μMV algebras

an approach to fixed points in Łukasiewicz logic

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Overview

Motivations

- Introducing fixed points in many valued logic
- A deep understanding of the algebraic semantic of the logic turned out to be crucial.
- Introducing a particular class of algebras called μMV algebras.

Methods

- Kripke-style semantics have been studied for several important t-norm based logics our approach to fixed point is not *classical* (cf. μ-calculus).
- In order to give a semantic to the μ operator we reaped benefit from the functional semantics of many valued logics.
- In Łukasiewicz logic connectives can be considered as continuous functions from $[0,1]^n$ to $[0,1]$, therefore, once parametrized as function in only one variable.

Theorem (Brouwer, 1909)

Every continuous function from the closed unit ball D^n to itself has a fixed point.

- The advantage is that with this method any formula has fixed points, whereas in classical cases, one has to restrict to formulas on which the variable under the scope of μ only appears positively.
- On the other hand, the function giving the fixed point of a formula need not to be continuous in the remaining variables, whence we can not allow nested occurencies of μ.

Definition

A **MV algebra** is an algebra $\mathcal{A} = \langle A, \oplus, \neg, 0 \rangle$ that satisfies

1. $\langle A, \oplus, 0 \rangle$ is a commutative monoid
2. $\neg\neg x = x$
3. $x \oplus \neg 0 = \neg 0$
4. $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$

Other connectives can be defined starting form the ones above:

$$1 = \neg 0 \quad x \odot y = \neg(\neg x \oplus \neg y) \quad x \to y = \neg x \oplus y \quad x \ominus y = \neg(\neg x \oplus y)$$

Moreover, given a MV algebra $\mathcal{A}$, defining

$$x \wedge y = x \odot (x \to y) \qquad \text{and} \qquad x \vee y = (x \ominus y) \oplus y$$

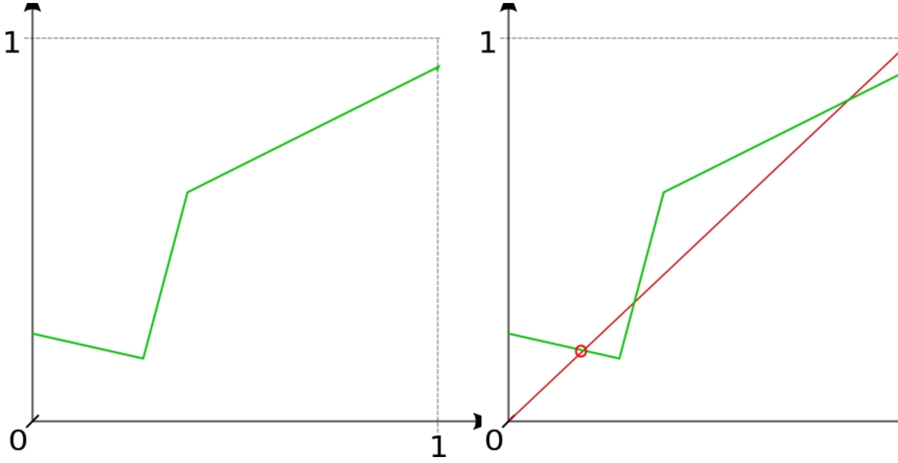
gives a lattice $\mathcal{A} = \langle A, \vee, \wedge \rangle$.

Definition

A μ**MV algebra** is a MV algebra, endowed with a function $\mu x.T(x)$ for any term $T(x)$ in the language of MV algebras, such that it satisfies the following conditions.

1. $\mu x.(T(x)) = T(\mu x.(T(x)))$
2. If $T(t) = t$ then $\mu x.(T(x)) \leq t$
3. If $\bigwedge_{i \leq n}(|p_i - q_i|) = 1$ then $\mu x.(T(p_1, ..., p_n)) = \mu x.(T(q_1, ..., q_n))$

Discontinuity of μ-functions



Proposition

Every μMV algebra is the subdirect product of linearly ordered μMV algebras.

Proof.

Axiom 3 in the definition of μMV algebras guarantees that μ functions are compatible with the congruences of MV algebra.

So the congruence of a μMV algebra are precisely the ones of its underlying MV algebra.

By Birkhoff representation theorem, every μMV algebra is the subdirect product of irreducible μMV algebras.

Irreducibility is characterized by the congruences lattice of the algebra. Since in this case they are identical, a μMV algebra is irreducible if, and only if, its underlying MV is linearly ordered.

□

Definition

A $\mathbf{MV_\Delta}$ **algebra** is a MV algebra with an operator Δ that satisfies:

1. $\Delta(1) = 1$.
2. $\Delta(x \Rightarrow y) \leq \Delta(x) \Rightarrow \Delta(y)$.
3. $\Delta(x) \sqcup \neg\Delta(x) = 1$.
4. $\Delta(x) \leq x$.
5. $\Delta(\Delta(x)) = \Delta(x)$.
6. $\Delta(x \sqcup y) = \Delta(x) \sqcup \Delta(y)$.

The presence of μ-functions allows to reintroduce the Δ operator as

$$\Delta(y) = \neg \mu x.(x \oplus \neg y)$$

With Δ it is possible to express quasi-equations as equations, so this proves:

Proposition

The class of μMV algebras is a variety.

Definition

A **divisible MV algebra** is an MV algebra with a family of operators δ_n such that:

1. $(n)\delta_n(x) = x$
2. $\delta_n(x) \odot (n-1)\delta_n(x) = 0$

Where $(n)x$ is a shorthand for $\underbrace{x \oplus \ldots \oplus x}_{n-times}$

Definition

A **divisible MV$_\Delta$algebra** is a structure $\mathcal{A} = \langle A, \oplus, \neg, 0, \delta_n, \Delta \rangle$ such that:

- $\langle A, \oplus, \neg, 0, \Delta \rangle$ is a MV$_\Delta$algebra
- $\langle A, \oplus, \neg, 0, \delta_n \rangle$ is a divisible MV algebra

For any n, the operators δ_n, of divisible MV algebras, are also definable by fixed points:

$$\delta_n(x) = \mu y.(x \ominus (n-1)y)$$

Lemma

For every μMV algebra, the operator defined above satisfies:

1. $(n)\delta_n(x) = x$
2. $\delta_n(x) \odot (n-1)\delta_n(x) = 0$

So we have reached the following result:

Proposition

Every μMV algebra contains a definable divisible MV_{Δ}algebra.

The other direction also holds but it require a little more work to be proved

Lemma

For every term $T(x)$ in the language of MV algebras, there exist:

- $c_1, ..., c_I$ *terms of the form* $(m)x \oplus k$, $(m)x \ominus k$, $\neg((m)x \oplus k)$ *or* $\neg((m)x\ominus)k$ *where* $m \in \mathbb{N}$ *and* k *is a term not containing* x
- $q_1, p_1, ..., q_I, p_I$ *terms not containing* x

such that for any evaluation $[\ ]^*$*:*

$$[T(x)]^* = \left[\bigvee_{i \leq I,} (\Delta(x \to q_i) \wedge \Delta(p_i \to x) \wedge c_i) \right]^*$$

- Every term of an MV algebra can be interpreted as a continuous pice-wise linear function with integer coefficents form $[0,1]^n$ to $[0,1]$.
- Parameterizing the function in all its variables but x this becomes of the form:

$$f(x) = \begin{cases} z_1 x \pm k_1 & \text{if } p_1 \leq x \leq q_1 \\ \vdots & \vdots \\ z_i x \pm k_i & \text{if } p_i \leq x \leq q_i \end{cases}$$

where $z_i \in \mathbb{Z}$ and k_i, p_i, q_i are polynomials in the variable parameterized.

Such a function is the interpretation of a term of the form:

$$\bigvee_{i \leq I,} (\Delta(x \to q_i) \wedge \Delta(p_i \to x) \wedge c_i)$$

where c_i are the terms corresponding to $z_i x \pm k_i$

Theorem

μMV algebras and divisible MV_Δ algebras are term-wise equivalent.

First of all we find the minimum fixed points of some basic term. Let us define:

$$
\begin{array}{rcr}
\bar{\mu}x.c = \neg\Delta(\neg k) & \text{if} & c = (m)x \oplus k \\
\bar{\mu}x.c = \delta_{m-1}(k) & \text{if} & c = (m)x \ominus k \\
\bar{\mu}x.c = \delta_{m+1}(\neg k) & \text{if} & c = \neg((m)x \oplus k) \\
\bar{\mu}x.c = \delta_{m+1}(k) \oplus \delta_{m+1}(1) & \text{if} & c = \neg((m)x \ominus k)
\end{array}
$$

Where we have put $\delta_0(x) = 0$ for every x. It is easy to see that in all four cases $\bar{\mu}x(c)$ is the minimum fixed point of c.

To give the fixed point function associated to any term $T(x)$ we first find a term equivalent to $T(x)$ in which all the linear components are explicitly present, let it be

$$\bigwedge_{i\leq I}(\Delta(x \to q_i) \wedge \Delta(p_i \to x) \wedge c_i)$$

By continuity of the function a fixed point for this term must exist and it will be among the fixed point of the functions c_i. So we define:

$$\mu x.T(x) = \bigwedge_{i\leq I}[\neg\Delta(T(\bar{\mu}x.c_i) \leftrightarrow \bar{\mu}x.c_i) \oplus \bar{\mu}x.c_i]$$

Once this equivalence is established it becomes fairly easy to extend known results (and techniques) about divisible MV algebras and MV_Δalgebra to μMV algebras.

Definition

A δ-lattice ordered group (δ-ℓ-**group**, for short) is a structure $\mathcal{G} = \langle G, +, -, \wedge, \vee, \delta, 0, 1\rangle$ where $\langle G, +, -, \wedge, \vee, 0, 1\rangle$ is an abelian lattice ordered group and δ is a unary operation satisfying:

$$\delta(x) \leq |x| \wedge 1 \qquad \delta(\delta(x)) = \delta(x) \qquad \delta(x) = \delta(x \wedge 1)$$

$$\delta(1) = 1 \qquad \delta(x) \vee (1 - \delta(x)) = 1 \qquad 0 \leq \delta(x)$$

$$\delta(x) \wedge \delta(y^+ + (1 - |x|)^+) \leq \delta(y)$$

where $|x| = x \vee (-x)$ and $x^+ = x \vee 0$

Theorem (Montagna, 2001)

There is a functor Γ_Δ *(extending Mundici's functor) between the category of* MV_Δ*algebra and the category of* δ-ℓ-*groups which, together with its inverse, forms an equivalence of category.*

Proposition

Each linearly ordered μMV algebra is isomorphic to the unitary interval of a linearly ordered divisible δ-group.

Proof.

Given a linearly ordered μMV algebra consider its equivalent linearly ordered divisible MV_Δalgebra $\mathcal{A}$.
Its divisible MV reduct is the interval algebra of a linearly ordered δ-group $\mathcal{G}$.
We only need to prove that $\mathcal{G}$ is divisible.
For every $x \in \mathcal{G}$ there exist $n \in \mathbb{N}$ such that $nu \geq x \geq (n-1)u$, hence $x' = x - nu$ belong to the unitary interval of $\mathcal{G}$.
Hence for every $m \in \mathbb{N}$ there exist y in the same interval such that $my = x' = x - nu$.
Let u' be such that $mu' = u$, then the element $y + nu'$ satisfies $m(y + nu') = x$. Hence $\mathcal{G}$ is divisible. □

Theorem

The μMV algebra $\langle [0,1], \oplus, \neg, 0, 1, \mu x.\varphi(x) \rangle$ generates the variety of μMV algebras.

Proof.

For the non trivial direction suppose that an equation φ fails in some μMV algebra.
Then it fails in a linearly ordered one.
Call $\mathcal{G}$ the linearly ordered δ-group in which the linealry ordered algerba embeds.
Then φ fails in $\mathcal{G}$.
In particular $\mathcal{G}$ is an abelian ordered group, so, by Gurevich-Kokorin theorem, this implies that φ fails in the reals, and hence in its interval algebra. $\square$

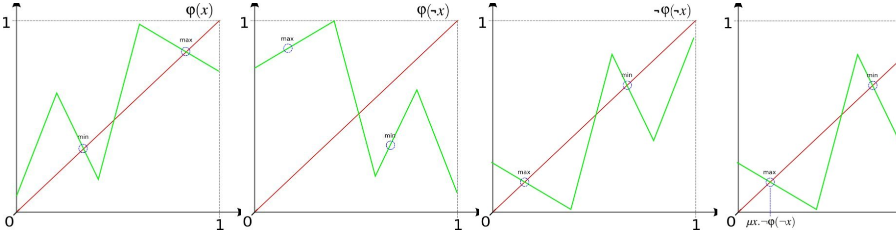
Proposition

For every μMV term, $\varphi(x)$, the term defined by $\nu x.\varphi =^{def} \neg\mu x.(\neg\varphi(\neg x))$ has the following properties:

- $\varphi(\nu x.\varphi(x)) = \nu x.\varphi(x)$
- *If* $\varphi(t) = t$ *then* $t \to \nu x.\varphi(x)$

Hence it interprets the maximum fixed point of $\varphi(x)$.

Proof.



□

Amalgamation

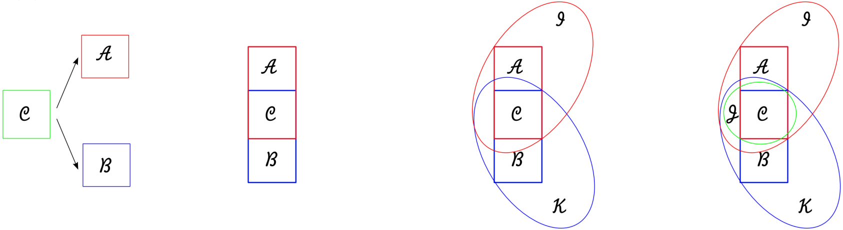
Lemma (Montagna, 2006)

Let **K** *a quasi-variety of BL algebras possibly with additional operators such that* $\mathbf{K}_{lin}$ *has the amalgamation property. Then* **K** *has the amalgamation property.*

Theorem

Linearly ordered μ*MV algebras enjoy amalgamation.*

Proof.



□

Further studies

- Describe the free μMV algebra (possibly, constructively).
- Does the forgetful functor have an adjoint?
- It is possible to find a subcategory of ℓ-groups which is categorical equivalent to μMV algebras?
- Explore the approach based on Kripke frames and find difference and analogies with the one given here.