The prime spectrum of MV-algebras

based on a joint work with A. Di Nola and P. Belluce

Luca Spada

Department of Mathematics and Computer Science
University of Salerno
www.logica.dmi.unisa.it/lucaspada

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MV-algebras were introduced by Chang in 1959 as the algebraic counterpart of Łukasiewicz logic.

**Definition**

An **MV-algebra** $\mathcal{A} = \langle A, \oplus, \neg, 0 \rangle$ is a commutative monoid $\mathcal{A} = \langle A, \oplus, 0 \rangle$ with an involution ($\neg \neg x = x$) such that for all $x, y \in A$,

$$x \oplus \neg 0 = \neg 0$$

$$\neg (\neg x \oplus y) \oplus y = \neg (\neg y \oplus x) \oplus x$$
They were rediscovered, in disguise, by Rodriguez and named Wajsberg algebras

**Definition**

An **Wajsberg algebra** $A = \langle A, \rightarrow, \neg, 1 \rangle$ is an algebra such that for all $x, y, z \in A$,

i) $1 \rightarrow x = x$

ii) $(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z))$

iii) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$

iv) $(\neg y \rightarrow \neg x) \rightarrow (x \rightarrow y)$

$x \rightarrow y = \neg x \oplus y$ and $1 = \neg 0$
An Example: the standard MV-algebra

The structure $[0, 1]_{MV} = \langle [0, 1], \oplus, \neg, 0 \rangle$ where the operation are defined as

$$x \oplus y := \min\{1, x + y\} \quad \text{and} \quad \neg x := 1 - x,$$

is an MV-algebra, called the standard MV-algebra.

**Theorem (Chang 1958)**

The algebra $[0, 1]_{MV} = \langle [0, 1], \oplus, \neg, 0 \rangle$ generates the variety of MV-algebras.
Definition
A partially ordered group is called **lattice ordered** (ℓ-group, for short) if its order is a lattice.

An ℓ-group \( G \) is called **unital** (ℓu-group, for short) if there exists an element \( u \in G \) (called the **strong unit**) such that for any positive \( x \in G \) there exists a natural number \( n \) such that \( u \oplus \ldots \oplus u \geq x \) \( n \) times.

Given any ℓ-group \( G = \langle G, +, −, \leq, 0 \rangle \) and a positive element \( u \in G \) the definable algebra

\[
\langle [0, u], \oplus, −, 0 \rangle \text{ with } x \oplus y = \min\{x + y, u\} \text{ and } \lnot x = u - x
\]

is an MV-algebra. Furthermore, every MV-algebra can be obtained in this way.
Categorical equivalence

The relationship between MV-algebras and abelian \( \ell \)-groups becomes an equivalence if one restricts to \( ul \)-groups

**Theorem (Mundici 1986)**

There exists a categorical equivalence between MV-algebras and abelian \( ul \)-groups.

or to **perfect**\(^1\) MV-algebras

**Theorem (Di Nola and Lettieri 1994)**

There exists a categorical equivalence between perfect MV-algebras and abelian \( \ell \)-groups.

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\(^1\)An MV-algebra is called **perfect** if it is generated by the intersection of all its maximal ideals.
Any MV-algebra has an underlying lattice structure, defined by:

\[ x \lor y = \neg(\neg x \oplus y) \oplus y \quad \text{and} \quad x \land y = \neg(\neg x \lor \neg y). \]

If \( A \) is any MV-algebra then:

i) The underlying lattice of \( A \) is **distributive**.

ii) The (definable) \((\lor, \land, \neg)\)-reduct is a **Kleene algebra** (so also a **DeMorgan algebra**).

iii) Define \( x \odot y = \neg(\neg x \oplus \neg y) \). The algebra \( \langle A, \odot, \to, 0, 1 \rangle \) is a bounded, commutative, **residuated lattice** (or even a bounded commutative **BCK-algebra**).
**Definition**

A function $[0, 1]^m$ to $[0, 1]$ is called **McNaughton function** if it is:

1. continuous,
2. piece-wise linear
3. with integer coefficients.

**Theorem (McNaughton 1951)**

The **free MV-algebra** over $m$ generators is isomorphic to the algebra McNaughton functions, where the MV operations are defined point-wise.
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Ideals

Given an MV-algebra $A$, a non-empty subset $I$ of $A$ is an **ideal** if it is

1. **downward closed**, i.e. $y \leq x$ and $x \in I$ imply $y \in I$ for all $y \in A$,

2. **stable** with respect to the MV-algebraic sum: $x, y \in I$ implies $x \oplus y \in I$.

An ideal $I$ is called **proper** if $I \neq A$. So MV-ideals are also ideals of the lattice reduct (**lattice ideals**.)
**Definition**

An ideal $P$ of an MV-algebra $A$ is called **prime** if $A/P$ is linearly ordered.

**Lemma**

An ideal $P$ of an MV-algebra $A$ is prime iff it satisfies the following equivalent conditions:

1. For all $a, b \in A$, $a \rightarrow b \in P$ or $b \rightarrow a \in P$;
2. For all $a, b \in A$, if $a \land b \in P$ then $a \in P$ or $b \in P$;
3. For all $I, J$ ideals of $A$, if $I \cap J \subseteq P$ then $I \subseteq P$ or $J \subseteq P$. 
An important insight in the class of MV-algebras is given by Chang subdirect representation theorem.

**Theorem**

Let $A$ be an MV-algebra and $\text{Spec } A$ the set of its prime ideals. Then $A$ is a subdirect product of the family $\{A/P\}_P$ with $P$ ranging among prime ideals of $A$. 
Spectrum of MV-algebras

The set $\text{Spec } A$ of all the prime ideals of $A$ is called the spectrum of $A$. As in the case of lattices, for any MV-algebra $A$, the spectral topology of $A$ is defined by means of its family of open sets $\{\tau(I) \mid I \text{ ideal of } A\}$ where $\tau(I) = \{P \in \text{Spec } A \mid I \nsubseteq P\}$.

**Definition**

A topological space is an **MV-space** if it is, up to homeomorphisms, the spectral space of an MV-algebra.
It is easy to see that there are examples of different MV-algebras with the same Spec.
**Definition**

A topological space $X$ is called **spectral** if

1. $X$ is a compact, $T_0$ space,
2. every non-empty irreducible closed subset of $X$ is the closure of a unique point ($X$ is **sober**),
3. and the set $\Omega$ of compact open subsets of $X$ is a basis for the topology of $X$ and is closed under finite unions and intersections.

Since a spectral space is $T_0$, it is partially ordered by the so-called **specialisation order**: $x \leq y$ iff $x \in \text{cl}(y)$ where $\text{cl}(y)$ is the closure of $y$. 
Since MV-space are spectral, one may be tempted to try to characterise them through inverse limit of finite spaces. However in 2004 Di Nola and Grigolia characterised the pro-finite MV-spaces and proved that they do not coincide with the full category of MV-space.

**Theorem**

An MV-space is pro-finite if and only if it is a completely normal dual Heyting space.

**Theorem**

There are MV-spaces, as well as completely normal spectral spaces, which are not pro-finite.
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2. **Ideals of MV-algebras**

3. **(Priestley) Dualities for MV-algebras**

4. **The Belluce functor**

5. **Properties of MV-spaces**

6. **Reduced MV-algebras**

7. **MV-spring**
**Priestley space**

**Definition**

Recall that the triple $\langle X, \leq, \tau \rangle$, where

1. $\langle X, \leq \rangle$ is a poset and
2. $\langle X, \tau \rangle$ is a topological space,

is called a **Priestley space** if

a) $\tau$ is a Stone space and

b) for any $x, y \in X$ such that $x \not\leq y$ there is a clopen decreasing set $U$ such that $y \in U$ and $x \not\in U$.

Priestley spaces, together with Priestley maps, i.e. continuous and order preserving maps, form the category Pries.

Note that each closed subset of a Priestley space is in turn a Priestley space with respect to the inherited topology.
Consider the controvariant functor $\Delta : \text{BDLat} \to \text{Pries}$ which assigns to Priestley space the lattice of clopen downward sets and $\Delta(f)(U) = f^{-1}(U)$.

Consider also the functor $\Xi$, assigning to each bounded distributive lattice $L$ its set of prime ideals, ordered by set inclusion and topologised by the basis given by the sets $\tau(a) = \{P \in \text{Spec}(L) \mid a \not\in P\}$ and their complements for $a \in L$. Furthermore put $\Xi(h)(P) = h^{-1}(P)$.

**Theorem**

The pair $\Delta, \Xi$ is a categorical duality between $\text{BDLat}$ and $\text{Pries}$.
In the nineties Martinez developed a duality for Wajsberg algebras.
The idea is to think of a Wajsberg algebra as a **distributive lattice, enriched** with supplementary operations.
This allows to exploit Priestly duality and to build on it.
More precisely Martinez works on a **particular case of Priestley duality**, developed by Cornish and Fowler, characterising Kleene algebras (which in turn are particular De Morgan algebras.)
Definition

A tuple \( \langle X, \tau, \leq, g, \{ \phi_p \}_{p \in X} \rangle \) is called a **Wajsberg space** if:

1. \( \langle X, \tau, \leq, g \rangle \) is a De Morgan space,
2. \( \{ \phi_p \}_{p \in X} \) is a family of functions \( \phi_p : D_p \longrightarrow X \) where \( D_p = \{ q \in X \mid p \leq g(q) \} \) such that \( \forall p, q \in X \):
   - a. \( \phi_p \) is order-preserving and continuous in the upper topology,
   - b. \( p \leq g(q) \) implies \( p, q \leq \phi_p(q) \),
   - c. \( p \leq g(q) \) implies \( \phi_p(q) = \phi_q(p) \),
   - d. \( p \leq g(q) \) implies \( \phi_p(g(\phi_p(q))) \leq g(q) \),
   - e. \( p, p' \leq g(q) \) implies \( \phi_p(\phi_p'(q)) = \phi_p'(\phi_p(q)) \),
   - f. If \( U \in Up(X) \) and \( q \notin U \), there exists \( q_U \), the greatest \( p \in X \) such that \( p \leq g(q) \) and \( \phi_p(q) \notin U \); given \( U, V \in Up(X) \) if \( q \notin U \cup V \) then \( (q_V)_V \notin U \).
3. For every \( U, V \in Up(X) \), \( \bigcap_{p \in U} \left( D_p^c \cup \phi_p^{-1}(V) \right) \in Up(X) \).

Where \( Up(X) \) is the lattice of clopen increasing subsets of \( X \).
In 2007 Gehrke&Priestley generalise Martinez’ approach to any distributive lattice expansion with operations which satisfy some mild order theoretical conditions.

The turning point is to take a “modal” perspective and to study duality through canonical extensions.

This allows to realise that the failure of canonicity for MV-algebras lays on an “alternation” of operations in the terms defining the variety.

The problem is overcome by considering class of algebras with a signature doubled respect to the initial one and to consider equations as inequalities.
Gehrke-Priestley duality

Topology arises as a tool for telling when two operations in the doubled signature are two facets of the same starting operation.

Thanks to this approach the supplementary conditions on the Priestly space become first order.

\[
\forall y_1^+, \ldots, y_j^+, y_{j+1}^-, \ldots, y_k^-, z_1^+, \ldots, z_{\ell}^+, z_{\ell+1}^-, \ldots, z_m^- \text{ all in } X^\Diamond
\]

\[
\left[\left(\bigvee_{\rho_s(y_i^+) = \alpha_1} y_i^+ \right) \lor \left(\bigvee_{\rho_t(z_i^-) = \alpha_1} z_i^- \right) \right] \leq \left(\bigwedge_{\rho_s(y_i^-) = \alpha_1} y_i^- \right) \land \left(\bigwedge_{\rho_t(z_i^+) = \alpha_1} z_i^+ \right)
\]

& \ldots &

\[
\left(\bigvee_{\rho_s(y_i^+) = \alpha_n} y_i^+ \right) \lor \left(\bigvee_{\rho_t(z_i^-) = \alpha_n} z_i^- \right) \leq \left(\bigwedge_{\rho_s(y_i^-) = \alpha_n} y_i^- \right) \land \left(\bigwedge_{\rho_t(z_i^+) = \alpha_n} z_i^+ \right)
\]

\[\implies s'(y_1^+, \ldots, y_j^+, \overline{y}_{j+1}, \ldots, \overline{y}_k) \leq t'(z_1^+, \ldots, z_{\ell}^+, \overline{z}_{\ell+1}, \ldots, z_m^-).\]
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**The Belluce functor**

**Definition**

Let \( A \) be an MV-algebra and consider the equivalence relation \( \equiv \) defined by

\[
x \equiv y \text{ if and only if, for all } P \in \text{Spec } A, \ x \in P \iff y \in P.
\]

It is easy to see that \( \equiv \) is a congruence on the lattice reduct of \( A \) and it also preserves the MV-algebraic sum (indeed it equalises \( \lor \) and \( \oplus \): \( [x \lor y] \equiv = [x \oplus y] \equiv \) for all \( x, y \in A \)).

Let us call \( [A] \) the quotient set \( A/\equiv \) and \( [x] \) the equivalence class \( [x] \equiv \).
The Belluce functor

**Lemma**

The structure $[A] = \langle[A], \lor, \land, [0], [1] \rangle$ is a bounded distributive lattice, with $[x] \lor [y] := [x \lor y] = [x \oplus y]$ and $[x] \land [y] := [x \land y]$, for all $x, y \in A$.

**Lemma**

The map $[\cdot]$ is a functor from the category of MV-algebras to the category of bounded distributive lattice.

Knowing on which category such a functor is invertible would constitute a key step in the characterisation of MV-spaces.
The Belluce functor

Theorem

The map $\gamma : \text{Spec } A \rightarrow \text{Spec}[A]$, defined by $\gamma(P) = [P]$, is a (Priestley) homeomorphism between the MV-space $\text{Spec } A$ and the spectral space $\text{Spec}[A]$.

Theorem

Every bounded distributive lattice in the range of $[\cdot]$ is dual completely normal (i.e. the set of prime ideals containing a prime ideal is totally ordered.)

Corollary

Every MV-space is completely normal.$^a$

$^a$X is normal if any two disjoint closed subsets of $X$ are separated by neighbourhoods.
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**Definition**

$X \subseteq \text{Spec } A$ is an **MV-subspace** if $X$ with the induced topology is homeomorphic to $\text{Spec } A'$ for some MV-algebra $A'$, i.e. if it is an MV-space itself.

**Proposition**

All closed subspaces of an MV-space are MV-subspaces.
**Proposition**

Let $\tau(a)$ be compact open in $\text{Spec } A$. Then $\tau(a)$ is an MV-space.

**Lemma**

Every MV-algebra has the prime extension property (PEP), namely every ideal extending a prime ideal is itself prime.

**Proposition**

An MV-algebra $A$ is called **hyper-archimedean** if for each $x \in A$ $nx \in (A)$ for some integer $n$. Hyper-archimedean MV-algebras are exactly the ones for which $\text{Spec } A = \text{Max } A$. 
**The order in MV-spaces**

**Proposition**
If $X$ is a linearly ordered spectral space, then $X$ is an MV-space.

**Proposition**
MV-spaces are root systems with respect to the specialisation order.
**Definition**

A poset $\langle X, \leq \rangle$ is called a **spectral root** if:

1. $\langle X, \leq \rangle$ has a maximum.
2. No pair of incompatible elements of $X$ has a common lower bound.
3. Every linearly ordered subset of $X$ has inf and sup in $X$.
4. If $x, y \in X$ are such that $x < y$, then there exist $s, t \in X$ such that $x \leq s < t \leq y$ and there is no $z \in X$ such that $s < z < t$.

**Definition**

A poset $\langle X, \leq \rangle$ is called a **spectral root system** if it the disjoint union of spectral roots.
A characterisation of the order

Theorem

A poset is a spectral root system if, and only if, it is order isomorphic to some MV-space.

Definition

Recalling that an MV-algebra is called local if it has a unique maximal ideal, an MV-space is called local if it has a greatest element or, equivalently, if its corresponding MV-algebra is local.

Corollary

Every MV-space is a disjoint union of local MV-spaces.
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Let $A$ be a linearly ordered MV-algebra, and let $\text{Pr} A$ be the set of principal ideals of $A$.

For each $P \in \text{Pr} A$ let us choose a generator $u_P$ of $P$, and set $A_0 = \langle u_P \mid P \in \text{Pr} A \rangle$, the subalgebra generated by the $u_P$’s.

$A_0$ is called the **reduced subalgebra** of $A$.

A linearly ordered algebra $A$ is **reduced** if $A = \langle u_P \mid (u_P] = P \in \text{Pr} A \rangle$. In this case, the set $\{u_P \mid P \in \text{Pr} A\}$ will be called a set of **principal generators** of $A$. Note that a set of principal generators is not unique in general.
Lemma

Let $A$ be a reduced MV-algebra with principal generators $\{u_P \mid P \in \text{Pr } A\}$. Then for every $a \neq 1$ in $A$ there is some principal generator $u_P$, $\tau(a) = \tau(u_P)$.

Proposition

Any reduced MV-algebra is perfect.

Lemma

$\text{Spec } A_0 \cong \text{Spec } A$
Three simple results and one remark

**Corollary**

Let $X$ be a spectral space such that, for each $x \in X$, $\text{cl}(x)$ is an upward chain under the specialisation order (i.e., if $y, z \in \text{cl}(x)$, then $x \leq y \leq z$ or $x \leq z \leq y$) then there is a reduced MV-algebra $A_x$ such that $\text{Spec } A_x \cong \text{cl}(x)$.

**Lemma**

In a reduced MV-algebra $A$ there is a bijection between the set of proper compact open sets of $\text{Spec } A$ and $\text{Pr } A$.

**Remark**

Note that, in a reduced algebra, $u_P \leq u_Q$ iff $\tau(u_P) \subseteq \tau(u_Q)$ iff $(u_P] \subseteq (u_Q]$. 
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Remark

In the following $X$ is a spectral space such that, for each $x \in X$, $\text{cl}(x)$ is an upward chain under the specialisation order.

Starting from $X$ we seek for a construction that yields an MV-algebra $A$ such that $X$ and Spec $A$ are homeomorphic. The theory of springs below gives a partial solution to this problem.
Given $X$ as above, let $\Omega$ be the set of its compact open subsets. For any $x \in X$ we set $\Omega_x = \{ \omega \in \Omega \mid x \in \omega \}$.

**Lemma**

In a spectral space $X$, the intersection of a compact open set $U$ and a closed set $V$ is a compact open subset of $V$.

Taking, in particular $V = \text{cl}(x)$, this suggests an equivalence relation on $\Omega$ of being *indistinguishable over $x$*, namely

$$\omega \equiv_x \omega' \text{ if, and only if, } \omega \cap \text{cl}(x) = \omega' \cap \text{cl}(x).$$

This is an equivalence relation and, so let $[\omega]_x$ denote the class of $\omega$. 
Then $[\omega]_x$ corresponds to a unique principal ideal $u_{[\omega]}_x$ in $A_x$ via the homeomorphism between Spec $A_x$ and cl$(x)$. The correspondence is (strictly) order preserving. Observe that, if $x \notin \omega$, then $\omega \cap\text{cl}(x) = \emptyset$, so we may limit ourselves to $\omega \in \Omega_x$. 


**Definition**

A triple \( \langle X, \{A_x\}_{x \in X}, A \rangle \) is an **MV-spring** provided \( X \) is a spectral space, each \( A_x \) is a reduced MV-algebra and \( A \) is a subdirect product of the family of \( A_x \)'s.

**Example**

Let \( A_i \) be a family of reduced MV-algebras and let \( A \) be a subdirect product of the \( A_i \)'s. Then \( \langle \text{Spec } A, \{A_i\}_{i \in I}, A \rangle \) is an MV-spring.
Let $\Omega$ as above and $\mathcal{F}$ be the free MV-algebra generated by $\Omega$.

Let $\chi_\omega \in \mathcal{F}^X$ be defined by $\chi_\omega(x) = \begin{cases} \omega & \text{if } x \in \omega \\ 0 & \text{otherwise} \end{cases}$

Let $A_1 = \langle \chi_\omega \mid \omega \in \Omega \rangle$ be the subalgebra of $\mathcal{F}^X$ generated by the $\chi_\omega$, and, for each $x \in X$, let $F_x = \langle u_{[\omega]_x} \mid \omega \in \Omega_x \rangle$. 
Building an MV-spring

Fix an \( x \in X \), the algebra \( A_1 \) can be projected into \( F_x \) by the following function.

\[
\mu_x : A_1 \xrightarrow{ev_x} \mathcal{F}_x \xrightarrow{\eta_x} F_x,
\]

where:

1. \( \mathcal{F}_x = \langle \omega \mid \omega \in \Omega_x \rangle \subseteq \mathcal{F} \), i.e. the free algebra generated by \( \Omega_x \)
2. \( ev_x : A_1 \xleftarrow{} \mathcal{F}_x \) is the evaluation map, given by \( ev_x(f) = f(x) \). It is not hard to see that \( ev_x \) is onto.
3. \( \eta_x : \mathcal{F}_x \xrightarrow{} F_x \) is the unique epimorphic extension of the partial function \( f(\omega) = u_{[\omega]} \) (note that \( f \) is surjective on the generators).

Now consider \( J_1 = \bigcap_x \ker \mu_x \) and define \( \hat{A} = A_1 / J_1 \).
**Proposition**

The triple \( \langle X, \{ F_x \}_{x \in X}, \hat{A} \rangle \) is a spring.

Given a spring \( \langle X, \{ A_x \}_{x \in X}, A \rangle \), we have a family of projections \( \pi_x : A \rightarrow A_x \) and we can define a new map

\[
\varphi_A : X \leftrightarrow \text{Spec } A \quad \text{by setting} \quad \varphi_A(x) = \ker \pi_x.
\]

Since \( A_x \) is linearly ordered, \( \ker \pi_x \in \text{Spec } A \) and the mapping is well-defined.

**Definition**

An MV-spring \( \langle X, \{ A_x \}_{x \in X}, A \rangle \) will be called an affine MV-spring provided \( \varphi_A \) is continuous.
Given the MV-spring above, \( \langle X, \{ F_x \}_{x \in X}, \hat{A} \rangle \), we will write \( \varphi \) for \( \varphi_{\hat{A}} \); so we have the map \( \varphi : X \longleftrightarrow \text{Spec} \hat{A} \) given by \( \varphi(x) = \ker \pi_x \).

**Theorem**

In \( \langle X, \{ F_x \}_{x \in X}, \hat{A} \rangle \) the following properties hold.

(i) \( \varphi \) is injective;

(ii) \( \varphi^{-1} : \varphi(X) \longleftrightarrow X \) is continuous;

(iii) \( \varphi^{-1} \) is order preserving;

(iv) \( \varphi(X) \) is a dense subspace of \( \text{Spec} \hat{A} \).
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