Forcing in Łukasiewicz logic
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History of forcing


Motivations

The aim of our work is to generalize the classical model-theoretical notion of forcing to the infinite-valued Łukasiewicz predicate logic.

Łukasiewicz predicate logic is not complete w.r.t. standard models and, its set of standard tautologies is in $\Pi_2$. The Lindenbaum algebra of Łukasiewicz logic is not semi-simple.

In introducing our notions we will follow the lines of Robinson and Keisler.
The language of Łukasiewicz propositional logic \( \mathbb{L}_\infty \) is defined from a countable set \( \text{Var} \) of propositional variables \( p_1, p_2, \ldots, p_n, \ldots \), and two binary connectives \( \to \) and \( \neg \). \( \mathbb{L}_\infty \) has the following axiomatization:

- \( \varphi \to (\psi \to \varphi) \);
- \( (\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi)) \);
- \( ((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi) \);
- \( (\neg \varphi \to \neg \psi) \to (\psi \to \varphi) \).

where \( \varphi, \psi \) and \( \chi \) are formulas. Modus ponens is the only rule of inference. The notions of proof and theorem are defined as usual.
MV-algebras

A **MV-algebra** is structures $A = \langle A, \oplus, *, 0 \rangle$ satisfying the following equations:

- $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,
- $x \oplus y = y \oplus x$,
- $x \oplus 0 = x$,
- $x \oplus 0^* = 0^*$,
- $x^{**} = x$,
- $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$.

Other operations are definable as follows:

$$x \rightarrow y = x^* \oplus y \quad \text{and} \quad x \circ y = (x^* \oplus y^*)^*.$$  

MV-algebras form the equivalent algebraic semantics of the propositional Łukasiewicz logic, in the sense of Blok and Pigozzi.
Łukasiewicz predicate logic

The following are the axioms of Łukasiewicz predicate logic ($\mathcal{PL}_\infty$):

1. the axioms of $\infty$-valued propositional Łukasiewicz calculus $\mathcal{L}_\infty$;
2. $\forall x \varphi \rightarrow \varphi(t)$, where the term $t$ is substitutable for $x$ in $\varphi$;
3. $\forall x (\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \forall x \psi)$, where $x$ is not free in $\varphi$;
4. $(\varphi \rightarrow \exists x \psi) \rightarrow \exists x (\varphi \rightarrow \psi)$, where $x$ is not free in $\varphi$.

$\mathcal{PL}_\infty$ has two rules of inference:
- Modus ponens (m.p.): from $\varphi$ and $\varphi \rightarrow \psi$, derive $\psi$;
- Generalization (G): from $\varphi$, derive $\forall x \varphi$. 
The semantic of $\mathcal{P}\mathcal{L}_\infty$

Let $L$ be an MV-algebra. An $L$-structure of the language $\mathcal{P}\mathcal{L}_\infty$ has the form $\mathfrak{A} = \langle A, (P^\mathfrak{A})_P, (c^\mathfrak{A})_C \rangle$ where

- $A$ is a non-empty set (the universe of the structure);
- for any $n$-ary predicate $P$ of $\mathcal{P}\mathcal{L}_\infty$, $P^\mathfrak{A} : A^n \to L$ is an $n$-ary $L$-relation on $A$;
- for any constant $c$ of $\mathcal{P}\mathcal{L}_\infty$, $c^\mathfrak{A}$ is an element of $A$.

The notions of evaluations, tautology, etc. are defined as usual.
Forcing properties

Let $\mathcal{P}_\infty(C)$ be the language of $\mathcal{P}_\infty$, to which we add an infinite set $C$ of new constants. Let $E$ be set of sentences of $\mathcal{P}_\infty(C)$ and $At$ the set of atomic sentences of $\mathcal{P}_\infty(C)$.

Definition

A **forcing property** is a structure of the form $\mathbf{P} = \langle P, \leq, 0, f \rangle$ such that the following properties hold:

(i) $(P, \leq, 0)$ is a poset with a first element $0$;
(ii) Every well-orderd subset of $P$ has an upper bound;
(iii) $f : P \times At \to [0, 1]$ is a function such that for all $p, q \in P$ and $\varphi \in At$ we have $p \leq q \implies f(p, \varphi) \leq f(q, \varphi)$.

The elements of $P$ are called **conditions**.
Finite forcing

Definition
Let $< P, \leq, 0, f >$ be a forcing property. For any $p \in P$ and any formula $\varphi$ we define the real number $[\varphi]_p \in [0, 1]$ by induction on the complexity of $\varphi$:

1. if $\varphi \in At$ then $[\varphi]_p = f(p, \varphi)$;
2. if $\varphi = \neg \psi$ then $[\varphi]_p = \bigwedge_{p \leq q} [\psi]_q^*$;
3. if $\varphi = \psi \rightarrow \chi$ then $[\varphi]_p = \bigwedge_{p \leq q} ([\psi]_q \rightarrow [\chi]_p)$;
4. if $\varphi = \exists x \psi(x)$ then $[\varphi]_p = \bigvee_{c \in C} [\psi(c)]_p$.

The real number $[\varphi]_p$ is called the forcing value of $\varphi$ at $p$. 
Some properties of finite forcing

For any forcing property $P$, $p \in P$ and for any sentence $\varphi$, $\psi$ or $\forall x \chi(x)$ of $\text{PL}_\infty(C)$ we have:

1. If $p \leq q$ then $[\varphi]_p \leq [\varphi]_q$
2. $[\neg\neg\varphi]_p = \bigwedge_{p \leq q} \bigvee_{q \leq v} [\varphi]_v$;
3. $[\varphi]_p \leq [\neg\neg\varphi]_p$.
4. $[\neg\varphi]_p = [\neg\neg\neg\varphi]_p$.
5. $[\forall x \chi(x)]_p = \bigwedge_{p \leq q} \bigwedge_{c \in C} \bigvee_{q \leq r} [\chi(c)]_r$.
6. $[\varphi \rightarrow \psi]_p = [\neg\varphi]_p \oplus [\psi]_p$;
7. $[\varphi \oplus \psi]_p = [\neg\neg\varphi]_p \oplus [\psi]_p$;
Generic sets

Definition
A non-empty subset $G$ of $P$ is called generic if the following conditions hold

1. If $p \in G$ and $q \leq p$ then $q \in G$,
2. For any $p, g \in G$ there exists $v \in G$ such that $p, g \leq v$;
3. For any $\varphi \in E$ there exists $p \in G$ such that $[\varphi]_p \oplus [\neg \varphi]_p = 1$.

Definition
Given a forcing property $\langle P, \leq, 0, f \rangle$, a model $\mathcal{A}$ is generated by a generic set $G$ if for all $\varphi \in E$ and $p \in G$ we have $[\varphi]_p \leq \|\varphi\|_{\mathcal{A}}$. A model $\mathcal{A}$ is generic for $p \in P$ if it is generated by a generic subset $G$ which contains $p$. $\mathcal{A}$ is generic if it is generic for $0$. 
Generic model theorem

**Theorem**

Let $< P, \leq, 0, f >$ be a forcing property and $p \in P$. Then there exists a generic model for $p$.

**Sketch of the proof.**

For any $p \in P$ build by stages a generic set $G$ such that $p \in G$, proving that the condition $[\varphi]_q \oplus [\neg \varphi]_q < 1$ must fail for some $q \geq p_n$

Build a structure starting from the constants in the language and define an evaluation by $e(\varphi) = \bigvee_{p \in G} [\varphi]_p$. Such an enumerable model is generated by $G$. 

□
Generic model theorem

Corollary

If $p$ belongs to some generic set $G$ which has a maximum $g$, then there exists $\mathcal{M}$, generic model for $p$, such that $\lbrack \varphi \rbrack_g = \| \varphi \|_{\mathcal{M}}$

Corollary

For any $\varphi \in E$ and $p \in P$ we have

$$\lbrack \lnot \lnot \varphi \rbrack_p = \bigwedge \{ \| \varphi \|_{\mathcal{M}} \mid \text{\mathcal{M} is a generic structure for } p \}. $$
Infinite forcing

Henceforth all structures will be assumed to be members of a fixed inductive class $\Sigma$.

Definition

For any structure $\mathcal{A}$ and for any sentence $\varphi$ of $\mathbb{P}L_\infty(\mathcal{A})$ we shall define by induction the real number $[\varphi]_\mathcal{A} \in [0, 1]$:

1. If $\varphi$ is an atomic sentence then $[\varphi]_\mathcal{A} = ||\varphi||_\mathcal{A}$;
2. If $\varphi = \neg \psi$ then $[\varphi]_\mathcal{A} = \bigwedge_{\mathcal{A} \subseteq \mathcal{B}} [\psi]_\mathcal{B}^*$;
3. If $\varphi = \psi \rightarrow \chi$ then $[\varphi]_\mathcal{A} = \bigwedge_{\mathcal{A} \subseteq \mathcal{B}} ([\psi]_\mathcal{B} \rightarrow [\chi]_\mathcal{A})$;
4. If $\varphi = \exists x \psi(x)$ then $[\varphi]_\mathcal{A} = \bigvee_{a \in \mathcal{A}} [\psi(a)]_\mathcal{A}$.

$[\varphi]_\mathcal{A}$ will be called the forcing value of $\varphi$ in $\mathcal{A}$. 
A natural question is whether $[\varphi]_{\mathcal{A}} = 1$ for any formal theorem $\varphi$ of $\mathcal{P}\mathcal{L}_\infty$. The following example shows that the answer is negative: Let us consider a language of $\mathcal{P}\mathcal{L}_\infty$ with a unique unary predicate symbol $R$. We define two standard structures $\mathcal{A}$ and $\mathcal{B}$ by putting

$\mathcal{A} = \{a, b\}$, \quad $R^\mathcal{A}(a) = 1/2$, \quad $R^\mathcal{A}(b) = 1/3$
$\mathcal{B} = \{a, b, c\}$, \quad $R^\mathcal{B}(a) = 1/2$, \quad $R^\mathcal{B}(b) = 1/3$, \quad $R^\mathcal{B}(c) = 1$. 
An example

Of course $\mathcal{A}$ is a substructure of $\mathcal{B}$. Let us take $\Sigma = \{\mathcal{A}, \mathcal{B}\}$ and consider the following sentence of $\mathcal{P}L_\infty$

$$\exists x R(x) \rightarrow \exists x R(x).$$

This sentence is a formal theorem of $\mathcal{P}L_\infty$ (identity principle), but:

$$[\exists x R(x)]_{\mathcal{A}} = [R(a)]_{\mathcal{A}} \lor [R(b)]_{\mathcal{A}} = \max(1/2, 1/3) = 1/2$$

$$[\exists x R(x)]_{\mathcal{B}} = [R(a)]_{\mathcal{B}} \lor [R(b)]_{\mathcal{B}} \lor [R(c)]_{\mathcal{B}} = \max(1/2, 1/3, 1) = 1.$$

and

$$[\exists x R(x) \rightarrow \exists x R(x)]_{\mathcal{A}} = [\exists x R(x)]_{\mathcal{B}} \rightarrow [\exists x R(x)]_{\mathcal{A}} = 1 \rightarrow 1/2 = 1/2.$$
Properties of infinite forcing

For any structure $\mathcal{A}$ and for any sentences $\varphi$, $\psi$ and $\forall x \chi(x)$ of $\mathcal{P}L_\infty(\mathcal{A})$ the following hold:

1. If $\mathcal{A} \subseteq \mathcal{B}$ then $[\varphi]_\mathcal{A} \leq [\varphi]_\mathcal{B}$.
2. $[\neg\neg\varphi]_\mathcal{A} = \bigwedge_{\mathcal{A} \subseteq \mathcal{B}} \bigvee_{\mathcal{B} \subseteq \mathcal{C}} [\varphi]_\mathcal{C}$;
3. $[\varphi]_\mathcal{A} \leq [\neg\neg\varphi]_\mathcal{A}$.
4. $[\varphi \rightarrow \psi]_\mathcal{A} = [\neg\varphi]_\mathcal{A} \oplus [\psi]_\mathcal{A}$;
5. $[\varphi \oplus \psi]_\mathcal{A} = [\neg\neg\varphi]_\mathcal{A} \oplus [\psi]_\mathcal{A}$;
6. $[\forall x \chi(x)]_\mathcal{A} = \bigwedge_{\mathcal{A} \subseteq \mathcal{B}} \bigwedge_{b \in \mathcal{B}} \bigvee_{\mathcal{B} \subseteq \mathcal{C}} [\chi(b)]_\mathcal{C}$.
7. $[\varphi]_\mathcal{A} \odot [\neg\varphi]_\mathcal{A} = 0$. 
Generic structures

The following result characterizes the members \( \mathcal{A} \) of \( \Sigma \) for which \([\ ]_{\mathcal{A}}\) and \(\|\|_{\mathcal{A}}\) coincide.

**Proposition**

*For any \( \mathcal{A} \in \Sigma \) the following assertions are equivalent:*

1. \( \| \varphi \|_{\mathcal{A}} = [\varphi]_{\mathcal{A}} \), for all sentences \( \varphi \) of \( PL_{\infty}(\mathcal{A}) \);
2. \( \| \varphi \|_{\mathcal{A}} = [\neg\neg\varphi]_{\mathcal{A}} \), for all sentences \( \varphi \) of \( PL_{\infty}(\mathcal{A}) \);
3. \( [\varphi]_{\mathcal{A}} \oplus [\neg \varphi]_{\mathcal{A}} = 1 \), for all sentences \( \varphi \) of \( PL_{\infty}(\mathcal{A}) \);
4. \( [\neg \varphi]_{\mathcal{A}} = [\varphi]^*_{\mathcal{A}} \), for all sentences \( \varphi \) of \( PL_{\infty}(\mathcal{A}) \).
Generic structures

Definition
A structure $\mathcal{A} \in \Sigma$ which satisfies the equivalent conditions of the proposition above will be called $\Sigma$-generic.

Theorem
Any structure $\mathcal{A} \in \Sigma$ is a substructure of a $\Sigma$-generic structure.

Theorem
Any $\Sigma$-generic structure $\mathcal{A}$ is $\Sigma$-existentially-complete.
Let us denote by $\mathfrak{G}_\Sigma$ the class of $\Sigma$-generic structures.

**Proposition**

$\mathfrak{G}_\Sigma$ is an inductive class.

**Theorem**

$\mathfrak{G}_\Sigma$ is the unique subclass of $\Sigma$ satisfying the following properties:

1. it is model-consistent with $\Sigma$;
2. it is model-complete;
3. it is maximal with respect to (1) and (2).