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ŁΠ$_q$ algebras and Quasifields

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Preliminary Definitions

Definition 1. A PMV-*algebra* is an algebra $\mathcal{A} = \langle A, \oplus, \neg, \cdot, 0, 1\rangle$ such that:

$\langle A, \oplus, \neg, 0, 1\rangle$ is a MV-algebra.

$\langle A, \cdot, 1\rangle$ is a commutative monoid.

For all $x, y, z \in \mathcal{A}$ one has: $x \cdot (y \ominus z) = (x \cdot y) \ominus (x \cdot z)$, where $x \ominus y = \neg(\neg x \oplus y)$.

Definition 2. A *LΠ-algebra* is an algebra

$$\mathcal{A} = \langle A, \oplus, \neg, \cdot, \rightarrow_\pi, 0, 1\rangle$$

such that $\langle A, \oplus, \neg, \cdot, 0, 1\rangle$ is a PMV-algebra, $\langle A, \cdot, \rightarrow_\pi, 0, 1\rangle$ is a bounded hoop, and letting $\neg_\pi x = x \rightarrow_\pi 0$ and $\Delta(x) = \neg_\pi \neg x$, the following equations hold:

- $x \rightarrow_\pi y \leq x \rightarrow y$.
- $x \wedge \neg_\pi x = 0$
- $\Delta(x) \odot \Delta(x \rightarrow y) \leq \Delta(y)$
- $\Delta(x) \leq x$
- $\Delta(\Delta(x)) = \Delta(x)$
- $\Delta(x \vee y) = \Delta(x) \vee \Delta(y)$
- $\Delta(x) \vee \neg\Delta(x) = 1$
- $\Delta(x \rightarrow y) \leq x \rightarrow_\pi y$.

Definition 3. A *LΠ$\frac{1}{2}$-algebra* is a ŁΠ-algebra with an additional constant $\frac{1}{2}$ satisfying $\frac{1}{2} = \neg\frac{1}{2}$.

Definitions

Definition 1. [MS03]. A ŁΠ$_q$-algebra is a structure

$$\mathcal{A} = \langle A, \oplus, \neg, \cdot, \rightarrow_q, q, 0, 1\rangle$$

where $\langle A, \oplus, \neg, \cdot, 0, 1\rangle$ is a PMV-algebra, q is a constant, and $\rightarrow_q$ is a binary operation such that the following conditions hold:

(A1) $q \leq \neg q$

(A2) $x \rightarrow_q y = (x \vee q) \rightarrow_q y$

(A3) $(x \vee q)(x \rightarrow_q y) = (x \vee q) \wedge y$

(A4) $q \rightarrow_q (xq) = x$

(A5) If $x^2 = 0$ then $x = 0$

Notation. We use $u(a)$ to denote $(a \vee 0) \wedge 1$.

Definition 2. A *f-quasifield* is a structure

$$\langle K, +, -, \times, /_q, \vee, \wedge, 0, 1, q\rangle$$

where $\langle K, +, -, \times, \vee, \wedge, 0, 1, q\rangle$ is a c-s-u-f-ring with strong unit 1, q is a constant and $/_q$ is a binary operation such that the following conditions are satisfied:

(K1) $0 \leq q \leq 1 - q$

(K2) $x/_q y = u(x)/_q u(y) = u(x)/_q(u(y) \vee q)$.

(K3) $(u(x) \vee q) \times (u(y)/_q u(x)) = (u(x) \vee q) \wedge u(y)$

(K4) $(u(x) \times q)/_q q = u(x)$

(K5) If $x \times x = 0$ then $x = 0$.

Definition 3. Let $\mathbf{LP}_q$ and $\mathbf{FQ}$ denote the category of ŁΠ$_q$-algebras and the category of f-quasifields respectively, with morphisms the homomorphisms in the sense of Universal Algebra.

We define a functor Π_q from $\mathbf{FQ}$ into $\mathbf{LP}$ as follows:

(a) For every f-quasifield $\mathcal{F}$ we define a structure $\Pi_q(\mathcal{F})$ whose domain $\Pi_q(F)$ is $[0,1] = \{x \in \mathcal{F} : 0 \leq x \leq 1\}$, whose constants 0, 1 and q are those of $\mathcal{F}$, and whose operations $\oplus$, $\neg$, $\cdot$ and $\rightarrow_q$ of $\Pi_q(F)$ are defined as follows:

(a1) $x \oplus y = (x + y) \wedge 1$, $\neg x = 1 - x$, and $x \rightarrow_q y = y/_q x$.

(a2) The operation $\cdot$ is the restriction of $\times$ to $[0, 1]$.

(b) For every morphism Φ from a f-quasifield $\mathcal{F}$ into a f-quasifield $\mathcal{K}$, we define $\Pi_q(\Phi)$ to be the restriction of Φ to $\Pi_q(\mathcal{F})$.

Now we define a functor Π_q^{-1} $\mathbf{LP}$ into $\mathbf{FQ}$ as follows:

(a) For every ŁΠ$_q$-algebra $\mathcal{A}$, the c-s-u-f-ring subreduct of $\Pi_q^{-1}(\mathcal{A})$ is $\Gamma_{\mathbf{R}}^{-1}(\mathbf{F}(\mathcal{A}))$. Moreover the constant q is interpreted as $q_0 = i_{\mathbf{F}(\mathcal{A})}(q^{\mathcal{A}})$, where $q^{\mathcal{A}}$ is the interpretation of q in $\mathcal{A}$.
Note that the domain of $\Gamma_{\mathbf{R}}(\Gamma_{\mathbf{R}}^{-1}(\mathbf{F}(\mathcal{A})))$ is contained into the domain of $\Gamma_{\mathbf{R}}^{-1}(\mathbf{F}(\mathcal{A}))$, therefore $i_{\mathbf{F}(\mathcal{A})}(q^{\mathcal{A}}) \in \Pi_q^{-1}(\mathcal{A})$.
Moreover we define:

$$x/_q y = i_{\mathbf{F}(\mathcal{A})}((i_{\mathbf{F}(\mathcal{A})})^{-1}(u(y)) \rightarrow_q (i_{\mathbf{F}(\mathcal{A})})^{-1}(u(x))).$$

(b) If ϕ is a morphism of ŁΠ-algebras from $\mathcal{A}$ into $\mathcal{B}$, then $\Pi_q^{-1}(\phi) = \Gamma^{-1}(\mathbf{F}(\phi))$.

Main Results

Theorem 1. *Let $\mathcal{K} = \langle K, +, -, \times, /_q, \vee, \wedge, 0, 1, q\rangle$ be a linearly ordered quasifield. The following are equivalent:*

(i) There are no infinitesimal in $\mathcal{K}$ (i.e. $(\forall\varepsilon > 0)$ $(\exists n \in \mathbf{N})(n\varepsilon > 1 - \varepsilon)$)

(ii) $\mathcal{K}$ is Archimedean (i.e. $\forall b \forall a > 0 \exists n \in \mathbf{N}(na \geq b)$).

(iii) $\langle K, +, -, \times, 0, 1\rangle$ is a field.

Corollary 2. *If $\mathcal{F}$ is a f-quasifield, then the ring of rationals $\mathbf{Q}$ can be embedded into the ring-reduct of $\mathcal{F}$.*

Theorem 3. *The categories of ŁΠ$_q$-algebras and of f-quasifields are equivalent via the functors Π_q and Π_q^{-1}.*

Corollary 4. *Every f-quasifield is isomorphic to a subdirect product of a family of linearly ordered f-quasifields.*

Corollary 5. *f-quasifields constitute a quasivariety, but not a variety.*

Examples

Example. Let $\mathbf{R}^\star$ be any non-trivial ultrapower of the ordered field $\mathbf{R}$ of real numbers, and let ε be any strictly positive infinitesimal. Then for all $n \in \mathbf{N}$, $n < \frac{1}{\varepsilon}$. So 1 is not a strong unit and for any choice of $q \in (0, \frac{1}{2}]$, $\langle \mathbf{R}^\star, +, -, \times, /_q, \vee, \wedge, , 0, 1, q\rangle$ (where $\times$ denotes product and $x/_q y = \frac{u(x)}{u(y) \vee q}$) is not a f-quasifield although $\langle \mathbf{R}^\star, +, -, \times, 0, 1, \rangle$ is a field.
Example. Let $\mathbf{R}^\star$ be as before, let $q = \frac{1}{2}$ and let

$$\mathbf{R}^\star_{fin} = \{x \in \mathbf{R}^\star : \exists n \in \mathbf{N}(|\, x \,| \leq n)\}.$$

It is easy to see that $\mathbf{R}^\star_{fin}$ is a c-s-u-f-ring. Now let $x, y \in [\frac{1}{2}, 1]$ be such that $x \leq y$, and let $z = \frac{x}{y}$. Then $\frac{1}{2} \leq z \leq 1$, therefore $z \in \mathbf{R}^\star_{fin}$.
It follows that, letting $a/_q b = \frac{u(a)}{(u(b) \vee q)}$, $\mathbf{R}^\star_{fin}$ is closed under $/_q$, and $/_q$ makes $\mathbf{R}^\star_{fin}$ a f-quasifield. Nevertheless $\mathbf{R}^\star_{fin}$ is not a field, because if $\varepsilon \in \mathbf{R}^\star_{fin}$ is a strictly positive infinitesimal, then $\frac{1}{\varepsilon} \notin \mathbf{R}^\star_{fin}$.

Proofs

Proof of theorem 1

(i)⇒(ii) Let $h = q/_q(q+q)$. Then $h(q+q) = q$, which immediately implies that $2h = 1$. It follows that $2hz = z$ for every $z \in \mathcal{K}$. Now let $x \in \mathcal{K} \setminus \{0\}$, and let us prove that there is a $y \in \mathcal{K}$ such that $yx = 1$. Without loss of generality we may assume that $x > 0$. Let k be minimal such that $x \leq 2^k$ (such a k exists because $\mathcal{K}$ is Archimedean). Then $h^k x \leq 1$. Moreover by the minimality of k we have $h^{k-1}x > 1$ (where we put $h^{k-1} = 2$ if $k = 0$). Hence $q \leq h < h^k x \leq 1$, and by axiom (K3) there is a $z \in \mathcal{K}$ such that $h^k xz = h$. Now let $y = h^{k-1}z$. Then $yx = h^{k-1}zx = 2h^k zx = 2h = 1$. Hence y is the desired element.

(ii)⇒(i) Let by contradiction $\mathcal{K}$ be a linearly ordered f-quasifield such that for some $a, b \in \mathcal{K}$ one has $a > 0$ and $na < b$ for every $n \in \mathbf{N}$. Then for every $n \in \mathbf{N}$ we have $n < ba^{-1}$, against the fact that 1 is a strong unit of $\mathcal{K}$.

Proof of theorem ?? (i). That $i_{\mathbf{F}(\mathcal{A})}$ is an isomorphism of PMV-algebras follows from Lemma **??**. That $i_{\mathbf{F}(\mathcal{A})}$ preserves the constant q follows from the definition of Π_q and of Π_q^{-1}.
We prove that $i_{\mathbf{F}(\mathcal{A})}$ preserves $\rightarrow_q$. Let $\Rightarrow_q$ denote the interpretation of $\rightarrow_q$ in $\Pi_q(\Pi_q^{-1}(\mathcal{A}))$. Thus $a \Rightarrow_q b = b/_q a$, and since $u(a) = a$ and $u(b) = b$, from (★) we obtain:

$$a \Rightarrow_q b = i_{\mathbf{F}(\mathcal{A})}((i_{\mathbf{F}(\mathcal{A})})^{-1}(a) \rightarrow_q (i_{\mathbf{F}(\mathcal{A})})^{-1}(b)). \quad (1)$$

Now for $x, y \in \mathcal{A}$, substituting $i_{\mathbf{F}(\mathcal{A})}(x)$ for a and $i_{\mathbf{F}(\mathcal{A})}(y)$ for b in equation (1), we obtain:

$$i_{\mathbf{F}(\mathcal{A})}(x) \Rightarrow_q i_{\mathbf{F}(\mathcal{A})}(y) = i_{\mathbf{F}(\mathcal{A})}(x \rightarrow_q y),$$

and the claim is proved.

(ii). Let us denote $\Pi_q(\mathcal{F})$ by $\mathcal{B}$. That $j_{\mathbf{S}(\mathcal{F})}$ is an isomorphism of c-s-u-f-rings follows from Lemma **??**. In order to prove that $j_{\mathbf{S}(\mathcal{F})}$ preserves q, note that the interpretation of q is the same in $\mathcal{F}$ and in $\Pi_q(\mathcal{F}) = \mathcal{B}$. Moreover in $\Pi_q^{-1}(\mathcal{B})$, q is interpreted as $i_{\mathbf{F}(\mathcal{B})}(q^{\mathcal{B}})$, where $q^{\mathcal{B}}$ is the interpretation of q in both $\mathcal{B}$ and $\mathcal{F}$. Therefore we only need to prove that $i_{\mathbf{F}(\mathcal{B})}(q^{\mathcal{B}}) = j_{\mathbf{S}(\mathcal{F})}(q^{\mathcal{B}})$. Now by Lemma **??**, $i_{\mathbf{F}(\mathcal{B})}(q^{\mathcal{B}}) = \Gamma_{\mathbf{R}}(j_{\mathcal{F}}(q^{\mathcal{B}})) = j_{\mathcal{F}}(q^{\mathcal{B}})$, and the claim follows.
Finally we prove that $j_{\mathbf{S}(\mathcal{F})}$ preserves $/_q$. Let $//_q$ denote the interpretation of $/_q$ in $\Pi_q^{-1}(\Pi_q(\mathcal{F}))$ (and let us identify $/_q$ with its realization in $\mathcal{F}$). Let $x, y \in \mathcal{F}$, and let us prove that $j_{\mathbf{S}(\mathcal{F})}(x/_q y) = j_{\mathbf{S}(\mathcal{F})}(x)//_q j_{\mathbf{S}(\mathcal{F})}(y)$. Since $x/_q y = u(x)/_q u(y)$ and $j_{\mathbf{S}(\mathcal{F})}$ preserves the operation u, we may assume without loss of generality that $u(x) = x$ and $u(y) = y$. Thus letting $\mathcal{D} = \Pi_q(\mathcal{F})$, we have $x, y \in \mathcal{D}$, and $j_{\mathbf{S}(\mathcal{F})}(x) = \Pi_q(j_{\mathbf{S}(\mathcal{F})}(x)) = i_{\mathbf{F}(\mathcal{D})}(x)$. Similarly, $j_{\mathbf{S}(\mathcal{F})}(y) = i_{\mathbf{F}(\mathcal{D})}(y)$. Thus recalling the last claim of Lemma **??** and the definition of Π_q^{-1}, we obtain:

$$\begin{aligned} j_{\mathbf{S}(\mathcal{F})}(x)//_q j_{\mathbf{S}(\mathcal{F})}(y) &= i_{\mathbf{F}(\mathcal{D})}(x)//_q i_{\mathbf{F}(\mathcal{D})}(y) = i_{\mathbf{F}(\mathcal{D})}(y \rightarrow_q x) = \\ &= j_{\mathbf{S}(\mathcal{F})}(y \rightarrow_q x) = j_{\mathbf{S}(\mathcal{F})}(x/_q y), \end{aligned}$$

and (ii) is proved.

(iii). Set $\mathcal{F} = \Pi_q^{-1}(\mathcal{A})$, $\mathcal{K} = \Pi_q^{-1}(\mathcal{B})$, $\psi = \Gamma^{-1}(\mathbf{F}(\phi))$. That ψ is a homomorphism of c-s-u-f-rings follows from Lemma **??**. We prove that ψ preserves q. The interpretation of q in $\mathcal{F}$ is $q^{\mathcal{F}} = i_{\mathbf{F}(\mathcal{A})}(q^{\mathcal{A}})$, and the interpretation of q in $\mathcal{K}$ is $q^{\mathcal{K}} = i_{\mathbf{F}(\mathcal{B})}(q^{\mathcal{B}})$. Now by Lemma **??**, $\Gamma_{\mathbf{R}}(\psi) \circ i_{\mathbf{F}(\mathcal{A})} = i_{\mathbf{F}(\mathcal{B})} \circ \phi$, therefore

$$\begin{aligned} \psi(q^{\mathcal{F}}) &= \Gamma(\psi)(q^{\mathcal{F}}) = (\Gamma(\psi) \circ i_{\mathbf{F}(\mathcal{A})})(q^{\mathcal{A}}) = \\ &= (i_{\mathbf{F}(\mathcal{B})} \circ \phi)(q^{\mathcal{A}}) = i_{\mathbf{F}(\mathcal{B})}(q^{\mathcal{B}}) = \\ &= q^{\mathcal{K}}. \end{aligned}$$

Finally we prove that ψ preserves $/_q$. Let $//_q$ denote the interpretation of $/_q$ in $\mathcal{K}$, and let us identify the symbol $/_q$ and its realization in $\mathcal{F}$. Let $x, y \in \mathcal{F}$. Since $x/_q y = u(x)/_q u(y)$, and since ψ preserves u, we can assume without loss of generality that $x = u(x)$ and $y = u(y)$. Then by clause (★) in the definition of Π_q^{-1} we have:

$$x/_q y = i_{\mathbf{F}(\mathcal{A})}((i_{\mathbf{F}(\mathcal{A})})^{-1}(y) \rightarrow_q (i_{\mathbf{F}(\mathcal{A})})^{-1}(y)) \quad (2)$$

$$\psi(x)//_q \psi(y) = i_{\mathbf{F}(\mathcal{B})}((i_{\mathbf{F}(\mathcal{B})})^{-1}(\psi(y)) \rightarrow_q (i_{\mathbf{F}(\mathcal{B})})^{-1}(\psi(x))). \quad (3)$$

Note that by Lemma **??**, $\Gamma_{\mathbf{R}}(\Gamma_{\mathbf{R}}^{-1}(\phi))) = i_{\mathbf{F}(\mathcal{B})} \circ \phi \circ i_{\mathbf{F}(\mathcal{A})}^{-1}$. Therefore, for all $z \in \Gamma_{\mathbf{R}}(\mathcal{F})$, we have:

$$\psi(z) = (\Gamma_{\mathbf{R}}(\psi))(z) = (\Gamma_{\mathbf{R}}(\Gamma_{\mathbf{R}}^{-1}(\phi)))(z) = i_{\mathbf{F}(\mathcal{B})}(\phi(i_{\mathbf{F}(\mathcal{A})}^{-1}(z))). \quad (4)$$

In particular, $\psi(x) = i_{\mathbf{F}(\mathcal{B})}(\phi(i_{\mathbf{F}(\mathcal{A})}^{-1}(x)))$ and $\psi(y) = i_{\mathbf{F}(\mathcal{B})}(\phi(i_{\mathbf{F}(\mathcal{A})}^{-1}(y)))$, therefore

$$(i_{\mathbf{F}(\mathcal{B})})^{-1}(\psi(y)) = \phi(i_{\mathbf{F}(\mathcal{A})}^{-1}(y)) \text{ and } (i_{\mathbf{F}(\mathcal{B})})^{-1}(\psi(x)) = \phi(i_{\mathbf{F}(\mathcal{A})}^{-1}(x)). \quad (5)$$

Substituting in eq. (3), recalling that ϕ and $i_{\mathbf{F}(\mathcal{A})}^{-1}$ are homomorphisms of ŁΠ$_q$-algebras and using eq. (4) and eq. (2), we obtain:

$$\begin{aligned} \psi(x)//_q \psi(y) &= i_{\mathbf{F}(\mathcal{B})}(\phi(i_{\mathbf{F}(\mathcal{A})}^{-1}(y)) \rightarrow_q \phi(i_{\mathbf{F}(\mathcal{A})}^{-1}(x))) = \\ &= i_{\mathbf{F}(\mathcal{B})}(\phi(i_{\mathbf{F}(\mathcal{A})}^{-1}(y) \rightarrow_q i_{\mathbf{F}(\mathcal{A})}^{-1}(x))) \\ &= i_{\mathbf{F}(\mathcal{B})}(\phi(i_{\mathbf{F}(\mathcal{A})}^{-1}(y \rightarrow_q x))) \\ &= \psi(y \rightarrow_q x) \\ &= \psi(x/_q y). \end{aligned} \quad (6)$$

This concludes the proof of the lemma.

References

[BF00] W.J. Blok and I.M.A. Ferreirim, *On the structure of hoops*, Algebra Universalis **43** 2000, 233-257.

[BKW77] A. Bigard, K. Keimel and S. Wolfenstein, *Groupes at anneaux reticulés*, Lecture Notes in Mathematics, **608**, Springer Verlag, Berlin 1977.

[C02] P. Cintula, *The ŁΠ and ŁΠ$\frac{1}{2}$ propositional and predicate logics*, Technical Report 2001.

[DND01] A. Di Nola, A. Dvurecenskij, *Product MV-algebras*, Multi. Val. Logic, **6**,193-215, (2001). Theorem 4.2.

[EGM01] F. Esteva, L. Godo, F. Montagna, *ŁΠ and ŁΠ$\frac{1}{2}$: two fuzzy logics joining Lukasiewicz and Product logics*, Archive for Mathematical Logic **40** (2001), pp. 39-67.

[EGH96] F. Esteva, L. Godo, P. Hájek, *A complete may-valued logic with product conjunction*, Archive for Mathematical Logic **35** (1996), 191-208.

[H98] P. Hájek, *Metamathematics of Fuzzy Logic*, Kluwer, 1988.

[Mo00] F. Montagna, *An algebraic approach to propositional fuzzy logic*, Journal of Logic, Language and Information **9** (2000), 91-124.

[Mo02] F. Montagna, *Subreducts of MV-algebras with product and product residuation*, Preprint 2002.

[MS03] F. Montagna, L. Spada, *Continuous approximations of MV-algebras with product and product residuation*, Preprint 2003.

[Mu86] D. Mundici, *Interpretations of AF $C^{\star}$ algebras in Łukasiewicz sentential calculus*, J. Funct. Analysis **65**, (1986), 15-63.