µMV algebras
an approach to fixed points in Łukasiewicz logic

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The Logic of Soft Computing
Overview

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Motivations

- Introducing fixed points in many valued logic
- A deep understanding of the algebraic semantic of the logic turned out to be crucial.
- Introducing a particular class of algebras called $\mu$MV algebras.
Methods

- Kripke-style semantics have been studied for several important t-norm based logics our approach to fixed point is not classical (cf. $\mu$-calculus).
- In order to give a semantic to the $\mu$ operator we reaped benefit from the functional semantics of many valued logics.
- In Łukasiewicz logic connectives can be considered as continuous functions from $[0, 1]^n$ to $[0, 1]$, therefore, once parametrized as function in only one variable.
Theorem (Brouwer, 1909)

Every continuous function from the closed unit ball $D^n$ to itself has a fixed point.

- The advantage is that with this method any formula has fixed points, whereas in classical cases, one has to restrict to formulas on which the variable under the scope of $\mu$ only appears positively.

- On the other hand, the function giving the fixed point of a formula need not to be continuous in the remaining variables, whence we can not allow nested occurrences of $\mu$. 
Definition

A **MV algebra** is an algebra \( \mathcal{A} = \langle A, \oplus, \neg, 0 \rangle \) that satisfies

1. \( \langle A, \oplus, 0 \rangle \) is a commutative monoid
2. \( \neg
\neg x = x \)
3. \( x \oplus \neg 0 = \neg 0 \)
4. \( \neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x \)

Other connectives can be defined starting from the ones above:

1. \( 1 = \neg 0 \) \( x \oslash y = \neg(\neg x \oplus \neg y) \) \( x \rightarrow y = \neg x \oplus y \) \( x \ominus y = \neg(\neg x \oplus y) \)

Moreover, given a MV algebra \( \mathcal{A} \), defining

\[ x \land y = x \oslash (x \rightarrow y) \quad \text{and} \quad x \lor y = (x \ominus y) \oplus y \]

gives a lattice \( \mathcal{A} = \langle A, \lor, \land \rangle \).
Definition

A \( \mu \) \textbf{MV algebra} is a MV algebra, endowed with a function \( \mu x. T(x) \) for any term \( T(x) \) in the language of MV algebras, such that it satisfies the following conditions.

1. \( \mu x. (T(x)) = T(\mu x. (T(x))) \)
2. If \( T(t) = t \) then \( \mu x. (T(x)) \leq t \)
3. If \( \bigwedge_{i \leq n} (|p_i - q_i|) = 1 \) then
   \( \mu x. (T(p_1, ..., p_n)) = \mu x. (T(q_1, ..., q_n)) \)
Discontinuity of $\mu$-functions
Proposition

*Every $\mu$MV algebra is the subdirect product of linearly ordered $\mu$MV algebras.*

Proof.

Axiom 3 in the definition of $\mu$MV algebras guarantees that $\mu$ functions are compatible with the congruences of MV algebra. So the congruence of a $\mu$MV algebra are precisely the ones of its underlying MV algebra.

By Birkhoff representation theorem, every $\mu$MV algebra is the subdirect product of irreducible $\mu$MV algebras.

Irreducibility is characterized by the congruences lattice of the algebra. Since in this case they are identical, a $\mu$MV algebra is irreducible if, and only if, its underlying MV is linearly ordered.
**Definition**

A **$\text{MV}_\Delta$ algebra** is a MV algebra with an operator $\Delta$ that satisfies:

1. $\Delta(1) = 1.$
2. $\Delta(x \Rightarrow y) \leq \Delta(x) \Rightarrow \Delta(y).$
3. $\Delta(x) \sqcup \neg \Delta(x) = 1.$
4. $\Delta(x) \leq x.$
5. $\Delta(\Delta(x)) = \Delta(x).$
6. $\Delta(x \sqcup y) = \Delta(x) \sqcup \Delta(y).$
The presence of $\mu$-functions allows to reintroduce the $\Delta$ operator as

$$\Delta(y) = \neg\mu x.(x \oplus \neg y)$$

With $\Delta$ it is possible to express quasi-equations as equations, so this proves:

**Proposition**

*The class of $\mu$MV algebras is a variety.*
Definition
A **divisible MV algebra** is an MV algebra with a family of operators $\delta_n$ such that:

1. $(n)\delta_n(x) = x$
2. $\delta_n(x) \circ (n - 1)\delta_n(x) = 0$

Where $(n)x$ is a shorthand for $x \oplus \ldots \oplus x$ $n$-times

Definition
A **divisible MV$_{\Delta}$ algebra** is a structure $\mathcal{A} = \langle A, \oplus, \neg, 0, \delta_n, \Delta \rangle$ such that:

- $\langle A, \oplus, \neg, 0, \Delta \rangle$ is a MV$_{\Delta}$ algebra
- $\langle A, \oplus, \neg, 0, \delta_n \rangle$ is a divisible MV algebra
For any $n$, the operators $\delta_n$, of divisible MV algebras, are also definable by fixed points:

$$\delta_n(x) = \mu y.(x \ominus (n - 1)y)$$

**Lemma**

*For every $\mu$ MV algebra, the operator defined above satisfies:*

1. $(n)\delta_n(x) = x$
2. $\delta_n(x) \circ (n - 1)\delta_n(x) = 0$
So we have reached the following result:

**Proposition**

*Every $\mu MV$ algebra contains a definable divisible $MV_\Delta$ algebra.*

The other direction also holds but it require a little more work to be proved
Lemma

For every term $T(x)$ in the language of MV algebras, there exist:

c_1, \ldots, c_I terms of the form $(m)x \oplus k$, $(m)x \ominus k$, $\neg((m)x \oplus k)$
or $\neg((m)x \ominus k)$ where $m \in \mathbb{N}$ and $k$ is a term not containing $x$

$q_1, p_1, \ldots, q_I, p_I$ terms not containing $x$

such that for any evaluation $[\cdot]^*$:

$$[T(x)]^* = \left[ \bigvee_{i \leq I} (\Delta(x \rightarrow q_i) \land \Delta(p_i \rightarrow x) \land c_i) \right]^*$$
• Every term of an MV algebra can be interpreted as a continuous piece-wise linear function with integer coefficients form $[0, 1]^n$ to $[0, 1]$.

• Parameterizing the function in all its variables but $x$ this becomes of the form:

$$f(x) = \begin{cases} 
  z_1 x \pm k_1 & \text{if } p_1 \leq x \leq q_1 \\
  \vdots  & \vdots \\
  z_i x \pm k_i & \text{if } p_i \leq x \leq q_i 
\end{cases}$$

where $z_i \in \mathbb{Z}$ and $k_i, p_i, q_i$ are polynomials in the variable parameterized.
Such a function is the interpretation of a term of the form:

\[
\bigvee_{i \leq l} (\Delta(x \rightarrow q_i) \land \Delta(p_i \rightarrow x) \land c_i)
\]

where \(c_i\) are the terms corresponding to \(z_i \pm k_i\).
Theorem

$\mu$ MV algebras and divisible $\text{MV}_{\Delta}$ algebras are term-wise equivalent.

First of all we find the minimum fixed points of some basic term. Let us define:

\[
\bar{\mu}x.c = \neg \Delta(\neg k) \quad \text{if} \quad c = (m)x \oplus k
\]
\[
\bar{\mu}x.c = \delta_{m-1}(k) \quad \text{if} \quad c = (m)x \ominus k
\]
\[
\bar{\mu}x.c = \delta_{m+1}(\neg k) \quad \text{if} \quad c = \neg((m)x \ominus k)
\]
\[
\bar{\mu}x.c = \delta_{m+1}(k) \oplus \delta_{m+1}(1) \quad \text{if} \quad c = \neg((m)x \ominus k)
\]

Where we have put $\delta_0(x) = 0$ for every $x$. It is easy to see that in all four cases $\bar{\mu}x(c)$ is the minimum fixed point of $c$. 
To give the fixed point function associated to any term $T(x)$ we first find a term equivalent to $T(x)$ in which all the linear components are explicitly present, let it be

$$
\bigwedge_{i \leq l} (\Delta(x \rightarrow q_i) \land \Delta(p_i \rightarrow x) \land c_i)
$$

By continuity of the function a fixed point for this term must exist and it will be among the fixed point of the functions $c_i$. So we define:

$$
\mu x. T(x) = \bigwedge_{i \leq l} [\neg \Delta( T(\mu x. c_i) \leftrightarrow \mu x. c_i) \oplus \mu x. c_i]
$$
Once this equivalence is established it becomes fairly easy to extend known results (and techniques) about divisible MV algebras and $\text{MV}_\Delta$ algebra to $\mu\text{MV}$ algebras.

**Definition**
A $\delta$-lattice ordered group ($\delta$-ℓ-group, for short) is a structure $G = \langle G, +, -, \wedge, \vee, \delta, 0, 1 \rangle$ where $\langle G, +, -, \wedge, \vee, 0, 1 \rangle$ is an abelian lattice ordered group and $\delta$ is a unary operation satisfying:

\[
\begin{align*}
\delta(x) &\leq |x| \wedge 1 \\
\delta(1) &= 1 \\
\delta(x) \vee (1 - \delta(x)) &= 1 \\
\delta(x) \wedge \delta(y^+ + (1 - |x|)^+) &\leq \delta(y)
\end{align*}
\]

where $|x| = x \vee (-x)$ and $x^+ = x \vee 0$
Theorem (Montagna, 2001)

*There is a functor* $\Gamma_\Delta$ *(extending Mundici’s functor)* *between the category of* $\text{MV}_\Delta$ *algebra and the category of* $\delta$-$\ell$-*groups which, together with its inverse, forms an equivalence of category.*
**Proposition**

*Each linearly ordered $\mu$MV algebra is isomorphic to the unitary interval of a linearly ordered divisible $\delta$-group.*

**Proof.**

Given a linearly ordered $\mu$MV algebra consider its equivalent linearly ordered divisible $\text{MV}_{\Delta}$ algebra $\mathcal{A}$.

Its divisible MV reduct is the interval algebra of a linearly ordered $\delta$-group $\mathcal{G}$.

We only need to prove that $\mathcal{G}$ is divisible.

For every $x \in \mathcal{G}$ there exist $n \in \mathbb{N}$ such that $nu \geq x \geq (n - 1)u$, hence $x' = x - nu$ belong to the unitary interval of $\mathcal{G}$.

Hence for every $m \in \mathbb{N}$ there exist $y$ in the same interval such that $my = x' = x - nu$.

Let $u'$ be such that $mu' = u$, then the element $y + nu'$ satisfies $m(y + nu') = x$. Hence $\mathcal{G}$ is divisible.
Theorem

*The* $\mu$**MV algebra** $\langle [0, 1], \oplus, \neg, 0, 1, \mu x. \phi(x) \rangle$ *generates the variety of* $\mu$**MV algebras.**

**Proof.**

For the non trivial direction suppose that an equation $\phi$ fails in some $\mu$**MV algebra. Then it fails in a linearly ordered one. Call $G$ the linearly ordered $\delta$-group in which the linealry ordered algebra embeds. Then $\phi$ fails in $G$. In particular $G$ is an abelian ordered group, so, by Gurevich-Kokorin theorem, this implies that $\phi$ fails in the reals, and hence in its interval algebra.
**Proposition**

*For every $\mu MV$ term, $\varphi(x)$, the term defined by $\nu x.\varphi \overset{\text{def}}{=} \neg \mu x. (\neg \varphi(\neg x))$ has the following properties:

- $\varphi(\nu x.\varphi(x)) = \nu x.\varphi(x)$

- If $\varphi(t) = t$ then $t \rightarrow \nu x.\varphi(x)$

Hence it interprets the maximum fixed point of $\varphi(x)$. 
Proof.
Amalgamation

Lemma (Montagna, 2006)

Let $K$ a quasi-variety of BL algebras possibly with additional operators such that $K_{lin}$ has the amalgamation property. Then $K$ has the amalgamation property.

Theorem

Linearly ordered $\mu$MV algebras enjoy amalgamation.

Proof.
Further studies

- Describe the free $\mu$MV algebra (possibly, constructively).
- Does the forgetful functor have an adjoint?
- It is possible to find a subcategory of $\ell$-groups which is categorical equivalent to $\mu$MV algebras?
- Explore the approach based on Kripke frames and find difference and analogies with the one given here.