A uniform version of Di Nola Theorem

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MV-algebras

MV-algebras are the equivalent algebraic semantics of Lukasiewicz logic.

A structure $A = (A, \oplus, \neg, 0)$ is an MV-algebra if $A$ satisfies the following equations, for every $x, y, z \in A$:

(i) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$;   (ii) $x \oplus y = y \oplus x$;

(iii) $x \oplus 0 = x$;   (iv) $x \oplus \neg 0 = \neg 0$;

(v) $\neg \neg x = x$;   (vi) $\neg (\neg x \oplus y) \oplus y = \neg (\neg y \oplus x) \oplus x$. 
Representations of MV algebras

The main general tools in representation theory of MV-algebras are

- Chang representation Theorem,
- McNaughton Theorem and
- Di Nola representation Theorem.
The main result

The main result in this talk gives a non-standard representation of any MV-algebra $A$ depending only on the cardinality of $A$.

**Theorem:** For any infinite cardinal $\alpha$, there exists an ultrapower of $[0,1]^*$, such that all MV-algebras of cardinality smaller or equal than $\alpha$ embed in an MV-algebra of functions with values in $[0,1]^*$.
Controlling the cardinality

Lemma. Let $A$ be an MV-algebra and $(G,u)$ an lu-group such that $A \cong \Gamma(G,u)$. Let $\alpha$ be an infinite cardinal then $|A| = \alpha$ if, and only if, $|G| = \alpha$.

Lemma. Let $G$ be an abelian l-group and $\alpha$ be an infinite cardinal such that $|G| = \alpha$. Then $G$ can be embedded into an abelian divisible l-group $D_G$ such that $|D_G| = \alpha$. 
\(\alpha\)-regular filters

**definition.** Let \(\alpha\) be a cardinal. A proper filter \(D\) over \(I\) is said to be \(\alpha\)-regular if there exists a subset \(E\) of \(D\) such that \(|E| = \alpha\) and each \(i \in I\) belongs to only finitely many \(e \in E\).

**definition.** Given a cardinal \(\alpha\), we say that a model \(A\) is \(\alpha\)-universal if for every model \(B\) we have:

\[B \equiv A \quad \text{and} \quad |B| < \alpha \quad \text{implies} \quad B \rightarrow_{el.} A.\]
α⁺-universal ultrapowers

**Theorem.** [Chang-Keisler] Let \(|\mathcal{L}| \leq \alpha\) and \(D\) be an ultrafilter which is \(\alpha\)-regular. Then, for every model \(A\), the ultrapower \(\Pi_D A\) is \(\alpha^+\)-universal.

**Lemma.** For any sentence \(\psi\) the language of MV algebras there is a formula with only one free variable \(\phi(v)\) in the language of lu-group such that for any MV-algebra \(A\) we have:

\[ A \models \psi \] if, and only if, \( G \models \psi[u] \),

for any abelian l-group \(G\) and \(u > 0\) in \(G\) such that \(A \approx \Gamma(G,u)\).
The additive group of reals

Since any non-trivial divisible totally ordered \( l \)-group is elementarily equivalent to the real numbers seen as an additive group, from the previous result we get:

**Proposition.** Any non-trivial divisible MV-chain is elementarily equivalent to \( \Gamma(R,1) = [0,1] \).
Representing MV-chains

**Proposition.** Let $\alpha$ be an infinite cardinal and $A$ be an MV-chain such that $|A| = \alpha$. Then $A$ can be embedded into an ultrapower of the MV-algebra $[0,1]$ via an ultrafilter $\alpha$-regular over $\alpha$ which does not depend on $A$. 
A sketch of the proof

**Proof.** Let $A$ be an infinite MV-chain such that $|A| = \alpha$ and $A \cong \Gamma(G,u)$. Then $G$ is an ordered abelian group with strong unit $u$ and $|G| = \alpha$.

So $(G,u)$ can be embedded into a divisible ordered group $D_G$ with strong unit $u_D$; in addition $|D_G| = \alpha$.

Now let $A_d \cong \Gamma(D_G, u_D)$: then $A$ embeds in $A_d$ and $A_d$ is a divisible MV-algebra; so $A_d$ is elementarily equivalent to $[0,1]$.

Let $F$ be a $\alpha$-regular ultrafilter over $\alpha$; then $\Pi_F[0,1]$ is $\alpha^+$-universal, hence $A_d$ embeds in $\Pi_F[0,1]$. Combining the embeddings we get that $A$ can be embedded into the ultrapower $\Pi_F[0,1]$. 
The general case

**Proposition.** Let $A$ be an MV-algebra such that $|A| = \alpha$, with $\alpha$ an infinite cardinal. Then there exists a set $X$ such that $A$ can be embedded into an MV-algebra of functions from $X$ to an ultrapower of the MV-algebra $[0,1]$ via an $\alpha$-regular ultrafilter over $\alpha$ which does not depend on $A$.

**Proof.**

$$A \hookrightarrow \prod_{P \in \text{Spec}(A)} A/P.$$ 

**Corollary.** For any infinite cardinal $\alpha$ there exists a single MV-algebra of functions such that every MV-algebra of cardinality smaller or equal than $\alpha$ embeds into it.
It is also possible to give a sharp bound on the cardinality of the target algebra, based on following fact, which is part of the classical literature on the subject.

**Proposition.** [Chang-Keisler] Let $F$ be a $\alpha$-regular ultrafilter of $\alpha$, with $\alpha$ infinite cardinal, then $|\Pi F A| = |A|^\alpha$. 
A “canonical” MV-algebra

The above construction gives no information on the target algebra, it only asserts its existence.

We will see now how it is possible to construct such an algebra in ZFC.

The key tool in this construction are iterated ultrapowers.
Introducing Iterated Ultrapowers

An iterated ultrapower can be roughly described as a structure obtained from a linearly ordered set of ultrapowers and such that all these ultrapowers are embedded into it.

We sketch here such a construction:

- Let $A$ be a first order structure
- $I$ be a set
- $(X,<)$ a linear order
- $D=\langle D_x \rangle_{x \in X}$ a l.o. sequence of ultrafilters on $I$
Let $K=I^X$ be the set of all functions from $X$ to $I$. Let $Z$ be a subset of $X$. We say that a function $f$ with domain $K$ lives on $Z$ if, for every function $i \in K$, $f(i)$ depends only on $i|_Z$.

We say that a subset of $K$ lives on $Z$ if its characteristic function lives on $Z$.

Let hereafter $K$ be finite.
The ultrafilter associated

To any $Z$ we associate an ultrafilter $D_Z$ on $I^Z$ as follows:

$$D_Z = \{ s \subseteq B_Z : D_{x_1y_1} \ldots D_{x_ny_n} \{(x_1,y_1), \ldots, (x_n,y_n)\} \in s \},$$

where $D_{x,y}.\phi(y)$ means $\{y : \phi(y)\} \in D_x$.

Consider the set

$$E(D) = \{ s \subseteq K : \exists Z. \text{ s lives on } Z \text{ and } s \downarrow Z \in D_Z \},$$

where $s \downarrow Z$ is the set of all restrictions to $Z$ of the members of $s$. 
Proposition. [Chang-Keisler] The subsets of $K$, living on some finite subset of $X$, form a Boolean algebra $S$. The set $E(D)$ is an ultrafilter in $S$.

$E(D)$ can be considered as an infinitary product of the ultrafilters $D_x$ (although it is not an ultrafilter on $K$ as one could expect).
Iterated Ultrapowers II

Definition. Let $A$ be a first order structure, $I$ be a set, and $D$ be a linearly ordered sequence of ultrafilters on $I$ indexed by the linear order $(X,<)$. The iterated ultrapower of $A$ on $D$, denoted $\Pi_D A$, is a first order structure over the same language as $A$.

The domain of $\Pi_D A$ is the set of all functions $f$ from $K(=B^X)$ to $A$ which live on some finite subset of $X$, modulo the equivalence $=_D$ given by:

$$f =_D g \text{ if, and only if, } \{i \in K : f(i) = g(i)\} \in E(D).$$

If $R$ is any predicate symbol in the language of $A$ then

$$R^\Pi_D A(f_1,...,f_n) \text{ iff } \{i \in K : R(f_1(i),...,f_n(i))\} \in E(D).$$
A final stratagem

Lemma. For every $x \in X$, $\Pi_{Dx}A$ embeds elementarily in $\Pi_DT_A$.

So, for our aim, it is sufficient to find a definable linear order of all ultrafilter on a given ordinal.

A final stratagem

Let $P(\alpha)$ be the powerset of $\alpha$. $P(\alpha)$ has a natural “lexicographic” linear order: given $E,F \subseteq \alpha$, we let $E < F$ if $E$ and $F$ are different, and the least element of $\alpha$ where $E$ and $F$ differ belongs to $F$.

Let $X$ be the set of all maps $x : |P(\alpha)| \to P(\alpha)$ such that the image of $x$ is an ultrafilter on $\alpha$. Note that every ultrafilter on $\alpha$ appears as image of some (actually infinitely many) elements of $X$.

The set $X$ is totally ordered by setting $x < x'$ if there is an ordinal $\xi < |P(\alpha)|$ such that $x|\xi = x'|\xi$ (that is, $x$ and $x'$ coincide on all the ordinals less than $\xi$) and $x(\xi) < x'(\xi)$ in the lexicographic order of $P(\alpha)$.
A Canonical Representation

**Theorem.** For every infinite cardinal $\alpha$ there is an iterated ultrapower $\Pi_\alpha$ of $[0,1]$, definable in $\alpha$, where every MV-chain of cardinality $\alpha$ embeds.

**Corollary.** For every infinite cardinal $\alpha$ there is an iterated ultrapower $\Pi_\alpha$ of $[0,1]$, definable in $\alpha$, such that every MV-algebra of cardinality $\alpha$ embeds in an algebra of functions with values in $\Pi_\alpha$. 
Thank you!