Some consequences of compactness in Łukasiewicz Predicate Logic

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Abstract. The Łoś-Tarski Theorem and the Chang-Łoś-Susko Theorem, two classical results in Model Theory, are extended to the infinite-valued Łukasiewicz logic. The latter is used to settle a characterisation of the class of generic structures introduced in the framework of model theoretic forcing for Łukasiewicz logic [1].

Key-words: First Order Many-Valued Logic, Łukasiewicz logic, Model Theory.

1 Introduction

Łukasiewicz logic is just one possibility in the myriad of infinite-valued generalisations of classical logic. An infinite-valued generalisation of classical logic is obtained by simply picking up some functions on an infinite superset of \( \{0, 1\} \) which behave on 0 and 1 accordingly to the classical connectives which they generalise; in other words, by extending the truth tables of the classical connectives.

Among those generalisations some are meaningless, for they have very little in common with a logic. Yet, when one requires a few natural properties to be fulfilled, the systems arising allow deep mathematical investigations; this is the case of continuous t-norm based logics. In these systems the conjunction is interpreted in an associative, commutative and weakly-increasing continuous function from \([0, 1]^2\) to \([0, 1]\), which behaves accordingly to classical conjunction in the limit cases 0 and 1, namely a continuous t-norm.

As a matter of fact the most important many-valued logics studied in mathematics are based on continuous t-norms; this is the case, for instance, of Łukasiewicz logic or Gödel logic. The logical system BL, introduced in [2], encompasses all logics based on continuous t-norms.

The setting based on continuous t-norm\(^1\), or equivalently BL, has been quite successful, for it provides a general mathematical framework for investigations on many-valued logics and offers an utter bridge towards fuzzy set theory and fuzzy logic, as t-norms are a pivotal tool in fuzzy logic.

\(^{1}\) The requirement on continuity is sufficient for the existence of a residual operation, which plays the role of implication. Such a requirement can be relaxed to only left-continuity leading to a logical system called MTL [3].
Yet Łukasiewicz logic stands out among those logics because of some of its properties. Indeed, Łukasiewicz logic is the only one, among continuous t-norm based logics, with a continuous implication and therefore the only logic whose whole set of formulae can be interpreted as continuous functions. Furthermore the Łukasiewicz negation is involutive, namely it is such that \( \neg \neg \varphi \leftrightarrow \varphi \). Those two features, inherited from classical logic, makes Łukasiewicz logic a promising setting to test how far the methods of model theory can reach in the realm of many-valued logics.

A model theoretic study of many-valued logic is especially important in the light of the negative results already obtained in the first order theory of these logics: the predicate version BL has a (standard) tautology problem whose complexity is not arithmetical, the same problem is \( \Pi_2 \)-complete for Łukasiewicz logic. Thus the favourable duality between syntax and semantics vanishes when switching to t-norm based logics and new tools must be developed.

The results so far are encouraging: in [1] the Robinson finite and infinite forcing were generalised to Łukasiewicz logic; here some basic results for a model theory of Łukasiewicz logic are presented and used to settle an open problem left therein.

2 Preliminaries

The language of the infinite-valued Łukasiewicz propositional logic, \( L \), is built from a countable set of propositional variables, \( \text{Var} = \{p_1, p_2, \ldots, p_n, \ldots \} \), and two connectives \( \rightarrow \) and \( \neg \). The axioms of \( L \) are the following:

\[
\begin{align*}
\varphi &\rightarrow (\psi \rightarrow \varphi); \\
(\varphi \rightarrow \psi) &\rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi)); \\
((\varphi \rightarrow \psi) \rightarrow \psi) &\rightarrow ((\psi \rightarrow \varphi) \rightarrow \varphi); \\
(\neg \varphi &\rightarrow \neg \psi) \rightarrow (\psi \rightarrow \varphi),
\end{align*}
\]

Modus ponens is the only rule of inference. The notions of proof and tautology are defined as usual.

The equivalent algebraic semantics for \( L \) (in the sense of [4]) is given by the variety of MV-algebras [5]. An MV-algebra is a structure \( A = \langle A, \oplus, *, 0 \rangle \) such that \( A = \langle A, \oplus, 0 \rangle \) is a commutative monoid, \( * \) is an involution and the following equations hold: \((x * y) * y \oplus y = (y * x) * y \oplus x \) and \( x * 0 = 0 * \).

An \( L \forall \) language \( L \) is defined similarly to a language for classical first order logic, without functional symbols, taking as primitive the connectives: \( \rightarrow, \neg, \exists \). This allows the syntactical concepts of term, (atomic) formula, free or bounded variable, substitutable variable for a term, formal proof, formal theorems, etc. to be defined just as usual. The set \( V = \{x, y, z, \ldots \} \) is a fixed set of variables and Form will be used to indicate the set of formulae of \( L \).

The axioms of \( L \forall \) are:

\[
\text{(i) All the axioms of the infinite-valued propositional Łukasiewicz calculus;}
\]

\footnote{Functional symbols can be added, in principle, to the language, however this requires a discussion on how equality has to be treated which goes beyond the scope of this article.}
Let $A$ be an $L$-structure. An $A$-structure has the form

$$M = \langle M, P_1^M, ..., P_n^M, c_1^M, ..., c_m^M \rangle$$

where $M$ is a non-empty set (called the universe of the structure); if $P_i$ is a predicate symbol in $L$ of arity $k$ then $P_i^M$ is a $k$-ary $A$-valued relation on $A$, namely $P_i^M : M^k \rightarrow A$; if $c_j$ is a constant symbol in $L$ then $c_j^M$ is an element of $M$.

Let $M$ be an $A$-structure. An evaluation of $L$ in $M$ is a function $e : V \rightarrow M$. Given any two evaluations $e, e'$ of $L$ and for $x \in V$ let $e \equiv_x e'$ if $e|_{V \setminus \{x\}} = e'|_{V \setminus \{x\}}$. For any term $t$ of $L$ and any evaluation in $M$ let

$$t^M(e) = \begin{cases} e(x) & \text{if } t \text{ is a variable } x \\ c^M & \text{if } t \text{ is a constant } c \end{cases}$$

Given any evaluation in $M$, $e$ and any formula $\varphi$ of $L$, the element $\| \varphi(e) \|_M$ of $A$ is defined by induction, and it is called the truth value of $\varphi$:

- if $\varphi$ is of the form $P(t_1, ..., t_n)$ then $\| \varphi(e) \| = P^M(t_1^M(e), ..., t_n^M(e))$;
- if $\varphi = \neg \psi$ then $\| \varphi(e) \| = \| \psi(e) \|^*$;
- if $\varphi = \psi \land \chi$ then $\| \varphi(e) \| = \| \psi(e) \| \land \| \chi(e) \|$;
- if $\varphi = \exists x \psi$ then $\| \varphi(e) \| = \bigvee \{\| \psi(e') \| \mid e' \equiv_x e \}$.

An $A$-structure $M$ is called safe if for any evaluation $e : V \rightarrow M$ and for any formula $\psi$ of $L$, the supremum $\bigvee \{\| \psi(e') \| \mid e' \equiv_x e \}$ exists in $A$ (in this case the infimum $\bigwedge \{\| \psi(e') \| \mid e' \equiv_x e \}$ also exists).

If $\| \varphi \|_M = 1$ then $\varphi$ is said to be true in $M$, this can be alternatively written as $M \models \varphi$. A safe $A$-structure $M$ is a model of a theory $T$ if $M \models \varphi$ for all $\varphi \in T$. A standard structure is a $[0,1]$-structure, which is always safe. A standard model of a theory $T$ is a $[0,1]$-structure which is a model of $T$. If $\varphi$ every $A$-model of a theory $T$ is also an $A$-model of a formula $\varphi$ then $\varphi$ is said an $A$-logical consequence of $T$, in symbols $T \models_A \varphi$, in particular, when this is true for standard models then I write $T \models_{[0,1]} \varphi$.

**Definition 1.** A formula $\varphi$ is generally satisfiable if there exists a model $M$ such that $\| \varphi \|_M = 1$. If the model can be taken standard then $\varphi$ is called just satisfiable. The previous definitions naturally generalise to theories. A theory $T$ is consistent if $T \not\models \bot$.

All the results in the next section hinge on the following theorems.
Theorem 1 (Weak Completeness [6]). Any consistent theory $T$ of $L\forall$ has a standard model.

As can be easily guessed from the considerations at the end of Section 1, the notion of compactness in Lukasiewicz logic splits in two, furthermore, if one considers also the two classes of general and standard models, then four different re-statements of compactness emerge (see e.g. [2] for more details). The situation in $L\forall$ is fully described below.

Theorem 2 (Compactness). Let $T$ be a theory in $L\forall$:

(i) If $T$ is finitely generally satisfiable then $T$ is generally satisfiable.
(ii) If $T$ is finitely satisfiable then $T$ is satisfiable.
(iii) If $T \models \varphi$ then there exists a finite $T_0 \subseteq T$ such that $T_0 \models \varphi$.
(iv) If $T \models_{[0,1]} \varphi$ then in general it is false that there exists a finite $T_0 \subseteq T$ such that $T_0 \models_{[0,1]} \varphi$.

3 Main results

Henceforth $L$ is assumed to be a fixed language of $L\forall$ and all structures are standard.

In Lukasiewicz logic, all the connectives are continuous, thus each formula is equivalent to one in prenex form. This allows to define a total hierarchy on the lines of the arithmetical hierarchy in classical logic.

Definition 2. A formula of $L$ belongs to the set $\Sigma_n$ ($\Pi_n$, respectively) if it is equivalent to a formula with $n$ blocks of quantifier, where each block is either empty or constituted of an uninterrupted sequence of the same quantifier, $\exists$ or $\forall$, and the first block is made of $\exists$'s ($\forall$'s respectively).

As in the classical case one has $\Sigma_n \cup \Pi_n \subseteq \Sigma_{n+1} \cap \Pi_{n+1}$.

Let $M$ be an $A$-structure, $L(M)$ is the expansion of the language $L$ with a new constant symbol for each element of $M$. The diagram of $M$, i.e. the set of atomic formulae $\varphi$ in $L(M)$ such that $\|\varphi\|_M = 1$, is indicated by $D(M)$; Th$(M)$ is the set of formulae $\varphi$ such that $\|\varphi\|_M = 1$.

Definition 3. If $M_1 \subseteq M_2$ are two $A$-structures and for any $\varphi \in D(M_1)$, $M_1 \models_A \varphi$ iff $M_2 \models_A \varphi$ then $M_1$ is a substructure of $M_2$, in symbols $M_1 \leq M_2$. If the same is true for any sentence of $L(M_1)$ than $M_1$ is an elementary substructure of $M_2$, written $M_1 \preceq M_2$.

Proposition 1. Let $T$ be a theory, let $T_\forall$ be the set of logical consequences of $T$ which are in $\Pi_1$ and let $K$ be the class of all substructures of models of $T$. Then $K$ is the class of model of $T_\forall$.

Proof. It follows directly from the definition of the interpretation of universal quantifiers that if $M \in K$ then $M \models_{[0,1]} T_\forall$. Vice-versa, let $M$ be a model of $T_\forall$, then it follows that $D(M) \cup T$ is finitely satisfiable. Indeed if it is not then
there exist finite subsets $\Psi \subseteq D(M)$ and $\Phi \subseteq T$ such that $\bigwedge \Psi \models [0,1] - \bigwedge \Phi$, but $\neg \bigwedge \Phi$ is in $\Pi_1$ and this contradicts the fact that $\bigwedge \Phi \in D(M)$. By compactness, $D(M) \cup T$ has a model, say $N$. Obviously $M$ embeds in $N$ which is a model of $T$, hence $M \in K$.

**Corollary 1 (Łoś-Tarski Theorem for Łukasiewicz logic).**

A theory is preserved under substructure if, and only if, it is equivalent to a universal (i.e. $\Pi_1$) theory.

Let $\alpha$ be an ordinal and $(M_\lambda)_{\lambda \in \alpha}$ a family of $L$-structure. The structure $(M_\lambda)_{\lambda \in \alpha}$ is a **chain** if for any $\lambda_1 \leq \lambda_2 < \alpha$, $M_{\lambda_1} \preceq M_{\lambda_2}$. If for any $\lambda_1 \leq \lambda_2 < \alpha$, $M_{\lambda_1} \preceq M_{\lambda_2}$ then $(M_\lambda)_{\lambda \in \alpha}$ is called **elementary chain**.

**Lemma 1 ([1, Lemma 4.2]).** Let $(M_\lambda)_{\lambda \in \alpha}$ be an elementary chain. Then for every $\lambda \in \alpha$, $M_\lambda \preceq \bigcup_{\lambda \in \alpha} M_\lambda$

$T$ is an inductive theory if it is closed under unions of chains.

**Theorem 3 (Chang-Łoś-Suszko Theorem for Łukasiewicz logic).**

A theory is inductive if, and only if, it is equivalent to a $\Pi_2$ theory.

**Proof.** For the non-easy direction suppose that a theory $T$ is inductive. Let $M \models T_{\forall_3}$ then $T \cup \text{Th}_3(M)$ is satisfiable, for if it is not then there exist two finite sets $\Psi \subseteq \text{Th}_3(M)$ and $\Phi \subseteq T$ such that $\bigwedge \Phi \models [0,1] - \bigwedge \Psi$, but then $\neg \bigwedge \Phi \in T_{\forall_3}$, whence it holds in $M$, which is a contradiction. So $T \cup \text{Th}_3(M)$ has a model $N$, and $M \preceq N$. Every existential sentence of $L(M)$ which is true in $N$ holds in $M$, hence $D(N) \cup \text{Th}(M)$ is satisfiable, so it has a model $M_1$ which is an extension of $N$ and an elementary extension of $M$. Repeating such a construction countably many times an infinite chain is produced:

$$M \preceq N \preceq M_1 \preceq N_1 \preceq \ldots$$

Let $O$ be the limit of this chain. $O$ is a model of $T$, for $T$ is inductive; furthermore $O$ is an elementary extension of $M$, because the chain $(M_i)_{i \in \omega}$ is elementary. Therefore $M$ is a model of $T$.

The above characterisation is extremely useful, when dealing with model complete theories.

**Corollary 2.** When the model companion of a theory is axiomatisable, it is equivalent to a $\forall \exists$ theory.

**Proof.** In a model companion every chain is elementary.

From this it is also easy to see that

**Corollary 3.** There exists at most one model companion of a theory.
In [1] both the notions of finite and infinite model theoretic forcing [7,8] were extended to Łukasiewicz logic. Following the lines of Robinson, the family of generic models, $\mathfrak{G}_K$, of a given class $K$ was studied and proved to contain the class of models existentially closed in $K$. Theorem 3 enables to complete this result.

**Proposition 2.** Let $T$ be an inductive theory, then if $\mathfrak{G}_{\text{Mod}(T)}$ is axiomatisable then it is the class of existentially closed models of $T$.

**Proof.** One direction is given in [1, Proposition 5.7]. For the other let $\mathcal{M}$ be a existentially closed model of $T$, then, by [1, Theorem 5.10], it embeds in a model $\mathcal{N} \in \mathfrak{G}_{\text{Mod}(T)}$. By [1, Theorem 5.9] $\mathfrak{G}_{\text{Mod}(T)}$ is inductive, so if it is axiomatisable then by Theorem 3 it is equivalent to a $\Pi_2$ theory. Since $\mathcal{M}$ is existentially closed, it is easy to see that it satisfies the same $\Pi_2$ formulae of $\mathcal{N}$, whence $\mathcal{M} \in \mathfrak{G}_{\text{Mod}(T)}$.

**References**