

# $\mu$ MV-algebras: an approach to fixed points in Łukasiewicz logic

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## Abstract

We study an expansion of MV-algebras, called  $\mu$ MV-algebras, in which minimal and maximal fixed points are definable. The first result is that  $\mu$ MV-algebras are term-wise equivalent to divisible  $MV_{\Delta}$ -algebras, i.e. a combination of two known MV-algebras expansion: divisible MV-algebras and  $MV_{\Delta}$ -algebras. Using methods from the two known extensions we derive a number of results about  $\mu$ MV-algebras; among others: subdirect representation, standard completeness, amalgamation property and a description of the free algebra.

*Key words:* Fixed Point, MV-algebras, Many-Valued Logic

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## 1 Introduction

In a recent work (24) we studied an expansion of LII logic in which fixed points of  $\rightarrow_{\Pi}$ -free formulas are definable.

Although a number of interesting results can be established for this logic, the restriction to particular formulas does not seem to be extremely natural. Since the discriminant property of the other connectives is the interpretability as continuous functions in the standard algebra, we decide to address to any extension of BL whose whole set of connectives has a continuous interpretation. As a matter of fact, a simple inspection on the structure of ordinal sums of linearly ordered BL algebras (1) leads to the conclusion that the only logic with this property is Łukasiewicz logic.

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During our research a deep understanding of the algebraic semantic of this logic turned out to be crucial. With this experience in mind we approach the study of fixed points in Łukasiewicz logic by directly introducing a particular class of algebras called  $\mu$ MV-algebras.

The classical approach to fixed points for propositional logics stands on their Kripke semantic (cf.  $\mu$ -calculus (14)), as a matter of fact such a kind of semantics exists and has been studied for several important t-norm based logics (3; 20). Nevertheless our approach to fixed points differs from the classical one: in order to give a semantic to the  $\mu$  operator we rather reaped benefit from the functional semantics of many valued logics.

In several cases many valued connectives can be considered as continuous functions from  $[0, 1]^n$  to  $[0, 1]$ , therefore, once all variables but one are fixed, they can be seen as function in only one variable and the existence of their fixed point in  $[0, 1]$  is guaranteed by Brouwer Theorem:

**Theorem 1.1 (Brouwer 1909)** *Every continuous function from the closed unit ball  $D^n$  to itself has a fixed point.*

In Łukasiewicz logic all connectives are continuous so we can safely use this approach. The drawback is that with this method any formula has fixed points, whereas in classical cases, one has to restrict to formulas on which the variable under the scope of  $\mu$  only appears positively. On the other hand, the function giving the fixed point of a formula need not to be continuous in the remaining variables, whence we can not allow every kind of nested occurrences of  $\mu$ .

## 2 Preliminaries

Łukasiewicz logic was introduced by Łukasiewicz and Tarski in (15) as an infinite-valued generalization of classical propositional logic.

Łukasiewicz logic can be also seen as a member of the family of fuzzy logics based on triangular norms (12) (i.e. binary, associative, commutative, monotone operations over  $[0, 1]$  having 1 as a neutral element). Indeed, as shown in (12), Łukasiewicz logic is complete w.r.t. the residuated lattice  $\langle [0, 1], \odot, \rightarrow, \min, \max, 0, 1 \rangle$  in which the monoidal operation  $\odot$  corresponds to the Łukasiewicz t-norm  $x \odot y = \max(x + y - 1, 0)$ , and  $x \rightarrow y = \min(1 - x + y, 1)$  is the residual implication of  $\odot$ .

Łukasiewicz logic was originally shown to be complete by Rose and Rosser in (23), and, independently, by Chang in (4; 5). While Rose and Rosser's proof was basically syntactic, Chang showed that Łukasiewicz logic is complete w.r.t. the variety of MV-algebras. For a complete account on MV-algebras the reader

may see (6).

**Definition 2.1** A *MV-algebra* is an algebra  $\mathcal{A} = \langle A, \oplus, \neg, 0 \rangle$  that satisfies

- (1)  $\langle A, \oplus, 0 \rangle$  is a commutative monoid,
- (2)  $\neg\neg x = x$ ,
- (3)  $x \oplus \neg 0 = \neg 0$ ,
- (4)  $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$ .

A number of other connectives can be defined starting from the ones above:

$$1 = \neg 0, \quad x \odot y = \neg(\neg x \oplus \neg y), \quad x \rightarrow y = \neg x \oplus y, \quad x \ominus y = \neg(\neg x \oplus y).$$

Moreover, given a MV-algebra  $\mathcal{A}$ , defining

$$x \wedge y = x \odot (x \rightarrow y) \quad \text{and} \quad x \vee y = (x \ominus y) \oplus y$$

gives a lattice  $\mathcal{A} = \langle A, \vee, \wedge \rangle$ .

We now briefly introduce some structures fairly known to people working in many valued logic.

The  $\Delta$ , also known as Baaz operator, is an operator which notably increase the expressive power of a logic. In the setting of many-valued logic it can be seen as a modality which gives the crisp truth value of a formula, but because of its behavior with regard to the rule of contraction it can be also seen as very close to the *exponential* operator of linear logic.

**Definition 2.2** A *MV $_{\Delta}$ -algebra* is a MV-algebra with an operator  $\Delta$  that satisfies:

- (1)  $\Delta(1) = 1$ ,
- (2)  $\Delta(x \rightarrow y) \leq \Delta(x) \rightarrow \Delta(y)$ ,
- (3)  $\Delta(x) \vee \neg\Delta(x) = 1$ ,
- (4)  $\Delta(x) \leq x$ ,
- (5)  $\Delta(\Delta(x)) = \Delta(x)$ ,
- (6)  $\Delta(x \vee y) = \Delta(x) \vee \Delta(y)$ .

In (13) is introduced an expansion of Łukasiewicz logic with *root* operators and in (2) it is proved that such a logic enjoys interpolation and that it corresponds to continuous piece-wise linear functions with rational coefficients.

In (10) all these results are collected and the following variety is introduced as the algebraic counterpart of this logic.

In the following will be used the shorthand  $(n)x$  for  $\underbrace{x \oplus \dots \oplus x}_{n\text{-times}}$

**Definition 2.3 ((10))** A *divisible MV-algebra* is an MV-algebra with a family of operators  $\delta_n$  such that:

- (1)  $(n)\delta_n(x) = x$ ,
- (2)  $\delta_n(x) \odot (n-1)\delta_n(x) = 0$ .

Putting together these two last systems we get the following structures.

**Definition 2.4** A *divisible  $MV_\Delta$ -algebra* is a structure  $\mathcal{A} = \langle A, \oplus, \neg, 0, \delta_n, \Delta \rangle$  such that:

- $\langle A, \oplus, \neg, 0, \Delta \rangle$  is a  $MV_\Delta$ -algebra,
- $\langle A, \oplus, \neg, 0, \delta_n \rangle$  is a divisible MV-algebra.

Such structures can be seen as the algebraic counterpart of the logic introduced in (2) and there called  $L-\square-\Delta$ .

We will introduce now some conventions on notation.

Let us denote by  $Term_{MV}$  the set of terms in the language of MV algebras. When we write  $t(x)$  for some term  $t$  we mean that the variable  $x$  actually appears in the term  $t$  although there can be other variables also appearing in  $t$ .

The symbol  $\bar{y}$  will be used for any sequence of variables  $y_1, \dots, y_n$ ; the same notation will be used for elements of an algebra  $\mathcal{A}$ , so  $\bar{a} = \langle a_1, \dots, a_n \rangle \in \mathcal{A}^n$  for some natural number  $n$ .

Given a term  $t(x, y)$  and  $a \in \mathcal{A}$  we will indicate by  $t(x, [a/y])$  the object obtained by substituting every occurrence of  $y$  with  $a$ . We will think of such an object as a function from  $\mathcal{A}$  in itself, associating to every  $b \in \mathcal{A}$  the element  $t(b, [a/y])$ , this easily generalizes to higher arity of the term  $t$ .

Sometimes, given a term  $t(x, \bar{y})$ , if  $\bar{y} = \langle y_1, \dots, y_n \rangle$  and  $\bar{a} = \langle a_1, \dots, a_n \rangle$ , to lighten notation, we will write  $t(x, \bar{a})$  for the unary function

$$t(x, [a_1/y_1], [a_2/y_2], \dots, [a_n/y_n]).$$

We introduce now the class of algebras that we are interested to study. Basically they are MV-algebras endowed with a new function  $\mu x_{t(x)}(\bar{y})$  for any  $t(x, \bar{y}) \in Term_{MV}$ . The notation should help to remember both the arity of each of those functions, which equals the arity of the term  $t$  minus 1, and the variable which, loosely speaking, is under the scope of  $\mu$ . Given  $t(x, \bar{y}) \in Term_{MV}$ , the value of  $\mu x_{t(x)}(\bar{a})$  is the minimum fixed point of the function  $t(x, \bar{a})$ .

**Definition 2.5** A  $\mu MV^-$ -algebra is a structure

$$\mathcal{A} = \langle A, \oplus, \neg, 0, \{\mu x_{t(x)}\}_{t(x) \in \text{Term}_{MV}} \rangle$$

such that  $\mathcal{A} = \langle A, \oplus, \neg, 0 \rangle$  is a MV-algebra and for every  $s, t \in \text{Term}_{MV}$

- (1)  $\mu x_{t(x)}(\bar{y}) = t(\mu x_{t(x)}, \bar{y})$ ,
- (2) If  $t(s(\bar{y}), \bar{y}) = s(\bar{y})$  then  $\mu x_{t(x)}(\bar{y}) \leq s(\bar{y})$ ,
- (3)  $\bigwedge_{i \leq n} (\neg \mu x_{x \oplus \neg y}(y_i \leftrightarrow z_i)) \leq \mu x_{t(x)}(y_1, \dots, y_n) \leftrightarrow \mu x_{t(x)}(z_1, \dots, z_n)$ .

The first two axioms have a rather clear meaning: axiom 1 says that for any  $\bar{a} \in |CCA^n|$   $\mu x_{t(x)}(\bar{a})$  is a fixed point of the function  $t(x, \bar{a})$ . Axiom 2 guarantees that  $\mu x_{t(x)}(\bar{a})$  is, in fact, the minimum among fixed points of  $t(x, \bar{a})$ .

The third axiom may look weird at this stage, for the moment the reader may find easy to interpret such an axiom in the *standard*  $\mu MV^-$ -algebra, i.e. the  $[0, 1]$  interval of the reals endowed with the standard MV-algebra structure given at the beginning of the section, plus the functions which give the fixed point of any continuous piecewise linear function with integer coefficients. It is quiet easy to check that such a structure is indeed a  $\mu MV^-$ -algebra.

In this case the term  $\neg \mu x_{x \oplus \neg y}(y)$  has the following interpretation:

$$\neg \mu x_{x \oplus \neg y}(y) = \begin{cases} 1 & \text{if } y = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Such a behavior is typical of the  $\Delta$  operator, so in this particular case, axiom 3 can be seen as asserting that if two sequences of elements are  $\Delta$ -congruent than every  $\mu$ -term in which the first sequence appears is congruent to the same  $\mu$ -term where the first sequence is substituted by the second sequence.

The formal motivation for introducing such an axiom as well as the exact meaning of this explanation will come soon.

Note that although the MV-algebra on the  $[0, 1]$  interval of the reals can be endowed with the structure of a  $\mu MV^-$ -algebra, this is not the case for every MV-algebra. Think, for instance, to the boolean algebra  $\{0, 1\}$ , which is also a MV-algebra, then it obviously does not have the fixed point of many MV-terms, such, as, for instance, the term  $\neg x$ , whose fixed point can not be either 0 or 1. In general every finite MV-algebra can not be a  $\mu MV^-$ -algebra since such an algebra should contain, for instance, all constants of the form  $\mu x_{t(x)}$ , where  $t(x)$  is a term only containing the variable  $x$ , which are in bijective correspondence with the rational numbers.

We wish to stress the fact that if  $\bar{y}$  is a non empty sequence, the term  $\mu x_{t(x, \bar{y})}(\bar{y})$  is not the fixed point of the term  $t(x, \bar{y})$  but is the function that associates to

any interpretation  $\bar{a}$  of the variable in  $\bar{y}$  the fixed point of the unary function  $t(x, \bar{a})$ . Nevertheless, with an abuse of language, we will sometimes refer to  $\mu x_{t(x, \bar{y})}(\bar{y})$  as *the* fixed point of  $t(x, \bar{y})$ .

### 3 Subdirect representation

By Birkhoff's Representation Theorem, every algebra is the subdirect product of irreducible algebras. Very often the algebraic semantics of many valued logics enjoy a property which drastically helps their study: the irreducible algebras are exactly the linearly ordered ones.

Unfortunately we are not able to prove it for the class of  $\mu MV^-$ -algebras without further assumptions. Indeed we have introduced a wild family of functions to  $MV$ -algebras, making the theory of congruences quite hard to handle. Note for instance, that the classical correspondence between congruences, filters and ideals does not easily generalizes to this case.

At this stage we can only prove an equivalence that will tell us which are sufficient assumptions to get such a result:

**Proposition 3.1** *Every  $\mu MV^-$ -algebra is the subdirect product of linearly ordered  $\mu MV^-$ -algebras if, and only if, the term  $\neg \mu x_{x \oplus \neg y}(y)$  satisfies the axioms of the  $\Delta$  operator.*

**Proof.** The “if” direction is given by the content of Lemma 3 in (7), which states that every expansion of a  $MV_\Delta$  with functions  $\{f_j\}_{j \in J}$  satisfying

$$\bigwedge_{i \leq n} (\Delta(y_i \leftrightarrow z_i)) \leq f_j(y_1, \dots, y_n) \leftrightarrow f_j(z_1, \dots, z_n),$$

for every  $j \in J$ , is irreducible if, and only if, its underlying  $MV_\Delta$  algebra is irreducible. Since the irreducible  $MV_\Delta$ -algebra are precisely the linearly ordered one, the result follows.

For the other direction, suppose that every  $\mu MV^-$ -algebra is the subdirect product of linearly ordered  $\mu MV^-$ -algebras, then we only have to check that the term  $\neg \mu x_{x \oplus \neg y}(y)$  satisfies the axioms of  $\Delta$  for linearly ordered  $\mu MV^-$ -algebras. This comes easily from the fact that on linearly ordered  $\mu MV^-$ -algebras we have:

$$\neg \mu x_{x \oplus \neg y}(y) = \begin{cases} 1 & \text{if } y = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Indeed  $\neg \mu x_{x \oplus \neg y}(0) = \neg 0 = 1$ . Moreover  $\neg \mu x_{x \oplus \neg y}(v) = 0$  if, and only if,  $\mu x_{x \oplus \neg y}(v) = 1$ , if, and only if, 1 is the minimum fixed point of  $x \oplus \neg v$ , which is true if, and only if  $v \neq 1$ .  $\square$

Motivated by the above proposition we introduce now a new class of algebras whose behavior will be smoother than  $\mu MV^-$ -algebras.

**Definition 3.2** *A  $\mu MV$ -algebra is a  $\mu MV^-$ -algebra which satisfies the following additional axioms:*

4.  $\neg\mu y_{y\oplus\neg z}(1) = 1$ ,
5.  $\neg\mu y_{y\oplus\neg z}(x \rightarrow y) \leq \neg\mu y_{y\oplus\neg z}(x) \rightarrow \neg\mu y_{y\oplus\neg z}(y)$ ,
6.  $\neg\mu y_{y\oplus\neg z}(x) \vee \neg\neg\mu y_{y\oplus\neg z}(x) = 1$ ,
7.  $\neg\mu y_{y\oplus\neg z}(x) \leq x$ ,
8.  $\neg\mu y_{y\oplus\neg z}(\neg\mu y_{y\oplus\neg z}(x)) = \neg\mu y_{y\oplus\neg z}(x)$ ,
9.  $\neg\mu y_{y\oplus\neg z}(x \vee y) = \neg\mu y_{y\oplus\neg z}(x) \vee \neg\mu y_{y\oplus\neg z}(y)$ .

As an immediate corollary of Proposition 3.1, we get:

**Corollary 3.3** *Every  $\mu MV$ -algebra is the subdirect product of linearly ordered  $\mu MV$ -algebras.*

It is clear from the definition that  $\mu MV$ -algebras form a quasivariety. But, the presence of a term which exactly behaves as  $\Delta$  allows to define a *discriminator* (see (17) for the definition), namely  $d(x, y, z) = (\Delta(x \leftrightarrow y) \wedge z) \vee (\Delta(x \leftrightarrow y) \wedge x)$  and since a quasivariety with a discriminator is a variety (16), this proves:

**Proposition 3.4** *The class of  $\mu MV$ -algebras is a variety.*

One could wonder whether the introduction of maximum fixed points would have made any difference. The answer is negative.

**Proposition 3.5** *For every  $t(x, \bar{y}) \in Term_{MV}$  the term defined by  $\nu x_{t(x)}(\bar{y}) = \neg\mu x_{(\neg t(\neg x))}(\bar{y})$  has the following properties:*

- $t(\nu x_{t(x)}, \bar{y}) = \nu x_{t(x)}(\bar{y})$ ,
- If  $t(s(\bar{y}), \bar{y}) = s(\bar{y})$  then  $s(\bar{y}) \leq \nu x_{t(x)}(\bar{y})$ .

*Hence it interprets the maximum fixed point of  $t(x)$ .*

**Proof.** For sake of simplicity we omit the argument of the functions, so instead of writing  $\nu x_{t(x)}(\bar{y})$  we write  $\nu x_{t(x)}$ . Call  $m = \neg\mu x_{(\neg t(\neg x))}$ . Let us first prove that it is a fixed point of  $t$ . Let  $s(z) = \neg(t(\neg z))$ , then  $m = \neg\mu x_{s(x)}$  so  $\neg m = \mu x_{s(x)}$ , i.e.  $\neg m$  is a fixed point of  $s(x)$ :  $\neg m = s(\neg m)$ . Hence  $\neg m = \neg t(\neg\neg m)$  which implies  $m = t(m)$ . To show that  $m$  is the maximum among fixed points let  $v$  be such that  $t(v) = v$ , then  $s(\neg v) = \neg v$ , so  $\neg v \geq \mu x_{s(x)}$  therefore  $v \leq \neg\mu x_{s(x)} = \mu x_{\neg\neg s(x)} = m$ .  $\square$

## 4 Term-wise equivalence

Not surprisingly, for any  $n$ , the operators  $\delta_n$ , of divisible MV-algebras, are also definable by fixed points:

$$\delta_n(x) = \mu y_{(x \ominus (n-1)y)}(x).$$

**Lemma 4.1** *For every  $\mu$ MV-algebra, the operator defined above satisfies:*

- (1)  $(n)\delta_n(x) = x$ ,
- (2)  $\delta_n(x) \odot (n-1)\delta_n(x) = 0$ .

**Proof.**

- (1) By definition of  $\delta$  we have that  $x \ominus (n-1)\delta_n(x) = \delta_n(x)$  which implies  $\neg(n-1)\delta_n(x) \rightarrow \neg x = \neg\delta_n(x)$  by residuation we have  $\neg\delta_n(x) \odot \neg(n-1)\delta_n(x) = \neg x$ . Finally, applying De Morgan's law for  $\odot$  and  $\oplus$ , we have  $\delta_n(x) \oplus (n-1)\delta_n(x) = x$ .
- (2) It is easy to check that in every MV-algebra the equation  $(x \ominus y) \odot y = 0$  holds. So we have  $(x \ominus (n-1)\delta_n(x)) \odot (n-1)\delta_n(x) = 0$ . But  $\delta_n(x)$  is a fixed point of  $x \ominus (n-1)\delta_n(x)$ . So  $\delta_n(x) \odot (n-1)\delta_n(x) = 0$ .

□

So every  $\mu$ MV-algebra contains a definable divisible  $MV_\Delta$ -algebra. The other direction also holds, but its proof requires more work.

**Lemma 4.2** *For every term  $t(x)$  in the language of MV-algebras and for any evaluation  $[\ ]^*$ , there exist:*

- $\{c_i\}_{i \in I}$  terms of the form  $(m)x \oplus k$ ,  $(m)x \ominus k$ ,  $\neg((m)x \oplus k)$  or  $\neg((m)x \ominus k)$  where  $m \in \mathbb{N}$  and  $k$  is a term not containing  $x$ ,
- $\{p_i\}_{i \in I}, \{q_i\}_{i \in I}$  terms not containing  $x$ ,

where  $I$  is some finite set of indices; such that:

$$[t(x)]^* = \left[ \bigvee_{i \in I} (\Delta(x \rightarrow q_i) \wedge \Delta(p_i \rightarrow x) \wedge c_i) \right]^*.$$

**Proof.** Every term of an MV-algebra can be interpreted as a continuous piece-wise linear function with integer coefficients from  $[0, 1]^n$  to  $[0, 1]$  (this can easily be proved by induction on the number of connectives in the term). If a function  $f(\bar{y})$  is piecewise linear then there exists a finite partition  $P_i$  of the domain, indexed by a set  $I$ , and linear functions  $\{f_i\}_{i \in I}$  such that  $f(\bar{y})$  can be



decomposed as follows:

$$f(\bar{y}) = \begin{cases} f_{i_1}(\bar{y}) & \text{if } x \in P_{i_1}, \\ \vdots & \vdots \\ f_{i_n}(\bar{y}) & \text{if } x \in P_{i_n}. \end{cases}$$

Hence, once all variable but one are fixed, the function associated to the term can be described as:

$$f(x) = \begin{cases} z_1x \pm k_1 & \text{if } p_1 \leq x \leq q_1, \\ \vdots & \vdots \\ z_ix \pm k_i & \text{if } p_i \leq x \leq q_i. \end{cases}$$

where  $z_i \in \mathbb{Z}$  and  $k_i, p_i, q_i$  are polynomials in the variable parameterized. But such a function is the interpretation of a term of the form:

$$\bigvee_{i \leq I} (\Delta(x \rightarrow q_i) \wedge \Delta(p_i \rightarrow x) \wedge c_i),$$

where  $c_i$  are the terms corresponding to  $z_ix \pm k_i$ .  $\square$

**Theorem 4.3**  *$\mu MV$ -algebras and divisible  $MV_\Delta$ -algebras are term-wise equivalent.*

**Proof.** One direction is given by Proposition 3.5.

For the other direction we first find the minimum fixed points of some basic terms. Let us define:

$$\begin{array}{lll} \bar{\mu}x_c = \neg\Delta(\neg k) & \text{if} & c = (m)x \oplus k \\ \bar{\mu}x_c = \delta_{m-1}(k) & \text{if} & c = (m)x \ominus k \\ \bar{\mu}x_c = \delta_{m+1}(\neg k) & \text{if} & c = \neg((m)x \oplus k) \\ \bar{\mu}x_c = \delta_{m+1}(k) \oplus \delta_{m+1}(1) & \text{if} & c = \neg((m)x \ominus k) \end{array}$$

Where, as convention, we have put  $\delta_0(x) = 0$  for every  $x$ . A simple calculation shows that in all four cases  $\bar{\mu}x_c$  is the minimum fixed point of  $c$ . To find the fixed point function associated to any term  $t(x)$  we first use Lemma 4.2, to find a term, equivalent to  $t(x)$ , in which all the linear components are explicitly present, let it be  $\bigwedge_{i \leq I} (\Delta(x \rightarrow q_i) \wedge \Delta(p_i \rightarrow x) \wedge c_i)$ . By continuity of the functions which interpret  $t(x)$ , a fixed point for this term must exist and it will be among the fixed points of the functions  $c_i$ . So we define:

$$\mu x_{t(x)} = \bigwedge_{i \leq I} [\neg\Delta(t(\bar{\mu}x_{c_i}) \leftrightarrow \bar{\mu}x_{c_i}) \oplus \bar{\mu}x_{c_i}]$$

This term basically gives the meet of all fixed points of the functions  $c_i$ , namely  $\bar{\mu}x_{c_i}$ , which are also a fixed point of the term  $t(x)$ , namely the ones for which

$\Delta(t(\bar{\mu}x_{c_i}) \leftrightarrow \bar{\mu}x_{c_i}) = 1$ . The fact that this is the minimum fixed point of  $t(x)$  is readily seen.  $\square$

Once this equivalence is established it becomes fairly easy to extend known results (and techniques) about divisible MV-algebras and  $MV_{\Delta}$ -algebra to  $\mu MV$ -algebras.

We recall a number of results and definitions contained in (18).

**Definition 4.4** *A  $\delta$ -lattice ordered group ( $\delta$ -**l-group**, for short) is a structure  $\mathcal{G} = \langle G, +, -, \wedge, \vee, \delta, 0, 1 \rangle$  where  $\langle G, +, -, \wedge, \vee, 0, 1 \rangle$  is an abelian lattice ordered group and  $\delta$  is a unary operation satisfying:*

$$\begin{aligned} \delta(x) &\leq |x| \wedge 1, & \delta(\delta(x)) &= \delta(x), & \delta(x) &= \delta(x \wedge 1), \\ \delta(1) &= 1, & \delta(x) \vee (1 - \delta(x)) &= 1, & 0 &\leq \delta(x), \\ & & \delta(x) \wedge \delta(y^+ + (1 - |x|)^+) &\leq \delta(y), \end{aligned}$$

where  $|x| = x \vee (-x)$  and  $x^+ = x \vee 0$ .

**Theorem 4.5** (18) *There is a functor  $\Gamma_{\Delta}$  (extending Mundici's functor (21)) between the category of  $MV_{\Delta}$ -algebra and the category of  $\delta$ -l-groups which, together with its inverse, forms an equivalence of category.*

**Proposition 4.6** *Each linearly ordered  $\mu MV$ -algebra is isomorphic to the unitary interval of a linearly ordered divisible  $\delta$ -group.*

**Proof.** We only need to adapt some ideas contained in (9) to our case.

Given a linearly ordered  $\mu MV$ -algebra we consider its equivalent to a linearly ordered divisible  $MV_{\Delta}$ -algebra  $\mathcal{A}$ . Its divisible MV reduct is the interval algebra of a linearly ordered  $\delta$ -group  $\mathcal{G}$ . We only have to prove that  $\mathcal{G}$  is divisible. For every  $x \in \mathcal{G}$  there exists  $m \in \mathbb{N}$  such that  $(m-1)u \leq x \leq (m)u$ , hence  $x' = mu - x$  belongs to the unitary interval of  $\mathcal{G}$  so, since  $\mathcal{A}$  is divisible, for every  $n \in \mathbb{N}$  there exists  $y$  in the same interval such that  $ny = x' = mu - x$ . Let finally  $u'$  be such that  $nu' = u$ , then the element  $mu' - y$  satisfies  $n(mu' - y) = x$ . Hence  $\mathcal{G}$  is divisible.  $\square$

**Theorem 4.7** *The  $\mu MV$ -algebra  $\langle [0, 1], \oplus, \neg, 0, \{\mu x_{t(x)}\}_{t(x) \in Term_{MV}} \rangle$  generates the variety of  $\mu MV$ -algebras.*

**Proof.** We prove that an equation holds in the standard  $\mu MV$ -algebra if, and only if, it holds in all  $\mu MV$ -algebra. For the non trivial direction suppose the contrary, i.e. an equation  $\varphi$  fails in some  $\mu MV$ -algebra, then it fails in a linearly ordered one. Call  $\mathcal{G}$  the linearly ordered  $\delta$ -group in which the linearly ordered algebra embeds as in Proposition 4.6. Then  $\varphi$  fails in  $\mathcal{G}$ . In particular  $\mathcal{G}$  is an abelian ordered group, so, by Gurevich-Kokorin theorem (11), this

implies that  $\varphi$  fails in the reals, and hence in its interval algebra.  $\square$

Another important result which can be established using the equivalence that we proved above is the characterization of the free  $\mu$ MV-algebra. It is known (2) that the free divisible  $MV_{\Delta}$ -algebra is the algebra of piecewise linear functions with rational coefficients. This result gives us automatically the characterization of the free  $\mu$ MV-algebra.

**Theorem 4.8** *The free  $\mu$ MV-algebra is the algebra of piecewise linear functions with rational coefficients*

Finally we derive the amalgamation property for  $\mu$ MV-algebras from Proposition 4.6 and some results contained in (19). In particular in (19, Lemma 3.3 and 3.4), the following is proved.

**Lemma 4.9** *Let  $\mathbf{K}$  be a quasi variety of BL algebras possibly with additional operators such that  $\mathbf{K}_{lin}$  has the amalgamation property. Then  $\mathbf{K}$  has the amalgamation property.*

Where  $\mathbf{K}_{lin}$  are the linearly ordered members of  $\mathbf{K}$ . So to prove the amalgamation property for  $\mu$ MV-algebras, we only need to show

**Theorem 4.10** *Linearly ordered  $\mu$ MV-algebras enjoy amalgamation.*

**Proof.** Since  $\mu$ MV-algebras form a variety it is sufficient to show that for every  $\mathcal{A}, \mathcal{B}$  and  $\mathcal{C}$  linearly ordered  $\mu$ MV-algebras such that  $\mathcal{A} = \mathcal{B} \cap \mathcal{C}$ , there exists  $\mathcal{D}$  and embeddings  $h$  and  $k$  of  $\mathcal{B}$  and  $\mathcal{C}$  respectively into  $\mathcal{D}$  such that the restriction of  $h$  and  $k$  to  $\mathcal{A}$  coincide. Let  $\mathcal{I}, \mathcal{J}$  and  $\mathcal{K}$  be the linearly ordered divisible  $\delta$ -groups built respectively from  $\mathcal{A}, \mathcal{B}$  and  $\mathcal{C}$ , we can safely suppose  $\mathcal{I} = \mathcal{J} \cap \mathcal{K}$ . By the amalgamation property for linearly ordered divisible groups, which clearly extends to linearly ordered divisible  $\delta$ -groups, we know that there is a linearly ordered divisible  $\delta$ -group  $\mathcal{L}$  and embeddings  $h$  and  $k$  from  $\mathcal{J}, \mathcal{K}$  into  $\mathcal{L}$  such that  $h$  and  $k$  coincide on  $\mathcal{I}$ , but then  $h$  and  $k$  coincide also on  $\mathcal{A}$ . Hence the interval algebra of  $\mathcal{L}$  plus the restriction of  $h$  and  $k$  to  $\mathcal{J}$  and  $\mathcal{K}$  is the amalgam we were looking for.  $\square$

## 5 Further studies

We give a glance at some problems we are interested to study in the time to come.

- Is it possible to give a constructive proof of Theorem 4.8, in the style of (22)?

- Although we did not address the issue of complexity, it is known (8) that the complexity of the satisfiability problem for rational Łukasiewicz logic with  $\Delta$  is NP-complete. It would be interesting to study whether there exists an efficient translation of  $\mu$ MV formula in  $\delta$ - $\Delta$ MV formula.
- Is it possible to find a subcategory of  $\ell$ -groups which is categorical equivalent to  $\mu$ MV-algebras?

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