Omitting type theorems for Łukasiewicz Logic

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ManyVal10. Varese, 3 Maggio 2010.

Łukasiewicz logic

Łukasiewicz logic is one of the earliest generalisation of symbolic logic introduced to cope with many truth values.

In modern settings, Łukasiewicz logic can be framed in the body of **continuous t-norm based logics**.

Yet Łukasiewicz logic stands out among those logics because of some of its properties. Indeed,

- Łukasiewicz logic is the only one, among continuous t-norm based logics, with a **continuous implication** and therefore the only logic whose whole set of formulae has a standard continuous interpretation.
- Furthermore the Łukasiewicz negation is **involutive**, namely it is such that $\neg\neg\varphi \leftrightarrow \varphi$.

Those two features, inherited from classical logic, makes Łukasiewicz logic a promising setting to test how far the methods of model theory can reach, in the realm of many-valued logics.

A model theory inside many-valued logic

A model theoretic study of many-valued logic is especially important in the light of the **negative results** obtained in the first order theory of these logics:

- the predicate version BL has a standard tautology problem whose complexity **is not arithmetical** (Montagna, 2001),
- the same problem is Π_2-complete for Łukasiewicz logic (Ragaz, 1981).

Thus the favourable **duality between syntax and semantics vanishes** when switching to t-norm based logics and new tools must be developed.
The results so far are encouraging: recently the **Robinson finite and infinite forcing** were generalised to Łukasiewicz logic; here some results for a **basic model theory** of Łukasiewicz logic are presented and used to settle an open problem left therein.

Łukasiewicz logic

The **syntax** of the infinite-valued Łukasiewicz propositional logic, Ł, is exactly as the classical one: a countable set of propositional variables, $Var = \{p_1, p_2, \ldots, p_n, \ldots\}$, and two connectives $\rightarrow$ and $\neg$.

The **axioms** of Ł are the following:

$$\varphi \rightarrow (\psi \rightarrow \varphi); \qquad (\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi));$$

$$((\varphi \rightarrow \psi) \rightarrow \psi) \rightarrow ((\psi \rightarrow \varphi) \rightarrow \varphi); \qquad (\neg\varphi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \varphi),$$

Modus ponens is the only **rule of inference**. The notions of proof and tautology are defined as usual.

MV-algebras

The equivalent algebraic semantics for Ł is given by the variety of **MV-algebras**.

Definition

An MV-algebra is a structure $\mathcal{A} = \langle A, \oplus, ^*, 0\rangle$ such that:

- $\mathcal{A} = \langle A, \oplus, 0\rangle$ is a commutative monoid,
- * is an involution
- the interaction between those two operations is described by the following two axioms:
  - $x \oplus 0^* = 0^*$
  - $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$

Predicate Łukasiewicz logic

The **syntax** of predicate Łukasiewicz logic (Ł$\forall$) is again the same as classical first order logic (without functional symbols). The primitive connectives are : $\rightarrow, \neg, \forall$.

So all syntactical concepts like term, (atomic) formula, free or bounded variable, ... are **defined just as usual**.

The axioms of Ł$\forall$ are:

(i) All the axioms of the infinite-valued propositional Łukasiewicz calculus;

(ii) $\forall x \varphi \rightarrow \varphi(t)$, where the term t is substitutable for x in φ;

(iii) $\forall x(\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \forall x \psi)$, where x is not free in φ;

The inference rules are *Modus ponens*: from φ and $\varphi \rightarrow \psi$, derive ψ; *Generalisation*: from φ, derive $\forall x \varphi$.

Structures for predicate Łukasiewicz logic

Let $\mathcal{L} = \langle P_1, ..., P_n, c_1, ..., c_m \rangle$ be a Ł∀ language with n predicate symbols and m constant symbols.
Let A be an MV-algebra. An ***A*-structure** has the form

$$\mathcal{M} = \langle M, P_1^{\mathcal{M}}, ..., P_n^{\mathcal{M}}, c_1^{\mathcal{M}}, ..., c_m^{\mathcal{M}} \rangle$$

where M is a non-empty set (called the **universe of the structure**).

If P is a predicate symbol in $\mathcal{L}$ of arity k then $P^{\mathcal{M}}$ is a k-ary ***A*-valued relation** on M, namely a function

$$P^{\mathcal{M}} : M^k \to A;$$

if c is a constant symbol in $\mathcal{L}$ then $c^{\mathcal{M}}$ is an **element** of M.

Evaluations

Let $\mathcal{M}$ be an A-structure. An **evaluation** of $\mathcal{L}$ in $\mathcal{M}$ is a function $e : V \to M$. For any term t of $\mathcal{L}$ and any evaluation in $\mathcal{M}$ let

$$t^{\mathcal{M}}(e) = \begin{cases} e(x) & \text{if } t \text{ is a variable } x \\ c^{\mathcal{M}} & \text{if } t \text{ is a constant } c \end{cases}$$

Truth values

Given any evaluation in $\mathcal{M}$, e and any formula φ of $\mathcal{L}$, the element $\|\varphi(e)\|_{\mathcal{M}}$ of A is defined by induction, and it is called the **truth value** of φ:

if $\varphi = P(t_1, ..., t_n)$ then
$\|\varphi(e)\| = P^{\mathcal{M}}(t_1^{\mathcal{M}}(e), ..., t_n^{\mathcal{M}}(e))$;
if $\varphi = \neg\psi$ then $\|\varphi(e)\| = \|\psi(e)\|^*$;
if $\varphi = \psi \to \chi$ then $\|\varphi(e)\| = \|\psi(e)\| \Rightarrow \|\chi(e)\|$;
if $\varphi = \forall x \psi$ then $\|\varphi(e)\| = \bigwedge\{\|\psi(e')\| \mid e' \equiv_x e\}$.

An evaluation $e : V \to M$ is called **safe** if for any formula ψ of $\mathcal{L}$, the infimum $\bigwedge\{\|\psi(e')\| \mid e' \equiv_x e\}$ exists).

If $\|\varphi\|_{\mathcal{M}}^{A} = 1$ then φ is said **to be true in** $\mathcal{M}$, this can be alternatively written as $\mathcal{M} \models_A \varphi$. An A-structure $\mathcal{M}$ is a **model** of a theory T if $\mathcal{M} \models_A \varphi$ for all $\varphi \in T$.

Logical consequence and satisfiability

Definition

A **standard structure** is a $[0,1]$-structure, any valuation is safe on a standard structure.
A **standard model** of a theory T is a $[0,1]$-structure which is a model of T.
A formula φ is called ***A*-logical consequence** of a theory T, in symbols $T \models_A \varphi$, if every A-model of T is also an A-model of φ. In particular, when this is true for standard models then I write $T \models_{[0,1]} \varphi$ or $T \models \varphi$.

Definition

A formula φ is **generally satisfiable** if there exists an A-model $\mathcal{M}$ such that $\|\varphi\|^A_{\mathcal{M}} = 1$. If the model can be taken standard then φ is called just **satisfiable**. This naturally generalises to theories. A theory T is **consistent** if $T \not\models \bot$.

Compactness

All the main results in this talk hinge on the following theorems.

Theorem (Hay 1963)

Any consistent theory T of Ł$\forall$ has a standard model.

Theorem (Compactness)

Let T be a theory in Ł$\forall$:

(i) *If T is finitely generally satisfiable then T is generally satisfiable.*

(ii) *If T is finitely satisfiable then T is satisfiable.*

(iii) *If for any MV-algebra A, $T \models_A \varphi$ then there exists a finite $T_0 \subseteq T$ such that for any MV-algebra A $T_0 \models_A \varphi$*

(iv) *If $T \models_{[0,1]} \varphi$ then* **in general it is false** *that there exists a finite $T_0 \subseteq T$ such that $T_0 \models_{[0,1]} \varphi$.*

A hierarchy on formulae

Henceforth $\mathcal{L}$ is assumed to be a fixed language of Ł∀ and all structures are standard.

Definition

A formula of $\mathcal{L}$ belongs to the set Σ_n (Π_n, respectively) if it is equivalent to a formula with n blocks of quantifier, where each block is either empty or constituted of an uninterrupted sequence of the same quantifier, $\exists$ or $\forall$, and the first block is made of $\exists$'s ($\forall$'s respectively).

As in the classical case one has $\Sigma_n \cup \Pi_n \subseteq \Sigma_{n+1} \cap \Pi_{n+1}$.

Relations among models

Let $\mathcal{M}$ be a structure, $\mathcal{L}(\mathcal{M})$ is the expansion of the language $\mathcal{L}$ with a new constant symbol for each element of M.

The **diagram** of $\mathcal{M}$, i.e. the set of atomic formulae φ in $\mathcal{L}(\mathcal{M})$ such that $\|\varphi\|_{\mathcal{M}} = 1$, is indicated by $D(\mathcal{M})$; $\mathrm{Th}(\mathcal{M})$ is the set of formulae φ such that $\|\varphi\|_{\mathcal{M}} = 1$.

Definition

If $\mathcal{M}_1 \subseteq \mathcal{M}_2$ are two structures and for any $\varphi \in D(\mathcal{M}_1)$, $\mathcal{M}_1 \models \varphi$ iff $\mathcal{M}_2 \models \varphi$ then $\mathcal{M}_1$ is a **substructure of** $\mathcal{M}_2$, in symbols $\mathcal{M}_1 \leq \mathcal{M}_2$. If the same is true for *any* sentence of $\mathcal{L}(\mathcal{M}_1)$ than $\mathcal{M}_1$ is an **elementary substructure of** $\mathcal{M}_2$, written $\mathcal{M}_1 \preceq \mathcal{M}_2$

Łoś-Tarski Theorem for Łukasiewicz logic

Theorem

A theory is preserved under substructure if, and only if, it is equivalent to a universal (i.e. Π_1) theory.

Proof.

We prove something stronger:
claim: *$T_\forall$, the set of logical consequences of T which are in Π_1, axiomatises the class of all substructures of models of T.*

If $\mathcal{M} \leq \mathcal{M}' \models T$ then $\mathcal{M} \models T_\forall$ is straightforward.
Let $\mathcal{M} \models T_\forall$, then $D_\forall(\mathcal{M}) \cup T$ is finitely satisfiable (if it were not then $\bigwedge \Psi \models \neg \bigwedge \Phi$, but $\neg \bigwedge \Phi \in \Pi_1$ fl.)
So there exists $\mathcal{N} \models D_\forall(\mathcal{M}) \cup T$ whence $\mathcal{M} \leq \mathcal{N} \models T$ □

(Elementary) chains

Definition

Let α be an ordinal and $(\mathcal{M}_\lambda)_{\lambda\in\alpha}$ a family of $\mathcal{L}$-structure. The structures $(\mathcal{M}_\lambda)_{\lambda\in\alpha}$ are a **chain** if for any $\lambda_1 \leq \lambda_2 < \alpha$, $\mathcal{M}_{\lambda_1} \leq \mathcal{M}_{\lambda_2}$.
If for any $\lambda_1 \leq \lambda_2 < \alpha$, $\mathcal{M}_{\lambda_1} \preceq \mathcal{M}_{\lambda_2}$ then $(\mathcal{M}_\lambda)_{\lambda\in\alpha}$ is called **elementary chain**.

Lemma

Let $(\mathcal{M}_\lambda)_{\lambda\in\alpha}$ be an elementary chain. Then for every $\lambda \in \alpha$, $\mathcal{M}_\lambda \preceq \bigcup_{\lambda\in\alpha} \mathcal{M}_\lambda$

T is an **inductive** theory if it is closed under unions of chains.

Chang-Łoś-Suszko Theorem for Łukasiewicz logic

Theorem

A theory is inductive if, and only if, it is equivalent to a Π_2 theory.

Proof.

Let T be inductive. If $\mathcal{M} \models T_{\forall_2}$ then $T \cup \mathrm{Th}_{\exists}(\mathcal{M})$ is finitely satisfiable (if not $\bigwedge \Phi \models \neg \bigwedge \Psi$, but then $\neg \bigwedge \Psi \in T_{\forall}$ fl.)
So there exists $\mathcal{N} \models T \cup \mathrm{Th}_{\exists}(\mathcal{M})$ s.t. $\mathcal{M} \leq \mathcal{N}$.
Every existential sentence of $L(\mathcal{M})$ which is true in $\mathcal{N}$ holds in $\mathcal{M}$, hence $D(\mathcal{N}) \cup \mathrm{Th}(\mathcal{M})$ is satisfiable, so it has a model $\mathcal{M}_1$ which is an extension of $\mathcal{N}$ and an elementary extension of $\mathcal{M}$.

$$\mathcal{M} \leq \mathcal{N} \leq \mathcal{M}_1 \leq \mathcal{N}_1 \leq \dots$$

If $\mathcal{O}$ is the limit of this chain, then $\mathcal{O} \models T$, for T is inductive and $\mathcal{M} \preceq \mathcal{O}$, (the chain $\{\mathcal{M}_i\}_{i \in \omega}$ is elementary). Hence $\mathcal{M} \models T$. □

Model companions

The above characterisation is extremely useful, when dealing with model complete theories.

Corollary

When the model companion of a theory is axiomatisable, it is equivalent to a $\forall\exists$ theory.

Proof.

In a model companion every chain is elementary. $\square$

From this it is also easy to see that

Corollary

There exists at most one model companion of a theory.

Generic models

The study of model theoretic forcing for Łukasiewicz logic, led to the study of the class of **generic models**, $\mathfrak{G}_{\mathbf{K}}$, contained in a given class **K**.

The class $\mathfrak{G}_{\mathbf{K}}$ was proved to contain the subclass of existentially closed models of **K**. The Chang-Łoś-Suszko theorem for Łukasiewicz logic enables to complete this result.

Proposition

Given a theory T, if $\mathfrak{G}_{\mathrm{Mod}(T)}$ is axiomatisable then **it is** *the class of existentially closed models of T.*

Proof.

Let $\mathcal{M}$ be a existentially closed model of T, then it embeds in a model $\mathcal{N} \in \mathfrak{G}_{\mathrm{Mod}(T)}$. The class $\mathfrak{G}_{\mathrm{Mod}(T)}$ is inductive, so if it is axiomatisable then it is equivalent to a Π_2 theory. Since $\mathcal{M}$ is existentially closed, it is easy to see that it satisfies the same Π_2 formulae of $\mathcal{N}$, whence $\mathcal{M} \in \mathfrak{G}_{\mathrm{Mod}(T)}$. □

n-forcing

Let T a theory of $\mathcal{L}$, for any n we will indicate with $\mathrm{Mod}(T, n)$ the class of models of $T_{\forall_{n+1}}$.
If $\mathcal{M} \subseteq \mathcal{M}'$ are structures, we will say that $\mathcal{M}$ is an ***n*-substructure** of $\mathcal{M}'$, in symbols $\mathcal{M} \preceq_{n+1} \mathcal{M}'$ if for any Π_n sentence φ, we have $\mathcal{M} \models \varphi$ iff $\mathcal{M}' \models \varphi$.

Remark

With the above notation we have $\mathcal{M} \in \mathrm{Mod}(T, n)$ iff there exists $\mathcal{M}'$ such that $\mathcal{M}' \models T$ and $\mathcal{M} \preceq_{n+1} \mathcal{M}'$

n-substructures

Definition

Let $\mathcal{M} \in \mathrm{Mod}(T, n)$ and φ any sentence in $\mathcal{L}(\mathcal{M})$, we define the **n-forcing value of φ at $\mathcal{M}$**, $[\varphi]^n_{\mathcal{M}}$ as follows:

1. if φ is atomic then $[\varphi]^n_{\mathcal{M}} = \|\varphi\|_{\mathcal{M}}$
2. if $\varphi = \neg\psi$ then $[\varphi]^n_{\mathcal{M}} = \bigwedge_{\mathcal{M} \preceq_n \mathcal{N}} ([\psi]^n_{\mathcal{N}})^*$
3. if $\varphi = \psi_1 \to \psi_2$ then $[\varphi]^n_{\mathcal{M}} = \bigwedge_{\mathcal{M} \preceq_n \mathcal{N}} ([\psi_1]^n_{\mathcal{N}} \Rightarrow [\psi_2]^n_{\mathcal{M}})$
4. if $\varphi = \exists\psi(x)$ then $[\varphi]^n_{\mathcal{M}} = \bigvee_{c \in \mathcal{M}} [\psi(c)]^n_{\mathcal{M}}$

Setting $n = 0$ gives back the infinite forcing in Łukasiewicz logic recently studied.

n-generic models

Definition

Let $\mathcal{M} \in \mathrm{Mod}(T, n)$ we will say that $\mathcal{M}$ is **Mod(T, n)-generic** if for any sentence φ of $L(\mathcal{M})$ we have $[\varphi]^n_{\mathcal{M}} \oplus [\neg\varphi]^n_{\mathcal{M}}$=1.

Theorem

Let $\mathcal{M} \in \mathrm{Mod}(T, n)$ then it exists $\mathcal{M}^* \in \mathrm{Mod}(T, n)$ such that $\mathcal{M} \preceq_n \mathcal{M}^*$ and $\mathcal{M}^*$ is $\mathrm{Mod}(T, n)$-generic.

The proof follows precisely the steps used in order to prove the existence of 0-generic models.

Some properties of n-generic models

Theorem

Let $\mathcal{M}, \mathcal{M}' \in \mathrm{Mod}(T, n)$, then

1. $\mathcal{M}$ *is* $\mathrm{Mod}(T, n)$*-generic iff for any sentence* φ *one has* $[\varphi]^n_{\mathcal{M}} = \|\varphi\|_{\mathcal{M}}$.
2. *If* $\mathcal{M}, \mathcal{M}'$ *are* $\mathrm{Mod}(T, m)$*-generic, and* $\mathcal{M} \preceq_n \mathcal{M}'$ *then* $\mathcal{M} \preceq \mathcal{M}'$
3. *If* $\mathcal{M}$ *is* $\mathrm{Mod}(T, n)$*-generic,* $\mathcal{M} \preceq_n \mathcal{M}'$ *and* φ *is a* Π_{n+2}*-sentence of* $L(\mathcal{M})$ *then* $\|\varphi\|_{\mathcal{M}'} \leq \|\varphi\|_{\mathcal{M}}$
4. $T_{\forall_{n+2}} \subseteq T^{(F,n)}$ *where* $T^{(F,n)}$ *is the set of all sentences of* $\mathcal{L}$ *valid in all the* $\mathrm{Mod}(T, n)$*-generic models.*

n-existential types

Henceforth the language $\mathcal{L}$ and the theory T over $\mathcal{L}$ are fixed .

Definition

A set Γ of formulae of $\mathcal{L}$ is called a **type** if it satisfies the following conditions:

1. All the formulae in Γ are consistent with T,
2. If $\gamma_1, \gamma_2 \in \Gamma$ then $\gamma_1 \wedge \gamma_2 \in \Gamma$.

A type Γ is called a **Σ_n-type** if all the formulae in Γ are equivalent to a Σ_n-formula.
If Δ, Γ are types, we write $\Delta \leq \Gamma$ if for any $T \cup \Delta \models \Gamma$.
A Σ_{n+1}-type Γ is called **$(n+1)$-existential type** if there exist no Σ_n-type Δ such that $\Delta \leq \Gamma$.

Main Lemma

Lemma

Let $\mathcal{M}$ be $\mathrm{Mod}(T, n)$ generic and Φ be a type of Σ_{n+2}-sentences of $\mathcal{L}(\mathcal{M})$, suppose that $\mathcal{M} \models \Phi$ then there exists a type of Σ_{n+1}-sentences Ψ of $\mathcal{L}(\mathcal{M})$ such that:

1. $\mathcal{M} \models \Psi$,
2. $T \cup \Psi \models \Phi$,
3. *all the constants of $\mathcal{M}$ which occur in Ψ already occur in Φ.*

Chang's Omitting Types Theorem

Theorem

If $\mathcal{M}$ is a $\text{Mod}(T, n)$-generic model and $\mathcal{M} \models T$, then $\mathcal{M}$ omits all the $(n+2)$-existential types.

Chang's Omitting Types Theorem

Let T be a theory of $\mathcal{L}$, such that $T \subseteq \Pi_{n+2}$. For any model $\mathcal{M}$ of T, there exists an extension $\mathcal{M}^*$ of $\mathcal{M}$ such that:

1. $\mathcal{M}^*$ is a model of T,
2. $\mathcal{M}^*$ realises every Σ_{n+1}-type,
3. $\mathcal{M}^*$ omits every type $(n+2)$-existential.

∞-universal types

Definition

Let Γ a set of formulae of $\mathcal{L}$. We will say that Γ is a **∞-universal type** if it satisfies the following conditions:

1. $\Gamma$is a type,
2. there exists an enumeration $(\gamma_k)_{k\in\omega}$ of Γ such that $T \models \gamma_{k+1} \rightarrow \gamma_k$ for any $k \in \omega$,
3. There exists a strictly increasing function f such that $\gamma_k \in \Pi_{f(k)} - \Sigma_{f(k)}$ for any $k \in \omega$
4. There exists no type Δ and integer k_0 such that: for any $k \geq k_0$ there exists a subset $\Delta_k \subseteq \Delta$ such that Δ_k is T-equivalent to a set of formulae in $\Sigma_{f(k)}$ and $T \cup \Delta_k \models \gamma_k$.

Second Chang's Omitting Types Theorem

Theorem

Let T be a theory of $\mathcal{L}$. For any model $\mathcal{M}$ of T, there exists an extension $\mathcal{M}^*$ of $\mathcal{M}$ such that:

- *$\mathcal{M}^*$ is a model of T,*
- *$\mathcal{M}^*$ omits every ∞-universal type.*

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