Projectivity and unification in many-valued logic

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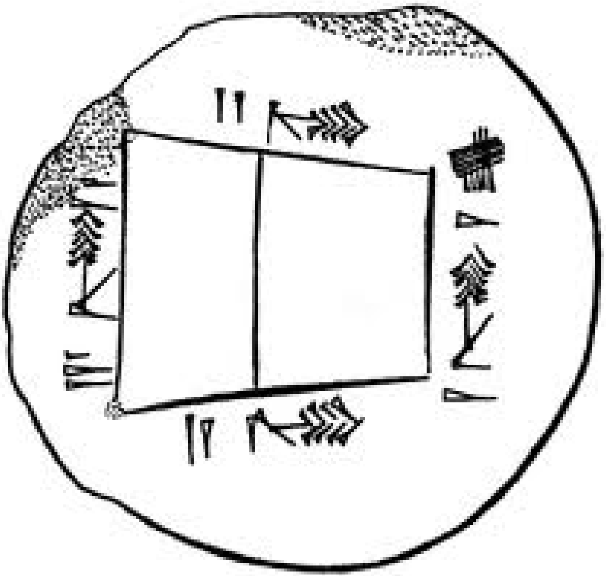
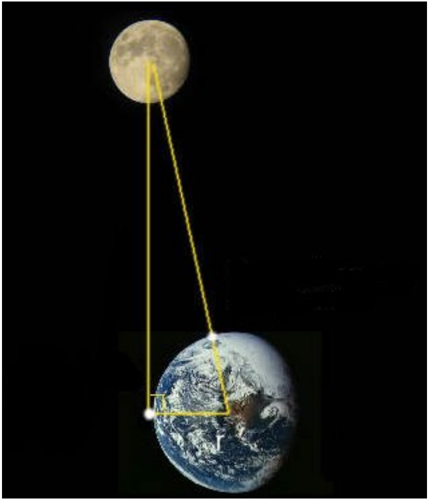
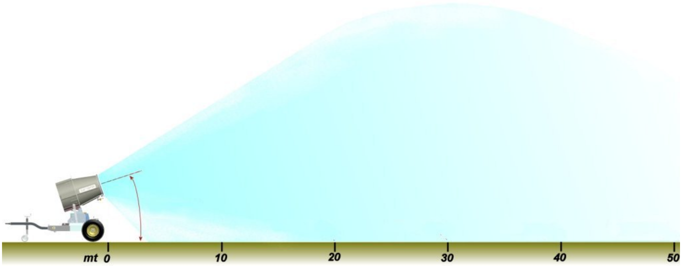
Prehistory of Mathematics



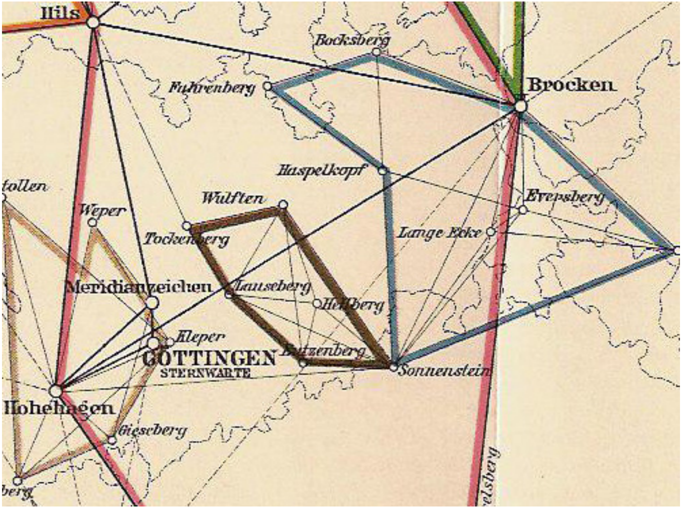
Prehistory of Mathematics



Prehistory of Mathematics



Prehistory of Mathematics



A first level of abstraction

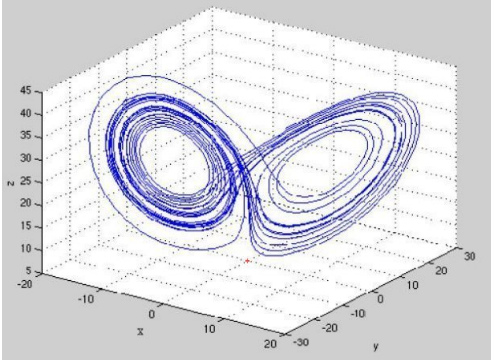
$$x^3 + 3x + 5 =$$
$$x?$$

A first level of abstraction

$$y^3 + 7x^2y + 5y = xy + 2?$$

A second level of abstraction

Consider the set of all solutions



Studying *all* solutions

One could be interested in properties which are common to all solutions.

Often the set of all solutions is quite large and complex.

The study can be simplified in presence of most representative solutions.

E-unifiers

Let s and t be term in some language $\mathcal{L}$. Let E be a set of equations in the language $\mathcal{L}$.
A substitution σ is an E-unifier for the pair (s, t) if

$$E \models \sigma(t) \approx \sigma(s)$$

Unifiers are functions from the set of variables of $\mathcal{L}$ ($Var(\mathcal{L})$) into the set of terms of $\mathcal{L}$ ($T(\mathcal{L})$). They can be easily extended to functions on $T(\mathcal{L})$.

It is often useful to keep control on the number of variables involved in the pair of terms (s, t) and in the image of the unifier, so if σ is a unifier for (s, t) one may indicate it by $\sigma : T_n \longrightarrow T_m$, where n is the number of distinct variables in (s, t) and m is the number of distinct variables in $(\sigma(s), \sigma(t))$.

E-equivalence

Given two unifiers σ, σ' we say σ is more general (mod E) than σ', written $\sigma \geq_E \sigma'$, if there is a substitution τ such that

$$E \models \sigma' \approx \tau \circ \sigma$$

$$\left[\text{i.e. } E \models \sigma'(x) \approx \tau \circ \sigma(x) \text{ for all } x \in Var(\mathcal{L})\right]$$

The relation $\leq_E$ is a preorder on $\{\sigma \mid T_n \longrightarrow T_n\}$. So it makes sense to say that two substitutions σ, σ' are E-equivalent, written $\sigma \sim_E \sigma'$,if

$$\sigma \leq_E \sigma' \text{ and } \sigma' \leq_E \sigma.$$

Most general unifiers

Unless otherwise stated $\mathcal{L}$ and E are arbitrary but fixed.
A substitution σ is a most general unifier for the pair (s, t), if

1. $E \models \sigma(t) \approx \sigma(s)$ and
2. if τ is a unifier for (s, t), $\sigma \leq_E \tau$ implies $\sigma \sim_E \tau$.

In other words, σ is a maximal element in the partial order induced by the equivalence $\sim_E$ on the preorder $\leq_E$.

Unification type

In full generality, when E varies among all possible set of equations, the following situations are encountered:
For any pair (s, t),

1. There is an E-unifier μ which is more general than any E-unifier of (s, t); unary.
2. There are finitely many most general E-unifiers $\mu_1, \ldots, \mu_n$ such that for any unifier σ, some μ_i is more general than σ; finitary.
3. There are infinitely many most general E-unifiers $\{\mu_i\}_{i \in I}$ such that for any unifier σ of (s, t) some μ_i is more general than σ; inifinitary.
4. There is a unifier σ such that no most general unifier of (s, t) is more general than σ; nullary.

Some technical considerations

In several cases every problem of unifying two terms reduces to find a substitution that identifies a term with a constant (usually either 0 or 1).

i.e. $t(x)$ is given and we look for σ such that $\sigma(t(x)) = 1$.

It makes sense then to speak of a single term as an unification problem, once the constant has been agreed.

In these cases, translated in logical terms, a unification problem amount to find a substitution that makes some propositional formula true.

Vector Spaces

Vector spaces over a field K can be regarded as an equational class with the usual vector space operations $+, -$, the constant 0, and a collection of unary operations f_k, for $k \in K$.

The set E contains equations that state that $+, -, 0$ are group operations, that f_k are linear and well behaved (i.e. $f_k(f_l(x)) = f_{kl}(x)$, $f_1(x) = x$ and $f_{k+l}(x) = f_k(x) + f_l(x)$).

The E-free algebra over n generators in this case is the familiar n-dimensional vector space over K, i.e. K^n and substitutions are linear maps.

Vector Spaces

This formulation still does not suffice to discuss linear equations, since the terms in the above language are homogeneous, i.e., of the form $\Sigma_{j=1}^{n} a_j x_j$. To remedy this one adds constants k, for $k \in K$, to the language, and the following axioms to E:

1. $f_k(k') \approx k \cdot k'$;
2. $-k_1 \approx k_2$ if this holds in K;
3. $k_1 + k_2 \approx k_3$ if this holds in K.

In this setting the E-free algebra F_n can be thought of as an $(n+1)$-dimensional vector-space over K, using the mapping

$$a_1 x_1 + \ldots + a_n x_n + b \mapsto (a_1, \ldots, a_n, b);$$

Vector Spaces

A particular solution to a $m \times n$ system of homogenous equations corresponds to a homomorphism $\sigma : K^n \longrightarrow K$.

A solution with paramenters equations corresponds to a homomorphism $\sigma : K^n \longrightarrow K^m$.

Gaussian elimination solves the unification problem and provides a most general unifier when such exists.

So the unification type of the theory of vector spaces is unitary

Semigroups

If we take E to be composed just by the associative law we get the theory of semigroups. In this case the unification type is infinitary.
Indeed elements of the free semigroup F_n can be thought of as strings on an alphabet, and any string s has attached a length $|s|$.
So, to a homomorphism $\sigma : F_n \longrightarrow F_n$ one associates a tuple of positive integers:

$$\#\sigma = (|\sigma(x_1)|, \ldots, |\sigma(x_n)|)$$

Then we have $|s| \leq |\sigma(s)|$, consequently $\sigma_1 \leq_E \sigma_2 \Rightarrow \#\sigma_1 \geq \#\sigma_2$.

So the more general a unifier is, the less symbols it contains.

Semigroups

Suppose that there exists an infinite, increasing sequence $\sigma_1 < \sigma_2 < \ldots$, then there must exists n_0 such that $i \geq n_0 \Rightarrow \#\sigma_i = \#\sigma_{n_0}$.
Take τ_i such that $\sigma_i = \tau_i \circ \sigma_{i+1}$, then for $i \geq n_0$, the function τ_i maps the variables of the range of σ_{i+1} to variables (otherwise $\#\sigma_i > \#\sigma_{i+1}$).
Such a τ_i cannot be one-to-one on the variables in the range of σ_{i+1}, for otherwise σ_i and σ_{i+1} would be equivalent under $\leq_E$.
This leads to the conclusion that the ranges of $\sigma_{n_0}, \sigma_{n_0+1}, \ldots$ have a strictly decreasing (finite) number of variables in them. This is impossible, and hence so is the existence of an infinite sequence like the above.

Gödel logic

In 1995 Wroński proved that Gödel logic has unitary unification type.

This should be compared to Ghilardi's result (1999) showing that the unification type of intuitionistic logic is finitary.

Unification for finitely-valued Łukasiewicz logic and BL

In 2008 Dzik considered, for any BL-algebra $\langle A, \cdot, \Rightarrow, \vee, \wedge, 0, 1\rangle$, the function

$$f(x) = (e \Rightarrow x) \cdot (\neg e \Rightarrow \tau(x))$$

where e is an idempotent and $\tau : A \to \{0, 1\}$ s.t. $\tau(e) = 1$.

He proved that f is an endomorphism of A and moreover f it a retraction, i.e. $f = f \circ f$.

Unification for finitely-valued Łukasiewicz logic and BL

From this he derives that

Theorem (Dzik 2008)

If a formula $\varphi(\bar{x})$ of a k-potent logic containing BL is unifiable, then it has a unifier of the form

$$\sigma(\bar{x}) = (\varphi^k \to x) \cdot (\neg\varphi^k \to \tau(x))$$

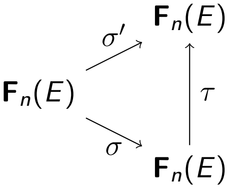
where τ is a ground unifier for φ. Furthermore σ is more general then any other unifier, hence the unification type of that logic is unitary.

Substitutions as homomorphisms

Given a set of equation E, let $\mathbf{F}_n(E)$ be the E-free algebra over n generators. The E-unifiers for some pair (s, t) are the endomorphisms

$$\{\sigma \in \mathrm{Hom}(\mathbf{F}_n(E), \mathbf{F}_n(E)) \mid \sigma(s) = \sigma(t)\}$$

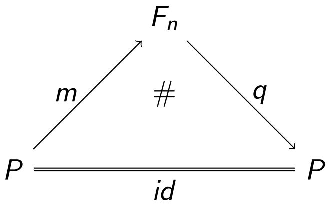
The relation $\sigma' \leq_E \sigma$ holds if there exists $\tau \in \mathrm{Hom}(\mathbf{F}_n(E), \mathbf{F}_n(E))$ such that the diagram below commutes:



Projective objects

The approach through homomorphisms of the free algebras gives a genuine algebraic perspective which has been used a number times in studying unification problems for equational theory.

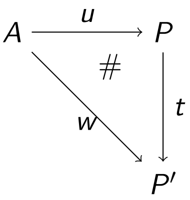
However in 1997 Ghilardi proposed an alternative approach to unification which has several advantages. The key concept is played by projective formulas and projective algebras.
An algebra is called projective in a variety if it is a retraction of some free algebra of that variety, i.e.,



Algebraic unifiers

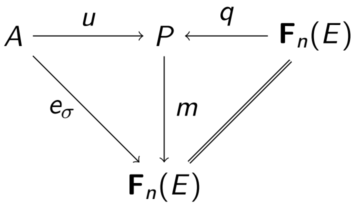
Given a pair of terms (s, t), an algebraic E-unifier is an arrow from the finitely presented algebra $A = F_n/\langle(s, t)\rangle$ into a finitely generated projective algebra P.

An algebraic E-unifier u, P is more general than w, P' if there exists an arrow t s.t.



Equivalence of the two approaches

The correspondence e between the two approaches works as follows.
To any unifier σ it is associated an *evaluation* morphism e_σ from A to $F_n(E)$ defined by $e_\sigma([t]) := [\sigma(t)]$.
Take any algebraic unifier u, P:



Define $[\sigma(x)] = m(u([x]))$. So we have $m \circ u = e_\sigma$ and $q \circ e_\sigma = u$.

Whence $e_\sigma \sim_E u$.

Commutative ℓ-groups

An ℓ-group is a partially ordered group in which the order is a lattice.
Note: In this talk all groups are Abelian.
These algebraic structures are the equivalent algebraic semantics of Casari's *Comparative logic*, aka Abelian logic.

In 1975 Beynon, expanding previous results by Baker, established a categorical duality which enabled a geometrical study of finitely presented ℓ-groups.

Theorem (1977 Beynon)

Finitely generated projective ℓ-groups are exactly the finitely presented ℓ-groups.

Commutative ℓ-groups

In the light of the previous result and Ghilardi's characterisation, one easily gets:

Theorem

The unification type of the theory of Abelian ℓ-groups, as well as the one of Abelian logic, is unitary.

In a forthcoming paper with V. Marra, we exploit Beynon proof and his geometrical duality to give an algorithm that, taken any term in the language of ℓ-groups, outputs its most general unifier.

MV-algebras

MV-algebras are the equivalent algebraic semantics for Łukasiewicz logic.

Definition

An MV-algebra is a structure $\mathcal{A} = \langle A, \oplus, ^*, 0\rangle$ such that:

- $\mathcal{A} = \langle A, \oplus, 0\rangle$ is a commutative monoid,
- * is an involution
- the interaction between those two operations is described by the following two axioms:
  - $x \oplus 0^* = 0^*$
  - $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$

Theorem (Mundici 1986)

The category of MV-algebras is equivalent to the category of ℓ-groups with strong unit.

Non unitarity of the unification in Łukasiewicz logic

Łukasiewicz logic has a *weak disjunction property*, namely: if $\varphi \vee \neg\varphi$ is derivable then either φ or $\neg\varphi$ must be derivable.

The unification type of Łukasiewicz logic is not unitary.

Indeed if σ is a unifier for $x \vee \neg x$, then it must unify either x, hence it is the substitution $x \mapsto 1$ or be unifier for $\neg x$, hence it must be the substitution $x \mapsto 0$.

McNaughton representation

Theorem (McNaughton 1951)

The free MV-algebra over n generators is isomorphic to the algebra of continuous, piece-wise linear map from $[0,1]^n$ *into* $[0,1]$ *with integer coefficients.* [McNaughton functions].

So, in the theory of MV-algebras unifiers are vector functions whose components are McNaugthon functions.

Note that a substitution is a unifier φ if its associated McNaughton function has its range included in the a subset of the points where the McNaughton function associated to φ is 1.

Substitution as McNaughton functions

Finitarity of the 1-variable fragment

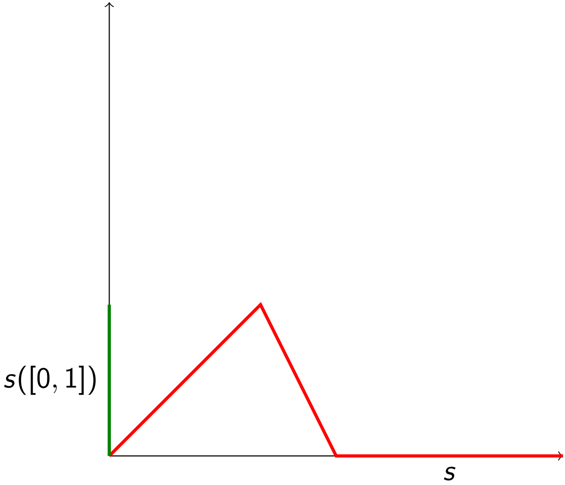
Theorem

The unification type of the 1-variable fragment of Łukasiewicz logic is finitary.

Proof. Suppose that φ is unifiable, then its associated McNaughton function must be one in an interval of the type $[0, p]$ or $[q, 1]$.
Claim: *For each such an interval there is a unique (modulo equivalence) most general unifier.*

To fix the ideas let us consider a Łukasiewicz formula φ whose associated McNaughton function is 1 exactly on the interval $[0, \frac{p}{q}]$. Let us call s the function which is the identity on the interval $[0, \frac{p}{q}]$ and then goes to 0.

[Proof cont'd]



[Proof cont'd]

The function σ has range included in $[0, \frac{p}{q}]$, so it corresponds to a unifier for φ.

To see that s yields indeed the m.g.u. of φ let us call σ its associated substitution and prove that for any unifier τ and its associated McNaughton function t, there exists a McNaughton function a such that $t(x) = s(a(x))$. This is a straightforward consequence of the following fact:

If f is a McNaughton function such that $\forall x \in [0, 1]\ f(x) \leq \frac{p}{q}$, then $s(f(x)) = f(x)$.

This holds because s is the identity on the interval $[0, \frac{p}{q}]$ and the range of the function f is completely contained in that interval.

Since the function associated to any unifier of φ must be bounded by $\frac{p}{q}$, the equality $t(x) = s(a(x))$, holds for any t by taking $t = a$. $\square$

The *n*-variable fragments and the full logic

Theorem (Marra-S.)

Łukasiewicz logic has nullary unification type.

The proof makes use of Ghilardi's characterisation together with the above mentioned duality for MV-algebras and a homotopy argument.

Conjecture (Marra-S.)

For any $n > 1$ the n-variable fragment of Łukasiewicz logic has nullary unification type.

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