The prime spectrum of MV-algebras

based on a joint work with A. Di Nola and P. Belluce

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Contents

MV-algebras were introduced by Chang in 1959 as the algebraic counterpart of Łukasiewicz logic.

Definition

An **MV-algebra** $\mathcal{A} = \langle A, \oplus, \neg, 0\rangle$ is a commutative monoid $\mathcal{A} = \langle A, \oplus, 0\rangle$ with an involution $(\neg\neg x = x)$ such that for all $x, y \in A$,

$$x \oplus \neg 0 = \neg 0$$
$$\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$$

They were rediscovered, in disguise, by Rodriguez and named Wajsberg algebras

Definition

An **Wajsberg algerba** $\mathcal{A} = \langle A, \to, \neg, 1 \rangle$ is an algebra such that for all $x, y, z \in A$,

i) $1 \to x = x$
ii) $(x \to y) \to ((y \to z) \to (x \to z))$
iii) $(x \to y) \to y = (y \to x) \to x$
iv) $(\neg y \to \neg x) \to (x \to y)$

$$x \to y = \neg x \oplus y \quad \text{and} \quad 1 = \neg 0$$

An Example: the standard MV-algebra

The structure $[0,1]_{MV} = \langle [0,1], \oplus, \neg, 0 \rangle$ where the operation are defined as

$$x \oplus y := \min\{1, x+y\} \qquad \text{and} \qquad \neg x := 1 - x,$$

is an MV-algebra, called the **standard** MV-algebra.

Theorem (Chang 1958)

The algebra $[0,1]_{MV} = \langle [0,1], \oplus, \neg, 0 \rangle$ generates the variety of MV-algebras.

Definition

A partially ordered group is called **lattice ordered** (ℓ-group, for short) if its order is a lattice.
An ℓ-group G is called **unital** (ℓu-group, for short) if there exists an element $u \in G$ (called the **strong unit**) such that for any positive $x \in G$ there exists a natural number n such that $\underbrace{u \oplus \ldots \oplus u}_{n \text{ times}} \geq x$

Given any ℓ-group $G = \langle G, +, -, \leq, 0 \rangle$ and a positive element $u \in G$ the definable algebra

$$\langle [0, u], \oplus, \neg, 0 \rangle \text{ with } x \oplus y = \min\{x + y, u\} \text{ and } \neg x = u - x$$

is an MV-algebra. Furthermore, **every** MV-algebra can be obtained in this way.

The relationship between MV-algebras and abelian ℓ-groups becomes an equivalence if one restricts to $u\ell$-**groups**

Theorem (Mundici 1986)

There exists a categorical equivalence between MV-algebras and abelian ℓu-groups.

or to **perfect**[1] MV-algebras

Theorem (Di Nola and Lettieri 1994)

There exists a categorical equivalence between perfect MV-algebras and abelian ℓ-groups.

[1] An MV-algebra is called **perfect** if it is generated by the intersection of all its maximal ideals.

Any MV-algebra has an underlying lattice structure, defined by:

$$x \vee y = \neg(\neg x \oplus y) \oplus y \text{ and } x \wedge y = \neg(\neg x \vee \neg y).$$

If A is any MV-algebra then:

i) The underlying lattice of A is **distributive**.

ii) The (definable) $(\vee, \wedge, \neg)$-reduct is a **Kleene algebra** (so also a **DeMorgan algebra**).

iii) Define $x \odot y = \neg(\neg x \oplus \neg y)$. The algebra $\langle A, \odot, \rightarrow, 0, 1\rangle$ is a bounded, commutative, **residuated lattice** (or even a bounded commutative **BCK-algebra**).

Definition

A function $[0, 1]^m$ to $[0, 1]$ is called **McNaughton function** if it is:

1. continuous,
2. piece-wise linear
3. with integer coefficients.

Theorem (McNaughton 1951)

The **free MV-algebra** *over m generators is isomorphic to the algebra McNaughton functions, where the MV operations are defined point-wise.*

Ideals

Given an MV-algebra A, a non-empty subset I of A is an **ideal** if it is

1. **downward closed**, i.e. $y \leq x$ and $x \in I$ imply $y \in I$ for all $y \in A$,
2. **stable** with respect to the MV-algebraic sum: $x, y \in I$ implies $x \oplus y \in I$.

An ideal I is called **proper** if $I \neq A$. So MV-ideals are also ideals of the lattice reduct (**lattice ideals**.)

Prime ideals

Definition

An ideal P of an MV-algebra A is called **prime** if A/P is linearly ordered.

Lemma

An ideal P of an MV-algebra A is prime iff it satisfies the following equivalent conditions:

1. *for all $a, b \in A$, $a \to b \in P$ or $b \to a \in P$;*
2. *for all $a, b \in A$, if $a \wedge b \in P$ then $a \in P$ or $b \in P$;*
3. *for all I, J ideals of A, if $I \cap J \subseteq P$ then $I \subseteq P$ or $J \subseteq P$.*

Chang representation theorem

An important insight in the the class of MV-algebras is given by Chang subdirect representation theorem.

Theorem

Let A be an MV-algebra and $\operatorname{Spec} A$ the set of its prime ideals. Then A is a subdirect product of the family $\{A/P\}_P$ with P ranging among prime ideals of A.

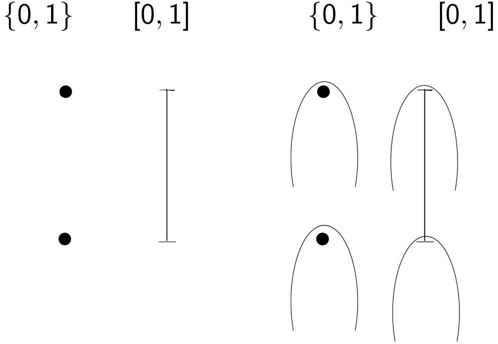
Spectrum of MV-algebras

The set Spec **A** of all the prime ideals of A is called the **spectrum** of A. As in the case of lattices, for any MV-algebra A, the **spectral topology of A** is defined by means of its family of open sets $\{\tau(I) \mid I \text{ ideal of } A\}$ where $\tau(I) = \{P \in \operatorname{Spec} A \mid I \nsubseteq P\}$.

Definition

A topological space is an **MV-space** if it is, up to homeomorphisms, the spectral space of an MV-algebra.

It is easy to see that there are examples of different MV-algebras with the same Spec.



Definition

A topological space X is called **spectral** if

1. X is a compact, T_0 space,
2. every non-empty irreducible closed subset of X is the closure of a unique point (X is **sober**),
3. and the set Ω of compact open subsets of X is a basis for the topology of X and is closed under finite unions and intersections.

Since a spectral space is T_0, it is partially ordered by the so-called **specialisation order**: $x \leq y$ iff $x \in \mathrm{cl}(y)$ where $\mathrm{cl}(y)$ is the closure of y.

Pro-finite MV-spaces

Since MV-space are spectral, one may be tempted to try to characterise them through inverse limit of finite spaces. However in 2004 Di Nola and Grigolia characterised the pro-finite MV-spaces and proved that they do not coincide with the full category of MV-space.

Theorem

An MV-space is pro-finite if and only if it is a completely normal dual Heyting space.

Theorem

There are MV-spaces, as well as completely normal spectral spaces, which are not pro-finite.

Definition

Recall that the triple $\langle X, \leq, \tau\rangle$, where

1. $\langle X, \leq\rangle$ is a poset and
2. $\langle X, \tau\rangle$ is a topological space,

is called a **Priestley space** if

a) τ is a Stone space and

b) for any $x, y \in X$ such that $x \not\leq y$ there is a clopen decreasing set U such that $y \in U$ and $x \notin U$.

Priestley spaces, together with Priestley maps, i.e. continuous and order preserving maps, form the category Pries.

Note that each closed subset of a Priestley space is in turn a Priestley space with respect to the inherited topology.

Consider the controvariant functor $\Delta : \mathsf{BDLat} \longrightarrow \mathsf{Pries}$ which assigns to Priestley space the lattice of clopen downward sets and $\Delta(f)(U) = f^{-1}(U)$.
Consider also the functor Ξ, assigning to each bounded distributive lattice L its set of prime ideals, ordered by set inclusion and topologised by the basis given by the sets $\tau(a) = \{P \in \mathrm{Spec}(L) \mid a \notin P\}$ and their complements for $a \in L$. Furthermore put $\Xi(h)(P) = h^{-1}(P)$.

Theorem

The pair Δ, Ξ is a categorical duality between BDLat and Pries.

In the nineties Martinez developed a duality for Wajsberg algebras.

The idea is to think of a Wajsberg algebra as a **distributive lattice, enriched** with supplementary operations.

This allows to exploit Priestly duality and to build on it.

More precisely Martinez works on a **particular case of Priestley duality**, developed by Cornish and Fowler, characterising Kleene algebras (which in turn are particular De Morgan algebras.)

Definition

A tuple $\langle X, \tau, \leq, g, \{\phi_p\}_{p\in X}\rangle$ is called a **Wajsberg space** if:

1. $\langle X, \tau, \leq, g\rangle$ is a De Morgan space,
2. $\{\phi_p\}_{p\in X}$ is a family of functions $\phi_p : D_p \longrightarrow X$ where $D_p = \{q \in X \mid p \leq g(q)\}$ such that $\forall p, q \in X$:
   a. ϕ_p is order-preserving and continuous in the upper topology,
   b. $p \leq g(q)$ implies $p, q \leq \phi_p(q)$,
   c. $p \leq g(q)$ implies $\phi_p(q) = \phi_q(p)$,
   d. $p \leq g(q)$ implies $\phi_p(g(\phi_p(q))) \leq g(q)$,
   e. $p, p' \leq g(q)$ implies $\phi_p(\phi_{p'}(q)) = \phi_{p'}(\phi_p(q))$,
   f. If $U \in Up(X)$ and $q \notin U$, there exists q_U, the greatest $p \in X$ such that $p \leq g(q)$ and $\phi_p(q) \notin U$; given $U, V \in Up(X)$ if $q \notin U \cup V$ then $(q_V)_V \notin U$.
3. For every $U, V \in Up(X)$, $\bigcap_{p\in U}\left(D_p^c \cup \phi^{-1}(V)\right) \in Up(X)$.

Where $Up(X)$ is the lattice of clopen increasing subsets of X.

Gehrke-Priestley duality

In 2007 Gehrke&Priestley generalise Martinez' approach to any distributive lattice expansion with operations which satisfy some mild order theoretical conditions.

The turning point is to take a "modal" perspective and to study duality through canonical extensions.

This allows to realise that the **failure of canonicity** for MV-algebras lays on an **"alternation" of operations** in the terms defining the variety.

The problem is overcome by considering class of algebras with a **signature doubled** respect to the initial one and to consider equations as inequalities.

Topology arises as a tool for telling when two operations in the doubled signature are two facets of the same starting operation.

Thanks to this approach the supplementary conditions on the Priestly space become **first order**.

$$\left(\forall y_1^+,\ldots,y_j^+,y_{j+1}^-,\ldots,y_k^-,z_1^+,\ldots,z_\ell^+,z_{\ell+1}^-,\ldots,z_m^- \textit{ all in } X^\Diamond\right)$$

$$\left[\left(\left(\bigvee_{\rho_{\mathfrak{s}}(y_i^+)=\alpha_1} \underline{y}_i^+\right)\vee\left(\bigvee_{\rho_{\mathfrak{t}}(z_i^-)=\alpha_1} \underline{z}_i^-\right)\right)\leqslant\left(\left(\bigwedge_{\rho_{\mathfrak{s}}(y_i^-)=\alpha_1} \overline{y}_i^-\right)\wedge\left(\bigwedge_{\rho_{\mathfrak{t}}(z_i^+)=\alpha_1} \overline{z}_i^+\right)\right)\right.$$

$$\&\ \ldots\ \&$$

$$\left.\left(\left(\bigvee_{\rho_{\mathfrak{s}}(y_i^+)=\alpha_n} \underline{y}_i^+\right)\vee\left(\bigvee_{\rho_{\mathfrak{t}}(z_i^-)=\alpha_n} \underline{z}_i^-\right)\right)\leqslant\left(\left(\bigwedge_{\rho_{\mathfrak{s}}(y_i^-)=\alpha_n} \overline{y}_i^-\right)\wedge\left(\bigwedge_{\rho_{\mathfrak{t}}(z_i^+)=\alpha_n} \overline{z}_i^+\right)\right)\right]$$

$$\Longrightarrow \mathfrak{s}'(\underline{y}_1^+,\ldots,\underline{y}_j^+,\overline{y}_{j+1}^-,\ldots,\overline{y}_k^-)\leqslant\mathfrak{t}'(\overline{z}_1^+,\ldots,\overline{z}_\ell^+,\underline{z}_{\ell+1}^-,\ldots,\underline{z}_m^-).$$

Definition

Let A be an MV-algebra and consider the equivalence relation $\equiv$ defined by

$$x \equiv y \text{ if and only if, for all } P \in \operatorname{Spec} A,\ x \in P \Leftrightarrow y \in P.$$

It is easy to see that $\equiv$ is a congruence on the lattice reduct of A and it also preserves the MV-algebraic sum (indeed it equalises $\vee$ and $\oplus$: $[x \vee y]_\equiv = [x \oplus y]_\equiv$ for all $x, y \in A$).
Let us call $[A]$ the quotient set $A/\equiv$ and $[x]$ the equivalence class $[x]_\equiv$.

Lemma

The structure $[A] = \langle [A], \vee, \wedge, [0], [1] \rangle$ *is a bounded distributive lattice, with* $[x] \vee [y] := [x \vee y] = [x \oplus y]$ *and* $[x] \wedge [y] := [x \wedge y]$*, for all* $x, y \in A$*.*

Lemma

The map $[\cdot]$ *is a functor from the category of MV-algebras to the category of bounded distributive lattice*

Knowing on which category such a functor is invertible would constitute a key step in the characterisation of MV-spaces.

Theorem

The map $\gamma : \mathrm{Spec}\, A \to \mathrm{Spec}[A]$, *defined by* $\gamma(P) = [P]$, *is a (Priestley) homeomorphism between the MV-space* $\mathrm{Spec}\, A$ *and the spectral space* $\mathrm{Spec}[A]$.

Theorem

Every bounded distributive lattice in the range of $[\cdot]$ *is dual completely normal (i.e. the set of prime ideals containing a prime ideal is totally ordered.)*

Corollary

Every MV-space is completely normal.[a]

[a] X is normal if any two disjoint closed subsets of X are separated by neighbourhoods

Definition

$X \subseteq \operatorname{Spec} A$ is an **MV-subspace** if X with the induced topology is homeomorphic to $\operatorname{Spec} A'$ for some MV-algebra A', i.e. if it is an MV-space itself.

Proposition

All closed subspaces of an MV-space are MV-subspaces.

MV-subspaces

Proposition

Let $\tau(a)$ be compact open in $\operatorname{Spec} A$. Then $\tau(a)$ is an MV-space.

Lemma

Every MV-algebra has the prime extension property (PEP), namely every ideal extending a prime ideal is itself prime.

Proposition

An MV-algebra A is called **hyper-archimedean** if for each $x \in A$ $nx \in (A)$ for some integer n. Hyper-archimedean MV-algebras are exactly the ones for which $\operatorname{Spec} A = \operatorname{Max} A$.

The order in MV-spaces

Proposition

If X is a linearly ordered spectral space, then X is an MV-space.

Proposition

MV-spaces are root systems with respect to the specialisation order

Definition

A poset $\langle X, \leq \rangle$ is called a **spectral root** if:

1. $\langle X, \leq \rangle$ has a maximum.
2. No pair of incompatible elements of X has a common lower bound.
3. Every linearly ordered subset of X has inf and sup in X.
4. If $x, y \in X$ are such that $x < y$, then there exist $s, t \in X$ such that $x \leq s < t \leq y$ and there is no $z \in X$ such that $s < z < t$.

Definition

A poset $\langle X, \leq \rangle$ is called a **spectral root system** if it the disjoint union of spectral roots.

A characterisation of the order

Theorem

A poset is a spectral root system if, and only if, it is order isomorphic to some MV-space.

Definition

Recalling that an MV-algebra is called **local** if it has a unique maximal ideal, an MV-space is called **local** if it has a greatest element or, equivalently, if its corresponding MV-algebra is local.

Corollary

Every MV-space is a disjoint union of local MV-spaces.

Reduced MV-algebras

Let A be a linearly ordered MV-algebra, and let $\Pr A$ be the set of principal ideals of A.
For each $P \in \Pr A$ let us choose a generator u_P of P, and set $A_0 = \langle u_P \mid P \in \Pr A \rangle$, the subalgebra generated by the u_P's.

A_0 is called the **reduced subalgebra** of A.

A linearly ordered algebra A is **reduced** if $A = \langle u_P \mid (u_P] = P \in \Pr A \rangle$. In this case, the set $\{u_P \mid P \in \Pr A\}$ will be called a set of **principal generators** of A. Note that a set of principal generators is not unique in general.

Reduced algebra are sufficient

Lemma

Let A be a reduced MV-algebra with principal generators $\{u_P \mid P \in \operatorname{Pr} A\}$. Then for every $a \neq 1$ in A there is some principal generator u_P, $\tau(a) = \tau(u_P)$.

Proposition

Any reduced MV-algebra is perfect.

Lemma

$\operatorname{Spec} A_0 \cong \operatorname{Spec} A$

Three simple results and one remark

Corollary

Let X be a spectral space such that, for each $x \in X$, $\mathrm{cl}(x)$ is an upward chain under the specialisation order (i.e., if $y, z \in \mathrm{cl}(x)$, then $x \leq y \leq z$ or $x \leq z \leq y$) then there is a reduced MV-algebra A_x such that $\operatorname{Spec} A_x \cong \mathrm{cl}(x)$.

Lemma

In a reduced MV-algebra A there is a bijection between the set of proper compact open sets of $\operatorname{Spec} A$ and $\operatorname{Pr} A$.

Remark

Note that, in a reduced algebra, $u_P \leq u_Q$ iff $\tau(u_P) \subseteq \tau(u_Q)$ iff $(u_P] \subseteq (u_Q]$.

Remark

In the following X is a spectral space such that, for each $x \in X$, $\mathrm{cl}(x)$ is an upward chain under the specialisation order.

Starting from X we seek for a construction that yields an MV-algebra A such that X and $\operatorname{Spec} A$ are homeomorphic. The theory of springs below gives a partial solution to this problem.

Given X as above, let Ω be the set of its compact open subsets. For any $x \in X$ we set $\Omega_x = \{\omega \in \Omega \mid x \in \omega\}$.

Lemma

In a spectral space X, the intersection of a compact open set U and a closed set V is a compact open subset of V.

Taking, in particular $V = \mathrm{cl}(x)$, this suggests an equivalence relation on Ω of being *indistinguishable over* x, namely

$$\omega \equiv_x \omega' \text{ if, and only if, } \omega \cap \mathrm{cl}(x) = \omega' \cap \mathrm{cl}(x).$$

This is an equivalence relation and, so let $[\omega]_x$ denote the class of ω.

Then $[\omega]_x$ corresponds to a unique principal ideal $u_{[\omega]_x}$ in A_x via the homeomorphism between $\operatorname{Spec} A_x$ and $\operatorname{cl}(x)$.
The correspondence is (strictly) order preserving.
Observe that, if $x \notin \omega$, then $\omega \cap \operatorname{cl}(x) = \varnothing$, so we may limit ourselves to $\omega \in \Omega_x$.

Springs

Definition

A triple $\langle X, \{A_x\}_{x \in X}, A \rangle$ is an **MV-spring** provided X is a spectral space, each A_x is a reduced MV-algebra and A is a subdirect product of the family of A_x's

Example

Let A_i be a family of reduced MV-algebras and let A be a subdirect product of the A_i. Then $\langle \operatorname{Spec} A, \{A_i\}_{i \in I}, A \rangle$ is an MV-spring.

Building an MV-spring

Let Ω as above and $\mathfrak{F}$ be the free MV-algebra generated by Ω.

$$\text{Let } \chi_\omega \in \mathfrak{F}^X \text{ be defined by } \chi_\omega(x) = \begin{cases} \omega & \text{if } x \in \omega \\ 0 & \text{otherwise} \end{cases}$$

Let $A_1 = \langle \chi_\omega \mid \omega \in \Omega \rangle$ be the subalgebra of $\mathfrak{F}^X$ generated by the χ_ω, and, for each $x \in X$, let $\mathbf{F}_x = \langle u_{[\omega]_x} \mid \omega \in \Omega_x \rangle$.

Fix an $x \in X$, the algebra A_1 can be projected into $\mathbf{F}_x$ by the following function.

$$\mu_x : A_1 \overset{\mathrm{ev}_x}{\twoheadrightarrow} \mathfrak{F}_x \overset{\eta_x}{\twoheadrightarrow} \mathbf{F}_x,$$

where:

1. $\mathfrak{F}_x = \langle \omega \mid \omega \in \Omega_x \rangle \subseteq \mathfrak{F}$, i.e. the free algebra generated by Ω_x
2. $\mathrm{ev}_x : A_1 \longleftrightarrow \mathfrak{F}_x$ is the evaluation map, given by $\mathrm{ev}_x(f) = f(x)$. It is not hard to see that ev_x is onto.
3. $\eta_x : \mathfrak{F}_x \longleftrightarrow \mathbf{F}_x$ is the unique epimorphic extension of the partial function $f(\omega) = u_{[\omega]_x}$ (note that f is surjective on the generators).

Now consider $J_1 = \bigcap_x \ker \mu_x$ and define $\hat{A} = A_1 / J_1$.

Proposition

The triple $\langle X, \{\mathbf{F}_x\}_{x\in X}, \hat{A}\rangle$ is a spring.

Given a spring $\langle X, \{A_x\}_{x\in X}, A\rangle$, we have a family of projections $\pi_x : A \longrightarrow A_x$ and we can define a new map

$$\varphi_A : X \longleftrightarrow \operatorname{Spec} A \qquad \text{by setting} \qquad \varphi_A(x) = \ker \pi_x.$$

Since A_x is linearly ordered, $\ker \pi_x \in \operatorname{Spec} A$ and the mapping is well-defined.

Definition

An MV-spring $\langle X, \{A_x\}_{x\in X}, A\rangle$ will be called an **affine MV-spring** provided φ_A is continuous.

Dense subset of $\operatorname{Spec} A$

Given the MV-spring above, $\langle X, \{\mathbf{F}_x\}_{x\in X}, \hat{A}\rangle$, we will write φ for $\varphi_{\hat{A}}$; so we have the map $\varphi : X \longleftrightarrow \operatorname{Spec} \hat{A}$ given by $\varphi(x) = \ker \pi_x$.

Theorem

In $\langle X, \{\mathbf{F}_x\}_{x\in X}, \hat{A}\rangle$ the following properties hold.

(*i*) *φ is injective;*

(*ii*) *$\varphi^{-1} : \varphi(X) \longleftrightarrow X$ is continuous;*

(*iii*) *φ^{-1} is order preserving;*

(*iv*) *$\varphi(X)$ is a dense subspace of $\operatorname{Spec} \hat{A}$.*

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