Free MV algebras as direct limit

a joint work with A. di Nola and R. Grigolia

Luca Spada
lspada@unisa.it
http://www.mat.unisi.it/∼lspada/

Department of Mathematics and Computer Science.
Università degli Studi di Salerno

Bratislava, 25 September 2008

Overview

Motivations

- MV-algebra are the equivalent algebraic semantic of Łukasiewicz logic.
- MV-algebras are categorical equivalent to unital ℓ-groups.
- The free MV-algebra is the algebra of all piece-wise linear functions with integer coefficients.
- Finding new characterizations of the free MV-algebras gives new insight in such a class.

MV-algebras

Recall that an algebra $A = (A; \oplus, \odot, \neg, 0, 1)$, is said to be a MV-algebra iff it satisfies the following equations:

$$(x \oplus y) \oplus z = x \oplus (y \oplus z); \qquad x \oplus y = y \oplus x;$$
$$x \oplus 0 = x; \qquad x \oplus 1 = 1;$$
$$\neg 0 = 1; \qquad \neg 1 = 0;$$
$$x \odot y = \neg(\neg x \oplus \neg y); \qquad \neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x.$$

The following abbreviations are very often used:

$$a^n = \underbrace{a \odot \cdots \odot a}_{n \text{ times}} \quad \text{and} \quad (n)a = \underbrace{a \oplus \cdots \oplus a}_{n \text{ times}}.$$

An example of MV-algebra

The unit interval of real numbers $[0,1]$ endowed with the following operations:

$$x \oplus y = \min(1, x+y) \quad x \odot y = \max(0, x+y-1)$$
$$\neg x = 1 - x,$$

is an MV-algebra.

Theorem

The MV-algebra $\mathcal{S} = ([0,1], \oplus, \odot, \neg, 0, 1)$ generates the variety $\mathbb{MV}$, in symbols $\mathcal{V}(S) = \mathbb{MV}$.

Łukasiewicz logic with n+1 truth values

The subvarieties $\mathbb{MV}_n \subset \mathbb{MV}$ are axiomatized by the extra axiom: $x^{n+1} = x^n$ (or $(n+1)x = nx$).
The subvarieties $\mathbb{MV}_n$ corresponds to Łukasiewicz logic with n+1 truth values.
Let $\omega_0 := \omega \setminus \{0\}$. For $n \in \omega_0$ we set $\mathcal{S}_n = (S_n; \oplus, \odot, \neg, 0, 1)$, where

$$S_n = \left\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\right\}$$

and the operations $\oplus, \odot, \neg$ are defined as in S.
Then we have that $\mathbb{MV}_n = \mathcal{V}(\{S_1, ..., S_n\})$.

Free MV$_n$ algebras

Let $F_{\mathbb{MV}_n}(m)$ be free m-generated MV-algebra in the variety $\mathbb{MV}_n$.
Let $F_{\mathbb{MV}}(m)$ be free m-generated MV-algebra in the variety $\mathbb{MV}$.
Define the function $v_m(x)$ as follows:

$$\begin{aligned} v_m(1) &= 2^m, \\ v_m(2) &= 3^m - 2^m, \\ &\vdots \\ v_m(n) &= (n+1)^m - (v_m(n_1) + \ldots + v_m(n_{k-1})), \end{aligned}$$

where $n_1 = 1, n_k = n$ and $n_2, \ldots, n_{k-1}$ are the strict divisors of n.

Proposition ([1])

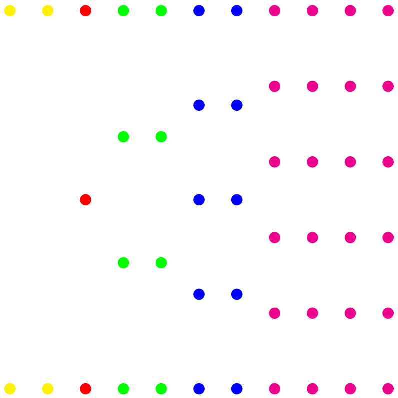
$$F_{\mathbb{MV}_n}(m) \cong S_1^{v_m(1)} \times \ldots \times S_n^{v_m(n)}.$$

[1] A. Di Nola , R. Grigolia, G. Panti, Finitely generated free MV-algebras and their automorphism groups, *Studia Logica*, **61**(1):65-78. 1998.

Some examples

The 1-generated case: $v(n) = (n+1) - (v(n_1) + ... + v(n_{k-1}))$
$v(1) = 2$, $v(2) = 3 - 2 = 1$, $v(3) = 4 - 2 = 2$,
$v(4) = 5 - 2 - 1 = 2$, $v(5) = 6 - 2 = 4$.

$F_{MV_1} =$



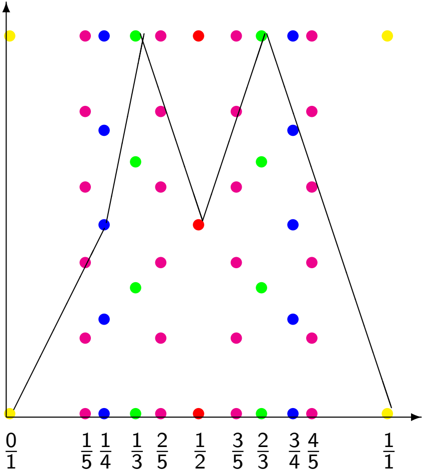
A characterization of the elements of F_{MV_n}

Proposition ([2])

Given a tuple $(a_1, ..., a_n)$ in F_{MV_k} there is a McNaughton function $f(x)$ such that the set $\{a_1, ..., a_n\}$ is exactly the range of $f(x)$ restricted to $\bigcup_{i=1}^{k} S_k$.

[2] A. Di Nola , R. Grigolia, G. Panti, Finitely generated free MV-algebras and their automorphism groups, *Studia Logica*, **61**(1):65-78. 1998.

An element of F_{MV_5}



Two different visualizations

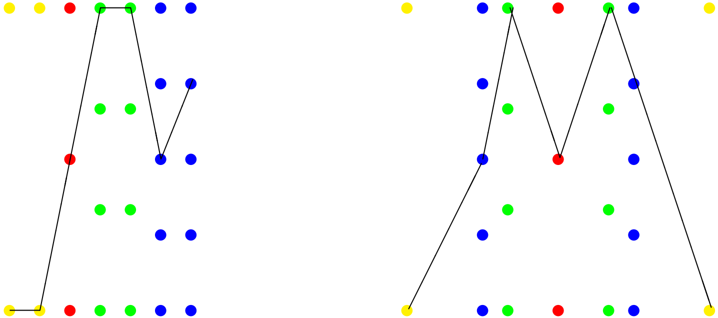


Figure: The black lines in the figures depict the same element of $F_{\mathbb{MV}_5}(1)(= S_1^2 \times S_2 \times S_3^2 \times S_4^2)$

Formalizing such visualizations

Definition

$\mathcal{Q}$ is the set of irreducible fractions between 0 and 1, endowed with the natural order, which we will indicate as usual with $<$.
$\mathcal{Q}^{\prec}$ has the same domain of $\mathcal{Q}$ but its linear order $\prec$ is given by

$$\frac{m}{n} \prec \frac{p}{q} \text{ if, and only if, } n < q \text{ or, if } n = q \text{ then } m < p$$

So the $\prec$-sorted listing of $\mathcal{Q}$ is $\{\frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \ldots\}$.

The direct system

The direct limit is a categorical construction:

Definition

Let $(I, \leq)$ be a directed set. Let $\{A_i \mid i \in I\}$ be a family of objects indexed by I and suppose we have a family of embeddings $\varepsilon_{ij} : A_i \to A_j$ for all $i \leq j$ with the following properties:

- ε_{ii} is the identity in A_i,
- $\varepsilon_{ik} = \varepsilon_{jk} \circ \varepsilon_{ij}$ for all $i \leq j \leq k$.

Then the pair (A_i, ε_{ij}) is called a direct system over I.

The direct limit of a system

Definition

The underlying set of the direct limit, A, of the direct system (A_i, ε_{ij}) is defined as the disjoint union of the A_i's modulo a certain equivalence relation $\sim$:

$$A = \varinjlim A_i = \coprod_i A_i \Big/ \sim .$$

Where, if $x_i \in A_i$ and $x_j \in A_j$, then $x_i \sim x_j$ if there is some $k \in I$ such that $\varepsilon_{ik}(x_i) = \varepsilon_{jk}(x_j)$.
One naturally obtains from this definition canonical morphisms $\varphi_i : A_i \to A$ sending each element to its equivalence class. The algebraic operations on A are defined via these maps in the obvious manner.

The embeddings between F_{MV_n}

We now define a family of embeddings $\varepsilon_k : F_k \to F_{k+1}$.

- Given a tuple $(a_1, ..., a_n)$ in F_k we know that there is a McNaughton function $f(x)$ such that the set $\{a_1, ..., a_n\}$ is exactly the range of $f(x)$ restricted to $\bigcup_{i=1}^{k} S_k$.
- Define $\varepsilon(a_1, ..., a_n)$ as the tuple given by the domain of $f(x)$ when restricted to $\bigcup_{i=1}^{k+1} S_{k+1}$.

But, how to chose f?

- In the 1-generated let's just chose the *simplest*.

Formalizing the idea

Definition

Let us define for any $\frac{n}{m} \in \mathcal{Q}$

$$(\frac{n}{m})^{+} = \max\{\frac{a}{b} \in \mathcal{Q} \mid \frac{a}{b} < \frac{n}{m} \text{ and } b < m\}$$

and

$$(\frac{n}{m})^{-} = \min\{\frac{a}{b} \in \mathcal{Q} \mid \frac{a}{b} > \frac{n}{m} \text{ and } b < m\}.$$

Formalizing the idea (cont'd)

Definition

Let $a = (a_{\frac{0}{1}}, a_{\frac{1}{1}}, ..., a_{\frac{i}{k}})$ be, for a suitable $\frac{i}{k} \in \mathcal{Q}$, an element of $F_{\mathbb{V}_k}$, then we define:

$$\varepsilon_k(a) = (a_{\frac{0}{1}}, a_{\frac{1}{1}}, ..., a_{\frac{i}{k}}, a_{\frac{1}{k+1}}, ..., a_{\frac{j}{k+1}})$$

where for all $\frac{j}{k+1} \in \mathcal{Q}$, we let $a_{\frac{j}{k+1}}$ be the solution of the linear equation:

$$\frac{\frac{j}{k+1} - (\frac{j}{k+1})^-}{(\frac{j}{k+1})^+ - (\frac{j}{k+1})^-} = \frac{a_{\frac{j}{k+1}} - a_{(\frac{j}{k+1})^-}}{a_{(\frac{j}{k+1})^+} - a_{(\frac{j}{k+1})^-}}$$

Lemma

ε_k is an embedding from F_k to F_{k+1}.

The snakes

Lemma (Characterization of the direct limit D)

For any element $a \in D$ there exists a unique $i \in \omega$ and a unique infinite sequences $(a^{(i)}, a^{(i+1)}, \ldots)$ such that

1. *for any $j \geq i$ there is exactly one $a^{(j)}$ in the sequence, such that $a^{(j)} \in F_j$;*
2. *$a^{(i)}$ has no inverse image with respect to ε_i;*
3. *$\varepsilon_{kj}(a^{(k)}) = a^{(j)}$ for any $k, j \geq i$:*
4. *for any $a^{(j)}$ in the sequence, the equivalence class of $a^{(j)}$ is a.*

Vice versa, given a sequence which satisfies the conditions (i)-(iii) above there exists a unique $a \in D$ for which the condition (iv) is satisfied.

The snakes (cont'd)

Definition

Given any element $a \in D$ we will call the sequence given by the lemma above, the snake of a.

Lemma

For every snake $a = (a^{(i)}, a^{(i+1)}, \dots)$, there exists a unique McNaughton function $f(x)$ such that for any $k \geq i$ there exists $p \in q$ such that $a^{(k)} = (f(q))_{q \prec p}$

Lemma

Let $a, b \in D$ and let $(a^{(i)}, a^{(i+1)}, \dots)$ and $(b^{(j)}, b^{(j+1)}, \dots)$ their respective snakes. If $i \leq j$ then for some l the sub-sequence $(a^{(j+l)} \oplus b^{(j+l)}, a^{(j+l+1)} \oplus b^{(j+l+1)}, \dots)$ of $(a_j \oplus b_j, a_{j+1} \oplus b_{j+1}, \dots)$ is a snake.

Reconstructing the MV-algebra

Definition

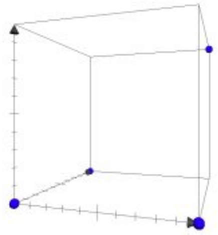
We define the operation $\oplus$ in D as follows: let $a, b \in D$ and let $(a^{(i)}, a^{(i+1)}, a^{(i+2)}, \ldots)$ and $(b^{(j)}, b^{(j+1)}, b^{(j+2)} \ldots)$ their respective snakes then $a \oplus b$ is defined as the element of D whose snake is inside $(a^{(j)} \oplus b^{(j)}, a^{(j+1)} \oplus b^{(j+1)}, \ldots)$.

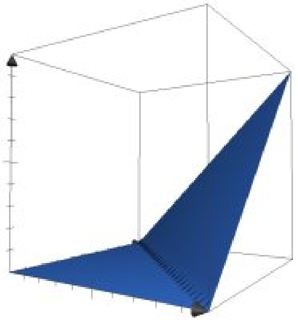
Theorem

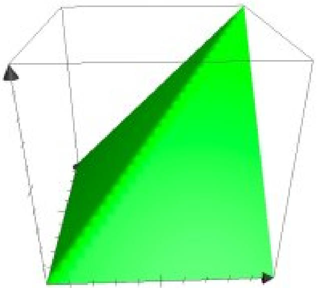
The algebra $\langle D, \oplus, \odot, \neg, 0, 1\rangle$ is isomorphic to the MV-algebra $\langle M, \oplus, \odot, \neg, 0, 1\rangle$ of all McNaughton function in one variable.

Note that even if we used the symbol $\oplus$ we have not proved that $\langle D, \oplus, \odot, \neg, 0, 1\rangle$ is an MV-algebra. Indeed the proof of the above theorem directly shows that such a structures is isomorphic to the 1-generated free MV-algebra.

The 2-generated case







How to chose f?