Continuous approximations of MV-algebras with product and product residuation

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Abstract. Recently, MV-algebras with product have been investigated from different points of view. In particular, in [EGM01], a variety resulting from the combination of MV-algebras and product algebras (see [H98]) has been introduced. The elements of this variety are called ŁΠ-algebras. Even though the language of ŁΠ-algebras is strong enough to describe the main properties of product and of Łukasiewicz connectives on $[0, 1]$, the discontinuity of product implication introduces some problems in the applications, because a small error in the data may cause a relevant error in the output. In this paper we try to overcome this difficulty, substituting the product implication by a continuous approximation of it. The resulting algebras, the $\text{Ł}\Pi_q$-algebras, are investigated in the present paper. In this paper we give a complete axiomatization of the quasivariety obtained in this way, and we show that such quasivariety is generated by the class of all $\text{Ł}\Pi_q$-algebras whose lattice reduct is the unit interval $[0, 1]$ with the usual order.

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1 Introduction

MV-algebras with product have been widely investigated by many authors, [DND01], [Mo00] , [EGM01], [M01] and [MP02] and with many motivations, arising from algebra, algebraic geometry and from the theory of many-valued control. While an axiomatization of the variety of MV-algebras with product generated by $[0, 1]$ with the Łukasiewicz operators and with product seems to be very problematic, the presence of product residuation simplifies the situation. Consider the structure $[0, 1]_{\text{Ł}\Pi} = \langle [0, 1], \oplus, \neg, \cdot, \to_\pi, 0, 1\rangle$, where $\cdot$ denotes ordinary product, and the remaining operations are defined as follows:

$$x \oplus y = \min\{x + y, 1\}, \quad \neg x = 1 - x, \quad x \to_\pi y = \begin{cases} \frac{y}{x} & \text{if } x > y \\ 1 & \text{otherwise} \end{cases}$$

Then $[0, 1]_{\text{Ł}\Pi}$ generates a variety which can be axiomatized by a finite number of equations. The members of such variety, called *ŁΠ-algebras* have been deeply investigated, [Mo00] , [EGM01], [M01] and [MP01].

A negative counterpart for the expressiveness of the language of ŁΠ-algebras is the loss of continuity of the truth functions of formulas, due to the fact that the truth function of $\to_\pi$ is not continuous in $(0, 0)$; this may cause problems when using ŁΠ in the treatment of approximate data, because a small error in the data may cause relevant errors in the output.
These observations constitute the main motivation for an investigation of a class of algebras in which the discontinuous product implication is replaced by a continuous approximation of it. The idea is the following: we fix a positive number q (the intuition is that q is greater than 0 but very close to 0), and we replace product implication $\to_\pi$ by the operation $\to_q$ defined by $x \to_q y = (x \vee q) \to_\pi y$. Note that $\to_q$ defined in this way is continuous.

The present paper is devoted to an investigation of the general properties of $Ł\Pi_q$-algebras. These algebras are introduced in Section 3. In this section we prove some general properties of these structures. For example, $Ł\Pi_q$-algebras constitute a quasivariety, but not a variety. Moreover, every $Ł\Pi_q$-algebra is isomorphic to a subdirect product of a family of linearly ordered $Ł\Pi_q$-algebras, and is a subalgebra of a $Ł\Pi_q$ algebra obtained from a ŁΠ-algebra letting $x \to_q y = (x \vee q) \to_\pi y$, where q denotes a suitably chosen constant. Finally, in Section 4 we prove that the class of $Ł\Pi_q$-algebras is generated as a quasivariety by the class of all $Ł\Pi_q$-algebras whose lattice reduct is $[0, 1]$ with the usual order.

2 Preliminaries

Definition 2.1 (see e.g. [BF00]). A *hoop* is an algebra $\langle H, \star, \to, 1\rangle$ such that $\langle H, \star, 1\rangle$ is a commutative monoid, and $\to$ is a binary operation such that the following identities hold:

$$x \to x = 1, \quad x \to (y \to z) = (x \star y) \to z \quad \text{and} \quad x \star (x \to y) = y \star (y \to x).$$

A *Wajsberg hoop* is a hoop satisfying the identity $(x \to y) \to y = (y \to x) \to x$.
A *bounded hoop* is a hoop equipped with a constant 0 such that $0 \to x = 1$.
A *Wajsberg algebra* is a bounded Wajsberg hoop.

The monoid operation of a Wajsberg algebra is usually denoted by $\odot$. In the sequel, given a Wajsberg algebra, we write $\neg x$ for $x \to 0$, $x \oplus y$ for $\neg x \to y$, $x \wedge y$ for $x \odot (x \to y)$, $x \vee y$ for $(x \to y) \to y$, and $x \leq y$ for $x \to y = 1$. Note that $\leq$ is a distributive lattice order, and $\vee$ and $\wedge$ are the corresponding operations of join and meet. We inductively define $(n)x$ and $x^{(n)}$ by:

$$(0)x = 0, \quad (n + 1)x = (n)x \oplus x, \qquad x^{(0)} = 1, \quad x^{(n+1)} = x^{(n)} \odot x.$$

Wajsberg algebras constitute a variety generated by $[0, 1]_W = \langle [0, 1], \odot, \to, 0, 1\rangle$.

If $\langle A, \odot, \to, 0, 1\rangle$ is a Wajsberg algebra, then the structure $\langle A, \oplus, \neg, 0, 1\rangle$ is called *a MV-algebra.* Every Wajsberg algebra is termwise equivalent to a MV-algebra ([H98]). Thus we will often identify a Wajsberg algebra and the corresponding MV-algebra.

In attempting to axiomatize the class of ŁΠ-algebras, in [Mo00] the concept of PMV-algebra has been introduced.

Definition 2.2 A PMV-*algebra* is an algebra $\mathcal{A} = \langle A, \oplus, \neg, \cdot, 0, 1\rangle$ such that:

$\langle A, \oplus, \neg, 0, 1\rangle$ is a MV-algebra.

$\langle A, \cdot, 1\rangle$ is a commutative monoid.

For all $x, y, z \in \mathcal{A}$ one has: $x \cdot (y \ominus z) = (x \cdot y) \ominus (x \cdot z)$, where $x \ominus y = \neg(\neg x \oplus y)$.

A *ŁΠ-algebra* is an algebra $\mathcal{A} = \langle A, \oplus, \neg, \cdot, \rightarrow_\pi, 0, 1\rangle$ such that $\langle A, \oplus, \neg, \cdot, 0, 1\rangle$ is a PMV-algebra, $\langle A, \cdot, \rightarrow_\pi, 0, 1\rangle$ is a bounded hoop, and letting $\neg_\pi x = x \rightarrow_\pi 0$ and $\Delta(x) = \neg_\pi \neg x$, the following equations hold:

$x \rightarrow_\pi y \leq x \rightarrow y.$

$x \wedge \neg_\pi x = 0$

$\Delta(x) \odot \Delta(x \rightarrow y) \leq \Delta(y)$

$\Delta(x) \leq x$

$\Delta(\Delta(x)) = \Delta(x)$

$\Delta(x \vee y) = \Delta(x) \vee \Delta(y)$

$\Delta(x) \vee \neg\Delta(x) = 1$

$\Delta(x \rightarrow y) \leq x \rightarrow_\pi y.$

A *ŁΠ$\frac{1}{2}$-algebra* is a ŁΠ-algebra with an additional constant $\frac{1}{2}$ satisfying $\frac{1}{2} = \neg\frac{1}{2}$.

Notation. In the sequel we will omit the symbol $\cdot$ when there is no danger of confusion. Moreover we inductively define x^n by $x^0 = 1,\ x^{n+1} = x^n x$.

In [Mo00] , Lemma 2.11 and Theorem 5.1, the following is shown:

Proposition 2.3

(i) Every PMV-algebra is isomorphic to a subdirect product of a family of linearly ordered PMV-algebras.

(ii) A PMV-algebra and its underlying Wajsberg algebra have the same congruences.

Definition 2.4 (Cf [BKW77]). A *lattice-ordered ring* is a structure

$$\mathcal{R} = \langle R, +, -, \times, \vee, \wedge, 0\rangle$$

such that:

(i) $\mathcal{R} = \langle R, +, -, \times, 0\rangle$ is a ring.

(ii) $\mathcal{R} = \langle R, \vee, \wedge \rangle$ is a lattice.

(iii) Let $\leq$ denote the partial order induced by $\vee$ and $\wedge$. Then $x \leq y$ implies $x + z \leq y + z$, and $x, y \geq 0$ implies $x \times y \geq 0$.

An *f-ring* is a lattice-ordered ring which is isomorphic to a subdirect product of linearly ordered lattice-ordered rings.

A *strong unit* of a lattice-ordered ring $\mathcal{R}$ is an element $u \in \mathcal{R}$ such that $u \times u \leq u$, and for all $a \in \mathcal{R}$ there is $n \in \mathbf{N}$ such that $a \leq nu$, where $nu = \underbrace{u + \ldots + u}_{n \text{ times}}$.

A *commutative unitary f-ring with strong unit* (for short: a *c-s-u-f-ring*) is a commutative f-ring with a unit for product which is also a strong unit.

In [DND01] the authors define a functor $\Gamma_{\mathbf{R}}$ from the category of lattice-ordered rings with strong unit into a category of algebras, called *product MV-algebras*. Here we describe the restriction of $\Gamma_{\mathbf{R}}$ to c-s-u-f-rings, which turns-out to be a functor from the category of c-s-u-f-rings into the category of PMV-algebras.

Definition 2.5 The functor $\Gamma_{\mathbf{R}}$ is defined as follows:

(i) Let $\mathcal{R} = \langle R, +, -, \times, \vee, \wedge, 0 \rangle$ be a c-s-u-f-ring, and let u be the unit of $\mathcal{R}$ (which by definition is also a strong unit). Then $\Gamma_{\mathbf{R}}(\mathcal{R})$ denotes the structure $\langle [0, u], \oplus, \neg, \cdot, 0, u \rangle$, where $[0, u] = \{x \in \mathcal{R} : 0 \leq x \leq u\}$, $x \oplus y = (x + y) \wedge 1$, $\neg x = u - x$, and $\cdot$ is the restriction of $\times$ to $[0, u]$.

(ii) Let $\mathcal{R}$, $\mathcal{R}'$ be lattice-ordered rings, and let h be a morphism (i.e., a homomorphism) from $\mathcal{R}$ into $\mathcal{R}'$. Then $\Gamma_{\mathbf{R}}(h)$ is defined as the restriction of h to $\Gamma_{\mathbf{R}}(\mathcal{R})$. (Note that $\Gamma_{\mathbf{R}}(h)$ is a homomorphism from $\Gamma_{\mathbf{R}}(\mathcal{R})$ into $\Gamma_{\mathbf{R}}(\mathcal{R}')$).

In [Mo02], as a special case of a result contained in [DND01], Theorem 4.2, the following is shown:

Proposition 2.6 $\Gamma_{\mathbf{R}}$ *is an equivalence between the category of c-s-u-f-rings and the category of PMV-algebras.*

3 ŁΠ_q algebras

Definition 3.1 A ŁΠ_q-algebra is a structure $\mathcal{A} = \langle A, \oplus, \neg, \cdot, \rightarrow_q, q, 0, 1 \rangle$ where $\langle A, \oplus, \neg, \cdot, 0, 1 \rangle$ is a PMV-algebra, q is a constant, and $\rightarrow_q$ is a binary operation such that the following conditions hold:

(A1) $q \leq \neg q$

(A2) $x \rightarrow_q y = (x \vee q) \rightarrow_q y$

(A3) $(x \vee q)(x \rightarrow_q y) = (x \vee q) \wedge y$

(A4) $q \rightarrow_q (xq) = x$

(A5) If $x^2 = 0$ then $x = 0$

Examples. Let $\mathcal{A} = \langle A, \oplus, \neg, \cdot, \to_\pi, 0, 1\rangle$ be a linearly ordered ŁΠ-algebra with more than two elements, and let $q \in \mathcal{A} \setminus \{0\}$ with $q \leq \neg q$. Define $x \to_q y = (x \vee q) \to_\Pi y,$. Then $\mathcal{A}_q = \langle A, \oplus, \neg, \cdot, \to_q, 0, 1\rangle$ is a $Ł\Pi_q$-algebra. We call $\mathcal{A}_q$ the *q-reduct* of $\mathcal{A}$.

The next example shows that not all $Ł\Pi_q$-algebras are q-reducts of ŁΠ-algebras. Let $[0,1]^*$ be the unit interval of a non-trivial ultraproduct of $\mathbf{R}$, and let ε be a positive infinitesimal. Let A be the set of all elements of the form $\frac{p+\varepsilon^2 P(\varepsilon)}{r+\varepsilon^2 R(\varepsilon)}$ where $p, r \in [0,1]$, $r > 0$, P and Q are polynomials with integer coefficients, and $0 \leq \frac{p+\varepsilon^2 P(\varepsilon)}{r+\varepsilon^2 R(\varepsilon)} \leq 1$. Let $q = 1/2$. Then it is easily seen that A contains 0 and 1, and is closed under $\oplus$, $\neg$, $\cdot$ and $\to_q$ defined by $x \to_q y = (x \vee \frac{1}{2}) \to_\pi y$. We verify e.g. closure under $\cdot$ and under $\to_\pi$.
Let $\alpha, \beta \in A$, where $\alpha = \frac{p+\varepsilon^2 P(\varepsilon)}{q+\varepsilon^2 Q(\varepsilon)}$, and $\beta = \frac{r+\varepsilon^2 R(\varepsilon)}{s+\varepsilon^2 S(\varepsilon)}$. Then $\alpha\beta = \frac{pr+\varepsilon^2 H(\varepsilon)}{sq+\varepsilon^2 K(\varepsilon)}$, where $H(x) = rP(x)+pR(x)+x^2P(x)R(x)$, $K(x) = qS(x)+sQ(x)+x^2Q(x)S(x)$. Hence A is closed under $\cdot$.
Now if $\alpha \vee \frac{1}{2} \leq \beta$, then $\alpha \to_q \beta = 1 \in A$. If $\beta < \alpha \leq \frac{1}{2}$, then $\alpha \to_q \beta = (2)\beta \in A$. Thus we may assume without loss of generality $\beta < \alpha$ and $\frac{1}{2} < \alpha$. In this case, $\alpha \to_q \beta = \frac{\beta}{\alpha} = \frac{rq+\varepsilon^2 T(\varepsilon)}{sp+\varepsilon^2 U(\varepsilon)}$, where $T(x) = qR(x) + rQ(x) + x^2Q(x)R(x)$, and $U(x) = sP(x) + pS(x) + x^2P(x)S(x)$.
Thus A is the domain of a $Ł\Pi_q$-algebra $\mathcal{A}$.

However, $\mathcal{A}$ is not a q-reduct of a ŁΠ-algebra, because A is not closed under $\to_\pi$. To see this, note that both ε^2 and ε^3 have the form $\frac{p+\varepsilon^2 P(\varepsilon)}{q+\varepsilon^2 Q(\varepsilon)}$: take $p = 0$, $q = 1$, and $Q(x) = 0$; then ε^2 is obtained letting $P(x) = 1$, and ε^2 is obtained letting $P(x) = x$. Hence ε^2 and ε^3 are elements of A. However, $\varepsilon^2 \to_\pi \varepsilon^3 = \varepsilon \notin A$. Indeed, $\varepsilon = \frac{p+\varepsilon^2 P(\varepsilon)}{q+\varepsilon^2 Q(\varepsilon)}$ would imply $p + \varepsilon^2 P(\varepsilon) = q\varepsilon + \varepsilon^3 Q(\varepsilon)$, and finally $p = q = 0$. But $q = 0$ is excluded by the definition of A.

From Definition 3.1 it follows that the $Ł\Pi_q$-algebras form a quasivariety. However:

Theorem 3.2 *The class of $Ł\Pi_q$-algebras does not constitute a variety.*

Proof. Let $[0,1]^*$, q, ε and $\to_q$ be as in the example above, and let us consider the structure $\mathcal{A} = \langle [0,1]^*, \oplus, \neg, \cdot, \to_q, 0, 1, q\rangle$ (with $\oplus, \neg, \cdot$ defined in the obvious way). It is readily seen that $\mathcal{A}$ is a $Ł\Pi_q$-algebra.
Define for $x, y \in \mathcal{A}$, $x\theta y$ iff there is $k \in \mathbf{N}$ such that $\mid x - y \mid \leq (k)\varepsilon^2$, where $\mid x - y \mid = (x \ominus y) \vee (y \ominus x)$. It is easy to see that θ is a congruence of PMV-algebras. We show that θ is compatible with $\to_q$. It is sufficient to prove that if $\mid x - y \mid \leq (k)\varepsilon^2$ then:

(a) $\mid (x \to_q z) - (y \to_q z) \mid \leq (4k)\varepsilon^2$ and (b) $\mid (z \to_q x) - (z \to_q y) \mid \leq (4k)\varepsilon^2$.

The proof of (a) splits into the following cases:

If $x \vee q \leq z$ and $y \vee q \leq z$, the claim is trivial.

If $x \vee q \leq z$ and $y \vee q > z$, then $|\ (x \rightarrow_q z) - (y \rightarrow_q z)\ |=|\ 1 - \frac{z}{y \vee q}\ |= \frac{(y \vee q) - z}{y \vee q} \leq \frac{(y \vee q) - (x \vee q)}{y \vee q} \leq \frac{y - x}{y \vee q} \leq \frac{y - x}{q} = 2\ |\ x - y\ |\leq (2k)\varepsilon^2 \leq (4k)\varepsilon^2$.

If $x \vee q > z$ and $y \vee q \leq z$, we reason as in the previous case.

If $x \vee q > z$ and $y \vee q > z$, then $|\ (x \rightarrow_q z) - (y \rightarrow_q z)\ |=|\ \frac{z}{x \vee q} - \frac{z}{y \vee q}\ |= \frac{z|(x \vee q) - (y \vee q)|}{(x \vee q)(y \vee q)} \leq \frac{z|x - y|}{(x \vee q)(y \vee q)} \leq \frac{|x - y|}{q^2} \leq (4k)\varepsilon^2$.

The proof of (b) splits into the following cases:

If $z \vee q \leq x$ and $z \vee q \leq y$, the claim is trivial.

If $z \vee q \leq x$ and $z \vee q > y$, then $|\ (z \rightarrow_q x) - (z \rightarrow_q y)\ |=|\ 1 - \frac{y}{z \vee q}\ |\leq \frac{(z \vee q) - y}{z \vee q} \leq \frac{|x - y|}{q} \leq (2k)\varepsilon^2 \leq (4k)\varepsilon^2$.

If $z \vee q > x$ and $z \vee q \leq y$ we reason as in the previous case.

If $z \vee q > x$ and $z \vee q > y$, then $|\ (z \rightarrow_q x) - (z \rightarrow_q y)\ |=|\ \frac{x}{z \vee q} - \frac{y}{z \vee q}\ |\leq \frac{|x - y|}{z \vee q} \leq \frac{|x - y|}{q} \leq (4k)\varepsilon^2$

Let ε_θ^2 denote the equivalence class of ε modulo θ. Then $\mathcal{A}/\theta \models \varepsilon_\theta^2 = 0$ but $\mathcal{A}/\theta \not\models \varepsilon_\theta = 0$. Therefore $\mathcal{A}/\theta$ does not satisfy the axiom (A5) in Definition 3.1 It follows that the class of ŁΠ$_q$-algebras is not closed under quotients, hence it is not a variety. ∎

Lemma 3.3 *Let $\mathcal{A}$ be any ŁΠ$_q$-algebra. Then for any $x \in \mathcal{A}$ and for any $n, k \in \mathbf{N} \setminus \{0\}$, if $q^k x^n = 0$ then $x = 0$.*

Proof. Induction on k. For $k = 0$ the claim follows from (A5). Suppose that the claim holds for $k = m$, and let us prove it for $k = m + 1$. First note that letting $x = 0$ in axiom (A4) we get $q \rightarrow_q 0 = 0$. Hence if $qx = 0$ then, by axiom (A4) one has $x = q \rightarrow_q qx = q \rightarrow_q 0 = 0$. So we have:

$$qx = 0 \Rightarrow x = 0. \tag{1}$$

Now if $q^{m+1}x^n = 0$ then $0 = q^{m+1}x^n = q(q^m x^n)$. Thus replacing x by $q^m x^n$ in (1), we obtain $q^m x^n = 0$ and by the induction hypothesis, $x = 0$. ∎

Lemma 3.4

(i) In any non-trivial ŁΠ$_q$-algebra one has $q > 0$.

(ii) Any linearly ordered ŁΠ$_q$-algebra has no zero divisors, i.e. if $xy = 0$, then either $x = 0$ or $y = 0$.

Proof. Claim (i) follows from Lemma 3.3, and claim (ii) follows from axiom (A5) of ŁΠ$_q$-algebras. ∎

Theorem 3.5 *Every subdirectly irreducible* ŁΠ_q*-algebra is linearly ordered. Hence every* ŁΠ_q*-algebra* $\mathcal{A}$ *can be decomposed as a subdirect product of a family of linearly ordered* ŁΠ_q*-algebras.*

Proof. For any $a \in \mathcal{A} \setminus \{0\}$, consider the family $\mathcal{I}_a$ of all MV-ideals J such that for every $n, k > 0$, $q^k a^n \notin J$. $\mathcal{I}_a$ is non-empty, since by Lemma 3.3 $\{0\} \in \mathcal{I}_a$. Moreover $\mathcal{I}_a$ is closed under unions of chains, therefore $\langle \mathcal{I}_a, \subseteq \rangle$ is an inductive partially ordered set, and, by Zorn's lemma, it has a maximal element, call it J_a. Let $\mathcal{A}^-$ be the PMV-reduct of $\mathcal{A}$. Since a PMV-algebra and its MV-reduct have the same congruences, the congruence θ_a associated with J_a is a congruence of PMV-algebras, too. Therefore $\mathcal{A}^-/\theta_a$ is a PMV-algebra. To continue the proof we show the following lemmas.

Lemma 3.6 *For every* $b, c \in \mathcal{A}$, *either* $b \ominus c \in J_a$ *or* $c \ominus b \in J_a$.

Proof. Let by contradiction $b, c \in \mathcal{A}$ be such that $b \ominus c \notin J_a$ and $c \ominus b \notin J_a$. Let for any subset X of $\mathcal{A}^-$, $\overline{X}$ denote the ideal generated by X. By the maximality of J_a there exist $k, n, h, m > 0$ with $q^k a^n \in \overline{J_a \cup \{b \ominus c\}}$ and $q^h a^m \in \overline{J_a \cup \{c \ominus b\}}$. Thus there are $f, g \in J_a$ and $r, s \in \mathbf{N}$ such that

$$q^k a^n \leq f \oplus (r)(b \ominus c) \text{ and } q^h a^m \leq g \oplus (s)(c \ominus b).$$

Let $u = f \vee g$ and $t = \max\{k, n, h, m, r, s\}$. Then

$$q^t a^t \leq u \oplus (t)(b \ominus c) \text{ and } q^t a^t \leq u \oplus (t)(c \ominus b),$$

therefore $q^t a^t \leq u \oplus ((t)(b \ominus c) \wedge (t)(c \ominus b)) = u$ and $q^t a^t \in J_a$, which is a contradiction. ■

Lemma 3.7 *If* $bc \in J_a$ *then either* $b \in J_a$ *or* $c \in J_a$.

Proof. Let by contradiction, $b, c \in \mathcal{A}$ be such that $b \notin J_a$, $c \notin J_a$ and $bc \in J_a$. By the maximality of J_a there exist $h, k, m, n > 0$ such that

$$q^k a^n \in \overline{J_a \cup \{b\}} \text{ and } q^h a^m \in \overline{J_a \cup \{c\}}.$$

Thus there are $f, g \in J_a$ and $r, s \in \mathbf{N}$ such that $q^k a^n \leq f \oplus (r)b$ and $q^h a^m \leq g \oplus (s)c$.
Let $u = f \vee g$ and $t = \max\{h, k, m, n, r, s\}$. Then $q^t a^t \leq u \oplus (t)b$ and $q^t a^t \leq u \oplus (t)c$, therefore $q^{2t} a^{2t} \leq (u \oplus (t)b)(u \oplus (t)c) \leq u^2 \oplus ((t)uc) \oplus ((t)ub) \oplus ((t^2)bc)$. Now $u^2 \oplus ((t)uc) \oplus ((t)ub) \in J_a$, and $(t^2)bc \in J_a$, therefore $q^{2t} a^{2t} \in J_a$, and a contradiction has been reached. ■

We continue the proof of theorem 3.5. Since $a \notin J_a$, $\bigcap_{a \in \mathcal{A} \setminus \{0\}} J_a = \{0\}$, hence $\bigcap_{a \in \mathcal{A} \setminus \{0\}} \theta_a$ is the minimal congruence. It follows that the map

$$\Phi : \mathcal{A}^- \xrightarrow{\Phi} \prod_{a \in \mathcal{A} \setminus \{0\}} \mathcal{A}^-/\theta_a \text{ defined by } \Phi(b) = \langle b/\theta_a : a \in \mathcal{A} \setminus \{0\} \rangle$$

is a monomorphism from $\mathcal{A}^-$ to $\prod_{a \in \mathcal{A} \setminus \{0\}} \mathcal{A}/\theta_a$.
In other words $\mathcal{A}^-$ can be decomposed as a subdirect product of linearly ordered PMV-algebras. Moreover by Lemma 3.7, each component $\mathcal{A}/\theta_a$ has no zero divisors. Finally, $q/\theta_a \neq 0$, because $q \notin J_a$. Thus we have shown:

Lemma 3.8 *The PMV-reduct of any* $\text{Ł}\Pi_q$*-algebra can be decomposed as a subdirect product of a family of linearly ordered PMV-algebras* $\langle \mathcal{A}_i : i \in I \rangle$ *without zero divisors. Moreover, for every* $i \in I$, $q_i > 0$. ■

Lemma 3.9 *For any* $a, b \in \mathcal{A}$ *and for every* $i \in I$*, the following conditions hold:*

If $a_i \vee q_i \leq b_i$*, then* $(a \rightarrow_q b)_i = 1$.

Otherwise, $(a \rightarrow_q b)_i$ *is the unique* $z_i \in \mathcal{A}_i$ *such that* $(a_i \vee q_i) z_i = b_i$.

In particular $(a \rightarrow_q b)_i$ *depends on* a_i *and* b_i *but not on* a *and* b.

Proof. First of all recall that $(a \vee q)(a \rightarrow_q b) = (a \vee q)((a \vee q) \rightarrow_q b) = b \wedge (a \vee q)$. Hence for every $i \in I$ we have $(a_i \vee q_i)(a \rightarrow_q b)_i = b_i \wedge (a_i \vee q_i)$. Let $z_i = (a \rightarrow_q b)_i$. Then:

If $(a \vee q)_i \leq b_i$ then $(a \vee q)_i z_i = ((a \vee q) \wedge b)_i = (a \vee q)_i$. So $(a \vee q)_i \ominus (a \vee q)_i z_i = (a \vee q)_i (1 \ominus z_i) = 0$. Since $(a \vee q)_i > 0$ and $\mathcal{A}_i$ has no zero divisors (Lemma 3.8) we get $z_i = 1$.

If $(a \vee q)_i > b_i$, then $(a \vee q)_i z_i = b_i$. Moreover, z_i is the unique element with this property. Indeed if $(a \vee q)_i u = b_i$ then $(a \vee q)_i \mid u - z_i \mid = 0$, and since $\mathcal{A}_i$ has no zero divisors and $q_i > 0$ we conclude that $u = z_i$.

■

We conclude the proof of Theorem 3.5. Define for $a, b \in \mathcal{A}$ and for $i \in I$, $a_i \rightarrow_i b_i = (a \rightarrow_q b)_i$. By Lemma 3.9 this definition is admissible. By Lemma 3.8 and 3.9, $\mathcal{A}_i$ equipped by the additional operator $\rightarrow_i$ satisfies axioms (A1) ...(A3) and (A5) of $\text{Ł}\Pi_q$-algebras. Let us check axiom (A4). If $x = 1$ then $q_i \rightarrow_i q_i x = 1 = x$. Otherwise, $q_i x < q_i$ and by Lemma 3.9, $q_i \rightarrow_i q_i x$ is the unique z such that $q_i z = q_i x$. But $q_i z = q_i x$ implies $z = x$, therefore $q_i \rightarrow_i q_i x = x$. This concludes the proof. ■

Corollary 3.10 *Every* $\text{Ł}\Pi_q$*-algebra is a subalgebra of a q-reduct of a* $\text{Ł}\Pi$*-algebra.*

Proof. Let $\mathcal{A}$ be any $\text{Ł}\Pi_q$-algebra, and let $\mathcal{A}_i : i \in I$ be the linearly ordered factors in the subdirect representation of $\mathcal{A}$ according to Theorem 3.5, let $\mathcal{A}_i^-$ denote the PMV-reduct of $\mathcal{A}_i$, and let $\Gamma_{\mathbf{R}}$ be the functor defined in Section 2. Then by Proposition 2.6 for every $i \in I$ there is a c-s-u-f-ring $\mathcal{R}_i$ such that $\mathcal{A}_i^- = \Gamma_{\mathbf{R}}(\mathcal{R}_i)$. It is readily seen that $\mathcal{R}_i$ is linearly ordered (because $\Gamma_{\mathbf{R}}(\mathcal{R}_i)$ is linearly ordered). Moreover $\mathcal{R}_i$ has no zero divisors: if $x \times y = 0$, then letting $\mid x \mid = x \vee -x$, and $z = \min\{1, \mid x \mid, \mid y \mid\}$ we have $z \in \Gamma_{\mathbf{R}}(\mathcal{R}_i) = \mathcal{A}_i$, and $z^2 = 0$.

By axiom (A5) this implies $z = 0$. This is only possible if either $x = 0$ or $y = 0$. It follows that the ring reduct of $\mathcal{R}_i$ is an integral domain. Now let $\mathcal{F}_i$ be the fraction field of $\mathcal{R}_i$. Then $\mathcal{A}_i^-$ is a subalgebra of $\Gamma_{\mathbf{R}}(\mathcal{F}_i)$. For $x, y \in \Gamma_{\mathbf{R}}(\mathcal{F}_i)$, define

$$(x \to_\pi y)_i = \begin{cases} 1 & \text{if } x \leq y \\ yx^{-1} & \text{otherwise} \end{cases}$$

Then $\to_\pi$ makes $\Gamma_{\mathbf{R}}(\mathcal{F}_i)$ a ŁΠ-algebra (see [Mo00]), call it $\mathcal{LP}_i$. Moreover by Lemma 3.9 for all $x, y \in \mathcal{A}_i$ we have: $x \to_q y = (x \vee q) \to_\pi y$. Therefore $\mathcal{A}_i$ is a subalgebra of a q-reduct of $\mathcal{LP}_i$, and $\mathcal{A}$ is a subalgebra of a q-reduct of $\prod_{i\in} \mathcal{LP}_i$. ■

Definition 3.11 Let $\mathcal{A}$ be any $\text{Ł}\Pi_q$-algebra. We say that $\varepsilon \in \mathcal{A} \setminus \{0\}$ is an *infinitesimal* if for any natural number n one has: $(n)\varepsilon \leq \neg\varepsilon$.

The next corollary shows that any linearly ordered $\text{Ł}\Pi_q$-algebra which is not a q-reduct of a ŁΠ-algebra must have infinitesimals:

Corollary 3.12 *Let $\mathcal{A}$ be a linearly ordered $\text{Ł}\Pi_q$-algebra without infinitesimals. Then $\mathcal{A}$ is a q-reduct of an ŁΠ-algebra.*

Proof. We just need to check that product in $\mathcal{A}$ has a residual $\to_\pi$. This amounts to prove that for any x, y there is a z such that $zx = x \wedge y$. If $x \leq y$ then we can take $z = 1$. If $x = 1$, then we can take $z = y$. Otherwise, since there are no infinitesimals, there is $n \in \mathbf{N}$ such that $(n)x \geq \neg x$. Take n minimal with this property. Now recall that $\mathcal{A}$ embeds into a q-reduct of a linearly ordered ŁΠ-algebra $\mathcal{B}$ (Corollary 3.10), and that every linearly ordered ŁΠ-algebra embeds into an ultrapower of the ŁΠ-algebra $[0, 1]_{\text{Ł}\Pi}$ on $[0, 1]$ ([Mo02]). Hence the universal formula

$$\forall x \forall y(((n)x \geq \neg x)\&((n-1)x < \neg x)\&(y < x)) \Rightarrow (x \to_\pi y = (n)x \to_\pi (n)y))$$

(where & and ⇒ denote classical conjunction and classical implication respectively) being true in $[0, 1]_{\text{Ł}\Pi}$, is true in $\mathcal{B}$. Now $q \leq (n)x$ (because $q \leq \neg q$). Since $\mathcal{A}$ embeds into a q-reduct of $\mathcal{B}$, $(n)x \to_q (n)y = ((n)x \vee q) \to_\pi (n)y = (n)x \to_\pi (n)y = x \to_\pi y$. ■

4 Generation by standard $\text{Ł}\Pi_q$-algebras

This section is entirely devoted to the proof of the fact that the variety of $\text{Ł}\Pi_q$-algebras is generated as a quasivariety by its *standard* members, i.e., by those $\text{Ł}\Pi_q$-algebras whose lattice reduct is $\langle [0, 1], \max, \min \rangle$.

Definition 4.1 In the sequel, for every $0 < q \leq \frac{1}{2}$, $[0, 1]_q$ will denote the $\text{Ł}\Pi_q$-algebra $\langle [0, 1], \oplus, \neg, \cdot, \to_q, 0, 1, q \rangle$, where $\oplus$, $\neg$ and $\cdot$ are defined as usual, and $x \to_q y = (x \vee q) \to_\pi y = \begin{cases} \frac{y}{x \vee q} & \text{if } x \vee q > y \\ 1 & \text{otherwise} \end{cases}$

Theorem 4.2 *The class of $Ł\Pi_q$-algebras is generated as a quasivariety by the class* $\mathbf{S} = \{[0,1]_q : 0 < q \leq \frac{1}{2}\}$.

Proof. Let Φ be a quasi identity which is not valid in all $Ł\Pi_q$-algebras. Then Φ fails to hold in some subdirectly irreducible, hence (Theorem 3.5) linearly ordered, $Ł\Pi_q$-algebra $\mathcal{A}$. Now (Corollary 3.10) $\mathcal{A}$ embeds into a q-reduct $\mathcal{B}$ of a linearly ordered ŁΠ-algebra $\mathcal{D}$, and Φ fails in $\mathcal{B}$, too. Moreover, ([Mo01]) every linearly ordered ŁΠ-algebra embeds into an ultrapower $\mathcal{E}$ of the ŁΠ-algebra $[0,1]_{Ł\Pi}$ on $[0,1]$. At this point, we can observe that the existence of an evaluation e in $\mathcal{B}$ which invalidates Φ can be written as an existential formula (in the language of ŁΠ-algebras) of the form

$$\exists q \exists x_1 \ldots \exists x_n (0 < q \& q \leq \neg q \& \Psi(x_1, \ldots x_n, q))$$

where Ψ quantifier-free, and $x_1, \ldots, x_n$ are the variables occurring in Φ. Such a formula is preserved under taking superstructures, hence it is true in $\mathcal{E}$, and finally it is true in $[0,1]_{Ł\Pi}$. Let $q \in (0, \frac{1}{2}]$ and $a_1, \ldots, a_n \in [0,1]$ be such that $\Psi(a_1, \ldots, a_n, q)$ is true in $[0,1]_{Ł\Pi}$, and let e be the evaluation defined by $e(x_i) = a_i$ for $i = 1, \ldots, n$. Then Φ is invalidated by e in $[0,1]_q$. ■

Corollary 4.3 *Let $\mathcal{A}$ be a linearly ordered $Ł\Pi_q$-algebra with more than two elements. Then the PMV-reduct $\mathcal{A}^-$ of $\mathcal{A}$ has a subalgebra isomorphic to* $\langle \mathbf{Q} \cap [0,1], \oplus, \neg, \cdot, 0, 1 \rangle$.

Proof. By Corollary 3.10, $\mathcal{B}$ is a subalgebra of a q-reduct of a linearly ordered ŁΠ-algebra $\mathcal{D}$. Hence it is sufficient to prove that for all $n \in \mathbf{N} \setminus \{0\}$ there is an element a of $\mathcal{B}$, denoted by $\frac{1}{n}$, such that $(n-1)a = \neg a$. Indeed if we prove this, then as in [Mo00] we can see that the map $\Phi : \frac{m}{n} \stackrel{\Phi}{\longrightarrow} (m)\frac{1}{n}$ is an embedding of $\langle \mathbf{Q} \cap [0,1], \oplus, \neg, \cdot, 0, 1 \rangle$ into the PMV-reduct of $\mathcal{B}$. Let $h = (q \oplus q) \rightarrow_q q$. Then $h = \neg h$, because this property can be expressed by an equation which is true in any q-reduct of $[0,1]_{Ł\Pi}$, hence by Theorem 4.2 it is true in any $Ł\Pi_q$-algebra. Let k be the minimum natural number such that $2^k \geq n$, and let $a = (n)h^k \rightarrow_q h^k$. Then for any choice of $0 < q \leq h$ we have that $q \leq h \leq (n)h^k$. Hence $a = (n)h^k \rightarrow_q h^k = (n)h^k \rightarrow_\pi h^k$. Now in $[0,1]_{Ł\Pi}$ if $h = \neg h$ and $a = (n)h^k \rightarrow_\pi h^k$, then $(n-1)a = \neg a$. Since this fact can be expressed by a universal Horn formula, it holds in any ŁΠ-algebra. Hence $(n-1)a = \neg a$, and we can take $\frac{1}{n} = a$. ■

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