The unification type of Łukasiewicz logic is nullary

Based on a joint work with V. Marra

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Topology, Algebra, and Categories in Logic
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Łukasiewicz (infinite-valued propositional) logic is a non-classical system going back to the 1920’s which may be axiomatised using the primitive connectives → (implication) and ¬ (negation)
Łukasiewicz (infinite-valued propositional) logic is a non-classical system going back to the 1920’s which may be axiomatised using the primitive connectives \( \to \) (implication) and \( \neg \) (negation) by the four axiom schemata:

1. \( \alpha \to (\beta \to \alpha) \),  
2. \( (\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma)) \),  
3. \( ((\alpha \to \beta) \to \beta) \to ((\beta \to \alpha) \to \alpha) \),  
4. \( (\neg \alpha \to \neg \beta) \to (\beta \to \alpha) \),

with *modus ponens* as the only deduction rule.
Łukasiewicz logic is a subsystem of classical logic and has a many-valued semantics: assignments $\mu$ to atomic formulæ range in the unit interval $[0, 1] \subseteq \mathbb{R}$. 
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\[
\begin{align*}
\mu(\neg \alpha) &= 1 - \mu(\alpha), \\
\mu(\alpha \rightarrow \beta) &= \min \{1, 1 - \mu(\alpha) + \mu(\beta)\}
\end{align*}
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Tautologies are defined as those formulæ that evaluate to 1 under every such assignment.
Chang first considered the Tarski-Lindenbaum algebras of Łukasiewicz logic, and called them MV-algebras.

**Definition**

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- the interaction between those two operations is described by the following two axioms:
  - \( x \oplus 0^* = 0^* \)
  - \( (x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x \)
Example 1: The standard MV-algebra

In modern terms one says that MV-algebras are the equivalent algebraic semantics of Łukasiewicz logic.
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**Example**

Consider the set of real number $[0, 1]$ endowed with the following operation:

- $\neg x = 1 - x$ and $x \oplus y = \min\{1, x + y\}$ (truncated sum).

Then $\langle [0, 1], \oplus, \neg, 0 \rangle$ is an MV-algebra.
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Then $\langle [0, 1], \oplus, \neg, 0 \rangle$ is an MV-algebra.

Actually the above algebra generates the variety of all MV-algebras. So the equations that hold for any MV-algebra are exactly the ones that hold in $[0, 1]$. 

Example 2: McNaughton functions

A McNaughton function is a function

\[ f: [0, 1]^n \rightarrow [0, 1] \]

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One can endow the set of McNaughton functions in $n$ variables with the structure of an MV-algebra by taking point-wise operation:

$$f \oplus g = (f \oplus g)(x) = f(x) \oplus g(x) \text{ (recall that } [0,1] \text{ is an MV-algebra)}$$
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These functions are named after McNaughton, who first found the following characterisation of free MV-algebras:

Theorem (McNaughton 1951)

The free MV-algebra over \( \kappa \) generators is isomorphic to the MV-algebra of McNaughton functions over \( [0,1]^\kappa \).
Example 3: lattice ordered groups

A \textit{ul-group} is a lattice-ordered group $G$ with an element $g$ such that

for any $g' \in G$ there exists $n \in \mathbb{N}$ such that $g + \ldots + g \geq g'$.

\textit{ul-groups}

\textit{MV-algebras}

\textit{Łukasiewicz logic}

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\textit{Related unification problems}

\textit{Rational polyhedra}

\textit{Main result}
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If one truncates the operations of an \textit{ul}-group to the interval $[0, g]$, the result is an MV-algebra.

\[\text{Theorem (Mundici 1986)}\]

The category of MV-algebras is equivalent to the category of Abelian \textit{ul}-groups (with \textit{ul}-morphisms preserving the strong unit).
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The category of MV-algebras is equivalent to the category of Abelian *ul*-groups (with *l*-morphisms preserving the strong unit).
Unitarity of finite-valued Łukasiewicz logic

Ghilardi himself noticed that finite-valued Łukasiewicz logic has unitary type.

Finite-valued logics are obtained by restricting the possible values of evaluations to some subchain \( \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\} \) of the \([0,1]\) algebra.
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MV-algebras
Łukasiewicz logic
MV-algebras
McNaughton functions
ℓl-groups
Related unification problems
Rational polyhedra
Main result

Commutative lattice-ordered groups (ℓ-groups)

Theorem (Beynon 1977)

Finitely generated projective ℓ-groups are exactly the finitely presented ℓ-groups.
Commutative lattice-ordered groups ($\ell$-groups)

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In the theory of $\ell$-groups all system of equations are solvable. In the light of the Beynon’s and Ghilardi’s results, one easily gets:

Corollary

The unification type of the theory of $\ell$-groups is unitary.
Commutative lattice-ordered groups (\(\ell\)-groups)

**Theorem (Beynon 1977)**

*Finitely generated projective \(\ell\)-groups are exactly the finitely presented \(\ell\)-groups.*

In the theory of \(\ell\)-groups all system of equations are solvable. In the light of the Beynon’s and Ghilardi’s results, one easily gets:

**Corollary**

*The unification type of the theory of \(\ell\)-groups is unitary.*

In a forthcoming paper with V. Marra, we exploit a geometrical duality for \(\ell\)-groups to give an algorithm that, taken any (system of) term in the language of \(\ell\)-groups, outputs its most general unifier.
Łukasiewicz logic has a weak disjunction property. Namely:

\[ \varphi \lor \neg \varphi \text{ is derivable then either } \varphi \text{ or } \neg \varphi \text{ must be derivable.} \]

(In other words the rule \( \frac{\varphi \lor \neg \varphi}{\varphi, \neg \varphi} \) is admissible.)
Non unitarity of the unification

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(Exercise: prove this by using McNaughton representation)

This entails the unification type of Łukasiewicz logic to be at least not unitary.
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This entails the unification type of Łukasiewicz logic to be at least not unitary.

Indeed if $\sigma$ is a unifier for $x \lor \neg x$, then it must unify either $x$ (hence it is the substitution $x \mapsto 1$) or $\neg x$ (hence it must be the substitution $x \mapsto 0$).
Rational polyhedral geometry

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**Definition**

A *rational polytope* is the convex hull of a finite set of *rational* points.
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\[ \frac{n}{m} \]
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Rational polyhedral geometry [Cont.d]

**Definition**

A *rational polyhedron* is the union of a finite number of rational polytopes.
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**Z-maps**
Another way of looking at the McNaughton theorem is as a characterisation of definable functions in the language of MV-algebras. The only difference here being the fact that we can operate from the \( m \)-dimensional \( \mathbb{R}^m \) spaces to the \( n \)-dimensional \( \mathbb{R}^n \) spaces.

**Definition**

A Z-map is a continuous piecewise linear function with integer coefficients.
Rational polyhedra and MV-algebras

Let $\mathcal{MV}_{fp}$ be the category of f.p. MV-algebras with their homomorphisms.
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I will define a pair of (contravariant) functors:

$$\mathcal{V} : \mathcal{MV}_{fp} \to \mathcal{P}_\mathbb{Z} \quad \text{and} \quad \mathcal{I} : \mathcal{P}_\mathbb{Z} \to \mathcal{MV}_{fp}.$$
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These functors operate very similarly to the classical ones in algebraic geometry that associate ideals with varieties.
F.p. MV-algebras and rational polyhedra: objects

Let $A = \frac{\text{Free}_n}{\theta} \in \mathcal{M}V_{fp}$. 
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Let \( \mathcal{V}(\theta) \) be the collection of all real points \( p \) in \([0, 1]^n\) such that

\[ s(p) = t(p) \text{ for all } (s, t) \in \theta \]
F.p. MV-algebras and rational polyhedra: objects

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Let $\mathcal{V}(\theta)$ be the collection of all real points $p$ in $[0, 1]^n$ such that $s(p) = t(p)$ for all $(s, t) \in \theta$.

The set $\mathcal{V}(\theta)$ is a rational polyhedron, so we set

$$\mathcal{V}(A) = \mathcal{V}(\theta).$$
F.p. MV-algebras and rational polyhedra: objects

Let $P \in \mathcal{P}_\mathbb{Z}$. 
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Let $\mathcal{I}(P)$ be the collection of all pair MV-terms $(s, t)$ such that

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$\mathcal{I}(P)$ is a congruence of the free MV-algebras on $n$ generators, so it makes sense to set

$$\mathcal{I}(P) = \frac{\text{Free}_n}{\mathcal{I}(P)}.$$
F.p. MV-algebras and rational polyhedra: arrows

Let $h: A \to B$ be a diagram in $\mathcal{MV}_{fp}$.

Suppose that $h$ sends the generators of $A$ into the elements $\{t_i\}_{i \in I}$ of $B$, then define

$$\mathcal{V}(h): \mathcal{V}(B) \to \mathcal{V}(A)$$
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\]

as

\[
p \in \mathcal{V}(A) \xrightarrow{\mathcal{V}(h)} \langle t_i(p) \rangle_{i \in I} \in \mathcal{V}(A).
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Let $h: A \rightarrow B$ be a diagram in $\mathcal{MV}_{fp}$.

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as

$$p \in \mathcal{V}(A) \mapsto \langle t_i(p) \rangle_{i \in I} \in \mathcal{V}(A).$$

Then, the function $\mathcal{V}(h): \mathcal{V}(B) \rightarrow \mathcal{V}(A)$ is a $\mathbb{Z}$-map.
Let $\zeta : P \to Q$ be a diagram in $\mathcal{P}_\mathbb{Z}$.

Define

$$\mathcal{I}(\zeta) : \mathcal{I}(Q) \to \mathcal{I}(P)$$
Let $\zeta: P \rightarrow Q$ be a diagram in $\mathcal{P}_\mathbb{Z}$.

Define

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$$f \in \mathcal{I}(Q) \xrightarrow{\mathcal{I}(\zeta)} f \circ \zeta \in \mathcal{I}(P).$$
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Define

$$I(\zeta): I(Q) \to I(P)$$

as

$$f \in I(Q) \xrightarrow{I(\zeta)} f \circ \zeta \in I(P).$$

Then, the function $I(\zeta)$ is a homomorphism of MV-algebras.
Duality for finitely presented MV-algebras

**Theorem (Folklore)**

The pair of functors

\[ I : \mathcal{M}V_{fp} \to \mathcal{P}_\mathbb{Z} \quad \text{and} \quad \mathcal{V} : \mathcal{P}_\mathbb{Z} \to \mathcal{M}V_{fp}. \]

constitutes a contravariant equivalence between the two categories.
Corollaries

As a corollaries of the above duality one immediately gets

**Corollary**

The rational polyhedron associated to the free algebra over \( n \) generators is the \( n \)-dimensional cube \([0, 1]^n\).
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The rational polyhedron associated to the free algebra over \( n \) generators is the *n*-dimensional cube \([0, 1]^n\).

**Corollary**

The rational polyhedron associated to any \( n \)-generated projective MV-algebra is a retraction of the \( n \)-dimensional cube \([0, 1]^n\).
As a corollary of the above duality one immediately gets

**Corollary**

The rational polyhedron associated to the free algebra over $n$ generators is the $n$-dimensional cube $[0, 1]^n$.

**Corollary**

The rational polyhedron associated to any $n$-generated projective MV-algebra is a retraction of the $n$-dimensional cube $[0, 1]^n$. 
Corollaries [Con.d]

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The fundamental group of any injective rational polyhedra is trivial.
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*The fundamental group of any injective rational polyhedra is trivial.*

**Proof.**

Let \( P \) an injective rational polyhedron corresponding to a \( n \)-generated MV-algebra, then

\[
\pi_1(P) \rightarrow \{ \pi_1([0, 1]^n) \} \rightarrow \pi_1(P)
\]
Corollaries [Con.d]

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The fundamental group of any injective rational polyhedra is trivial.

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\end{align*}
\]
Co-unification

Since Ghilardi’s approach is purely categorical one can speak of co-unification to refer to the dual problem in the dual category.
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- A co-unification problem as a rational polyedron $Q$. 


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Since Ghilardi’s approach is purely categorical one can speak of co-unification to refer to the dual problem in the dual category.

As we have seen that rational polyhedra are equivalent to the dual category of finitely presented MV-algebras, it makes sense to define:

- A co-unification problem as a rational polyedron $Q$.
- An co-unifier for the problem $Q$ as a pair $(P, u)$ where
  1. $P$ is an injective rational polyhedron,
  2. $u$ is a $\mathbb{Z}$-map from $P$ to $Q$, $u : P \rightarrow Q$. 

Finitarity result

**Proposition**

The unification type of the 1-variable fragment of Łukasiewicz logic is *finitary*. 
The unification type of Łukasiewicz logic is nullary.

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MV-algebras

Rational polyhedra

Main result

Prologue

Statement

A sequence of unifiers

A crucial lemma

Finitarity result

**Proposition**

The unification type of the 1-variable fragment of Łukasiewicz logic is **finitary**.

In particular the proof shows that there are at most **two** most general unifiers, for any given formula.
The unification type of Łukasiewicz logic is nullary.

**Proposition**

The unification type of the 1-variable fragment of Łukasiewicz logic is **finitary**.

In particular the proof shows that there are at most two most general unifiers, for any given formula. Indeed the most general form of a co-unification problem is

\[ 0 \rightarrow A \rightarrow B \rightarrow 1 \]

With \( A \) or \( B \) possibly empty or restricted to a point.
From 1-variable to the full calculus

The reason of the absence of a good unification theory for Łukasiewicz logic has to be found in the following geometrical phenomenon:
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\[ A \quad \quad B \]
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From 1-variable to the full calculus

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\[ x \lor x^* \]

\[
\begin{array}{cc}
0 & 1 \\
\end{array}
\]
From 1-variable to the full calculus

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The full Łukasiewicz logic has \textit{nullary} unification type.
The unification type of Łukasiewicz logic is nullary.

**Main result**

**Statement**

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**Proof.** It is sufficient to exhibit one problem with nullary type.
Nullarity of Łukasiewicz logic

**Theorem**

*The full Łukasiewicz logic has nullary unification type.*

**Proof.** It is sufficient to exhibit one problem with nullary type. As seen above the co-unification problem associated to

\[(x \lor x^* \lor y \lor y^*, 1)\]

is the rational polyhedron

\[A\]
Step 1.

Consider the following sequence of pair of maps and rational polyhedra,

\[
\begin{align*}
&\gamma_1 & \gamma_2 & \gamma_3 & \cdots \\
&\zeta_1 & \zeta_2 & \zeta_3 & \\
\end{align*}
\]

It can be proved (cfr. Cabrer and Mundici) that each \(\gamma_i\) is a retract of \([0; 1]_n\) for some \(n\), so the pairs \((\gamma_i; \zeta_i)\) are co-unifiers for \(A\).
Proof Cont.'d

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\begin{align*}
&\zeta_1 \downarrow \\
&\zeta_2 \downarrow \\
&\zeta_3 \downarrow \\
&\zeta_4 \downarrow \\
&\zeta_5 \downarrow \\
&\zeta_6 \downarrow \\
&\zeta_7 \downarrow \\
&\zeta_8 \downarrow \\
\end{align*}

It can be proved (cfr. Cabrer and Mundici) that each \( t_i \) is a retract of \([0, 1]^n\) for some \( n \), so the pairs \((t_i, \zeta_i)\) are co-unifiers for \( A \).
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![Diagram](image-url)
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Indeed $\iota_{ij}$ is just the embedding of $t_i$ in $t_j$
Step 3: The lifting of **definable** functions.

Proof Cont.'d
Step 3: The lifting of definable functions.

For any co-unifier \((u, P)\) of \(A\),

\[
\begin{array}{c}
A \xymatrix{ & \text{u} & P \ar[l]}
\end{array}
\]
Proof Cont.'d

Step 3: The lifting of defirable functions.

For any co-unifier \((u, P)\) of \(A\), there exists some \(t_i\):

\[
\begin{array}{c}
\chi_i \\
\downarrow \\
A \\
\end{array} 
\xleftarrow{u} \begin{array}{c}
\chi \\
P
\end{array} 
\xrightarrow{u} \begin{array}{c}
\chi_i \\
\downarrow \\
A \\
\end{array} 
\xleftarrow{t_i} 
\]
Step 3: The lifting of definable functions.

For any co-unifier \((u, P)\) of \(A\), there exists some \(t_i\) and an arrow \(\tilde{u}\) (called the lift of \(u\)) making the following diagram commute.

\[
\begin{array}{c}
A \\
\downarrow \zeta_i \\
\tilde{u} \\
\downarrow \approx u \\
P
\end{array}
\]
The above lemma is the piecewise linear version of the “Lifting of functions” Lemma, widely used in algebraic topology.
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The Definable Lifting Lemma has two important corollaries.
Proof Cont.'d

Step 4: Corollary 1.

Given any co-unifier \((P, u)\) for \(A\), there exists a co-unifier \((t_i, \zeta_i)\) in the above sequence such that \((P, u)\) is less general than \((t_i, \zeta_i)\).

(The sequence is cofinal in the poset of co-unifiers.)
**Proof Cont.'d**

**Step 4: Corollary 1.**

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A **lattice point** is a vector with integer coordinates.

**Step 5: Corollary 2.**

If \((P, u)\) is a co-unifier for \(A\) with **strictly fewer lattice points** than \(t_i\),
Proof Cont'd

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A lattice point is a vector with integer coordinates.

Step 5: Corollary 2.
If \((P, u)\) is a co-unifier for \(A\) with \textit{strictly fewer} lattice points than \(t_i\), then there is no arrow \(v: t_i \rightarrow P\) making the following diagram commute.

\[
\begin{array}{ccc}
A & \xrightarrow{u} & P \\
\downarrow{v} & & \downarrow{v} \\
\zeta_i & \xleftarrow{t_i} & \end{array}
\]
Step 6.

As a consequence of the last corollary we obtain:

1. The sequence of $t_i$ is strict; i.e., $t_i$ is not more general than $t_j$ if $i < j$.
2. The sequence admits no bound with a finite number of lattice elements. Therefore, no rational polyhedra can bound the sequence of $t_i$. 
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End of the Proof

Conclusion

Summing up, we have found a strictly linearly ordered, cofinal sequence of unifiers for $A$. Furthermore, the sequence is unbounded, hence the co-unification type of rational polyhedra is nullary (in a stronger sense). This proves that the Łukasiewicz calculus (as well as the theory of MV-algebras and $\ell$-groups with strong unit) has nullary unification type.
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This proves that the Łukasiewicz calculus (as well as the theory of MV-algebras and $\ell$-groups with strong unit) has nullary unification type.
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