]_T$
- $1 = [1]_T$
- $[\varphi]_T \star [\psi]_T = [\varphi \& \psi]_T$
- $[\varphi]_T \Rightarrow [\psi]_T = [\varphi \rightarrow \psi]_T$
- $[\varphi]_T \cap [\psi]_T = [\varphi \land \psi]_T$
- $[\varphi]_T \cup [\psi]_T = [\varphi \lor \psi]_T$

This algebra will be denoted as $L_T$
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- $[\varphi]_T \cup [\psi]_T = [\varphi \lor \psi]_T$

This algebra will be denoted as $L_T$

Lemma

$L_T$ is a BL algebra
Some Lemma

Definition

Given a lattice $L$, a filter $F$ is a non-empty subset of $L$ s.t.

- If $a, b \in F$ then $a \cap b \in F$
- If $a \in F$ and $a \leq b$ then $b \in F$

A filter is said to be prime if for any $x, y \in L$ either $(x \Rightarrow y) \in F$ or $(y \Rightarrow x) \in F$
Lemma

Let $L$ be a BL algebra and $F$ a filter. Let $x \sim_F y$ if, and only if, $(x \Rightarrow y) \in F$ and $(y \Rightarrow x) \in F$ then

- $\sim_F$ is a congruence and the corresponding quotient $L/\sim_F$ is a BL algebra
- $L/\sim_F$ is linearly ordered iff $F$ is prime
Some Lemma

Lemma

Let $L$ be a BL algebra and $F$ a filter. Let $x \sim_F y$ if, and only if, $(x \Rightarrow y) \in F$ and $(y \Rightarrow x) \in F$ then

- $\sim_F$ is a congruence and the corresponding quotient $L/\sim_F$ is a BL algebra
- $L/\sim_F$ is linearly ordered iff $F$ is prime

Lemma

Let $L$ be a BL algebra and $a \in L$, with $a \neq 1$, then there is a prime filter not containing $a$
Completeness

Theorem

Every BL algebra is the subdirect product of linearly ordered BL algebras
Completeness

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Theorem

BL algebras are the algebraic semantic for BL logic. Thus a formula $\varphi$ is provable in the logic BL if, and only if, it holds in every BL algebra
**Completeness**

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Summing up

BL logic is hence important for two reasons

1. It gives us a formal system to prove properties that are common to all t-norms.
Summing up

BL logic is hence important for two reasons

1. It gives us a formal system to prove properties that are common to all t-norms.

2. It generalizes the above mentioned three most important t-norm based logics. Indeed one can rescue any of the three logical systems just by adding one axiom to BL.
The system $G$

**Reminder**

Gödel t-norm is defined as:

$$x \ast y = \min\{x, y\}$$

and its residuum

$$x \Rightarrow y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$$
The system $G$

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$$x \Rightarrow y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$$

Definition

A Gödel algebra is a BL algebra that satisfies the following axiom

$$x = x \ast x$$
The system $G$

**Theorem (Completeness)**

The Logic $G$ is sound and complete w.r.t the class of Heiting algebras satisfying prelinearity.
The system G

Theorem (Completeness)

The Logic G is sound and complete w.r.t the class of Heiting algebras satisfying prelinearity.

Theorem (Standard Completeness)

The G is standard complete. In other words, a formula \( \varphi \) is true in \([0, 1]_G\) if, and only if, it can be proved in G.
The system $\Pi$

Reminder

*Product t-norm is defined as:*

$$x \ast y = x \cdot y$$

*and its residuum*

$$x \Rightarrow y = \begin{cases} 1 & \text{if } x \leq y \\ y/x & \text{otherwise} \end{cases}$$
The system $\Pi$

Reminder

*Product t-norm is defined as:*

$$x * y = x \cdot y$$

*and its residuum*

$$x \Rightarrow y = \begin{cases} 1 & \text{if } x \leq y \\ y/x & \text{otherwise} \end{cases}$$

Definition

A $\Pi$ algebra is a BL algebra that satisfies the following axiom

$$(y \Rightarrow 0) \lor ((y \Rightarrow x * y) \Rightarrow x)$$
The system $\Pi$

**Theorem (Completeness)**

*The Logic $\Pi$ is sound and complete w.r.t the class of $\Pi$ algebras.*
The system $\Pi$

Theorem (Completeness)

*The Logic $\Pi$ is sound and complete w.r.t the class of $\Pi$ algebras.*

Theorem (Standard Completeness)

*The $\Pi$ is standard complete. In other words, whenever a formula $\varphi$ is true in $[0, 1]_\Pi$ it can be proved in $\Pi$.***
The system Ł

Reminder

Łukasiewicz t-norm is defined as:

\[ x \ast y = \max \{0, x + y - 1\} \]

and its residuum

\[ x \Rightarrow y = \min \{1, 1 - x + y\} \]
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Definition (old style)
Łukasiewicz Logic has the following axioms:

- \( \varphi \rightarrow (\psi \rightarrow \varphi) \)
- \( (\varphi \rightarrow \theta) \rightarrow (\theta \rightarrow \psi) \rightarrow (\varphi \rightarrow \psi) \)
- \( (\neg \varphi \rightarrow \neg \psi) \rightarrow (\psi \rightarrow \varphi) \)
- \( (((\varphi \rightarrow \psi) \rightarrow \psi) \rightarrow (((\psi \rightarrow \varphi) \rightarrow \varphi)) \)
**Definition**

A Łukasiewicz algebra (bka. MV algebra or Wejsbergh algebra, or ...) is a BL algebra that satisfies the following axiom

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Theorem (Completeness)

The Logic Ł is sound and complete w.r.t the class of MV algebras.
MV algebra

**Definition**

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**Theorem (Completeness)**

The Logic Ł is sound and complete w.r.t the class of MV algebras.

**Theorem (Standard Completeness)**

The calculus Ł is standard complete. In other words, a formula φ is true in \([0, 1]_L\) if, and only if, it can be proved in Ł.
## Definition

A **lattice ordered group** with a **strong unit** is a structure \( G = \langle G, +, -, \lor, \land, 0, 1 \rangle \) such that:

1. \( \langle G, +, -, 0 \rangle \) is an abelian group
2. \( \langle G, \lor, \land \rangle \) is a lattice
3. If \( \leq \) denotes the partial order given by \( \land, \lor \), then: if \( x \leq y \) then \( x + z \leq y + z \)
4. For any \( x \in G \) there is \( n \in \mathbb{N} \) such that \( 1 + \ldots + 1 \geq x \) \( n \) times
Results about $\mathcal{L}$

Definition

A lattice ordered group with a strong unit is a structure $G = \langle G, +, -, \vee, \wedge, 0, 1 \rangle$ such that:

- $\langle G, +, -, 0 \rangle$ is an abelian group
- $\langle G, \vee, \wedge \rangle$ is a lattice
- If $\leq$ denotes the partial order given by $\wedge, \vee$ then: if $x \leq y$ then $x + z \leq y + z$
Results about Ł

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A lattice ordered group with a strong unit is a structure 
\( \mathcal{G} = \langle G, +, -, \lor, \land, 0, 1 \rangle \) such that:

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- $\langle G, \lor, \land \rangle$ is a lattice
- If $\leq$ denotes the partial order given by $\land, \lor$ then: if $x \leq y$ then $x + z \leq y + z$
- For any $x \in G$ there is $n \in \mathbb{N}$ such that $1 + \ldots + 1 \geq x$ $n$ times

**Theorem (Representation)**

*There is a categorical equivalence between lattice ordered groups with strong unit and MV algebras.*
Summing up

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2. There are important links with other fields of mathematics
Summing up

1. Connectives can be interpreted (in some case) as continuous functions.
2. There are important links with other fields of mathematics.
3. These links are important to prove standard completeness but they are also interesting in their own.
Fixed points

Our next aim is to introduce fixed point operators in some of the systems seen above. This can be done in two different way:

1. Use known result about Kripke-style semantic for the main t-norm based logic and introduce fixed points like in $\mu$-calculus.
2. Take advantage from the semantic given by continuous t-norms and their residua and use Brouwer theorem to guarantee the existence of fixed points for any formula.

(To start with) we chose the most expressive among t-norm based logic.
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(To start with) we chose the *most expressive* among t-norm based logic
The system ŁΠ

Definition

The Logic ŁΠ is axiomatized as following

- All the axioms of Ł

Theorem

ŁΠ logic faithfully interprets Ł, Π, and G.
The system ŁΠ

Definition

The Logic ŁΠ is axiomatized as following

- All the axioms of Ł
- All the axioms of Π
The system $\mathcal{L}\Pi$

Definition

The Logic $\mathcal{L}\Pi$ is axiomatized as following

- All the axioms of $\mathcal{L}$
- All the axioms of $\Pi$
- $\varphi \&\Pi (\psi \ominus \theta) \leftrightarrow_{L} (\varphi \&\Pi \psi) \ominus (\varphi \&\Pi \theta)$

Theorem $\mathcal{L}\Pi$ logic faithfully interprets $\mathcal{L}$, $\Pi$, and $G$. 

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The system $\mathcal{LP}$

**Definition**

The Logic $\mathcal{LP}$ is axiomatized as following:

- All the axioms of $\mathcal{L}$
- All the axioms of $\Pi$
- $\varphi \&_\Pi (\psi \Theta \theta) \leftrightarrow_{\mathcal{L}} (\varphi \&_\Pi \psi) \Theta (\varphi \&_\Pi \theta)$
- $\Delta (\varphi \rightarrow_{\mathcal{L}} \psi) \rightarrow_{\mathcal{L}} (\varphi \rightarrow_{\Pi} \psi)$
The system $\mathcal{LP}$

Definition

The Logic $\mathcal{LP}$ is axiomatized as following

- All the axioms of $\mathcal{L}$
- All the axioms of $\Pi$
- $\varphi \&_\Pi (\psi \ominus \theta) \leftrightarrow_\mathcal{L} (\varphi \&_\Pi \psi) \ominus (\varphi \&_\Pi \theta)$
- $\Delta (\varphi \rightarrow_\mathcal{L} \psi) \rightarrow_\mathcal{L} (\varphi \rightarrow_\Pi \psi)$
- The rules *Modus Ponens* and *Necessitation* $\varphi \over \Delta(\varphi)$
The system $\mathcal{LP}$

**Definition**

The Logic $\mathcal{LP}$ is axiomatized as following

- All the axioms of $\mathcal{L}$
- All the axioms of $\Pi$
- $\varphi \& \Pi (\psi \Theta \theta) \leftrightarrow L (\varphi \& \Pi \psi) \Theta (\varphi \& \Pi \theta)$
- $\Delta (\varphi \rightarrow L \psi) \rightarrow L (\varphi \rightarrow \Pi \psi)$
- The rules *Modus Ponens* and *Necessitation* $\frac{\varphi}{\Delta (\varphi)}$

**Theorem**

$\mathcal{LP}$ logic faithful interprets $\mathcal{L}$, $\Pi$ and $G$. 

ŁΠ with fixed points

Definition

The Fixed Point ŁΠ Logic for short) has the following theory:

1. All axioms and rules from ŁΠ Logic
2. \( \mu x. \phi(x) \leftrightarrow \phi(\mu x. \phi(x)) \)
3. If  \( \phi(p) \leftrightarrow p \) then  \( \mu x. \phi(x) \rightarrow p \)
4. If  \( \bigwedge_{i \leq n} (p_i \leftrightarrow q_i) \) then  \( \mu x. \phi(p_1, \ldots, p_n) \leftrightarrow \mu x. \phi(q_1, \ldots, q_n) \)
Definition

The Fixed Point $\mathcal{LP}$ Logic for short) has the following theory:

1. All axioms and rules from $\mathcal{LP}$ Logic
ŁΠ with fixed points

Definition
The Fixed Point ŁΠ Logic for short) has the following theory:

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2. $\mu x. \varphi(x) \leftrightarrow \varphi(\mu x. \varphi(x))$
ŁΠ with fixed points

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The Fixed Point ŁΠ Logic for short) has the following theory:

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2. $\mu x.\varphi(x) \leftrightarrow \varphi(\mu x.\varphi(x))$
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2. \( \mu x.\varphi(x) \leftrightarrow \varphi(\mu x.\varphi(x)) \)
3. If \( \varphi(p) \leftrightarrow p \) then \( \mu x.\varphi(x) \rightarrow p \)
4. If \( \bigwedge_{i \leq n} (p_i \leftrightarrow q_i) \) then \( \mu x.\varphi(p_1, ..., p_n) \leftrightarrow \mu x.\varphi(q_1, ..., q_n) \)
Results on $\mathcal{L}\Pi$ with fixed points

**Theorem**

*Every linearly ordered $\mu\mathcal{L}\Pi$ algebra is isomorphic to the interval algebra of some real closed field.*
Results on $\mu \mathcal{L} \mathcal{P}$ with fixed points

**Theorem**

*Every linearly ordered $\mu \mathcal{L} \mathcal{P}$ algebra is isomorphic to the interval algebra of some real closed field.*

**Theorem**

*$\mu \mathcal{L} \mathcal{P}$ is standard complete, i.e. a formula $\varphi$ is a $\mu \mathcal{L} \mathcal{P}$ tautology if, and only if, it is true on the $\mu \mathcal{L} \mathcal{P}$ algebra on $[0, 1]$.*
Results on $\mathcal{L}\Pi$ with fixed points

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*Every linearly ordered $\mu\mathcal{L}\Pi$ algebra is isomorphic to the interval algebra of some real closed field.*

**Theorem**

*$\mu\mathcal{L}\Pi$ is standard complete, i.e. a formula $\varphi$ is a $\mu\mathcal{L}\Pi$ tautology if, and only if, it is true on the $\mu\mathcal{L}\Pi$ algebra on $[0, 1]$*

**Theorem**

*The category of $\mu\mathcal{L}\Pi$ algebras and the category of subdirect products of real closed fields are equivalent.*
Suggested reading
