Continuous approximations of MV-algebras with product and product residuation: a category-theoretic equivalence

Luca Spada

Dipartimento di Matematica
Via del Capitano 15
53100 Siena (Italy)
spada@unisi.it

Abstract. In [9] a new class of MV-algebras with product, called ŁΠ_q-algebras, has been introduced. In these algebras, the discontinuous product residuation $\to_\pi$ defined in [4] is replaced by a continuous approximation of it. These algebras seem to be a good compromise between the need of expressiveness and the need of continuity of connectives.
Following a good tradition in many-valued logic (see e.g. [10], [7], [3] and [8]), in this paper we introduce a class of commutative f-rings with strong unit and with a sort of weak divisibility property, called f-quasifields, and we show that the categories of ŁΠ_q-algebras and of f-quasifields are equivalent.

1 Preliminaries

Definition 1. *(see e.g. [1]). A* ***hoop*** *is an algebra $\langle H, \star, \to, 1\rangle$ such that $\langle H, \star, 1\rangle$ is a commutative monoid, and $\to$ is a binary operation such that the following identities hold:*

$$x \to x = 1, \qquad x \to (y \to z) = (x \star y) \to z, \qquad x \star (x \to y) = y \star (y \to x).$$

A ***Wajsberg hoop*** *is a hoop satisfying $(x \to y) \to y = (y \to x) \to x$.*
A ***bounded hoop*** *is a hoop equipped with a constant 0 such that $0 \to x = 1$.*
A ***Wajsberg algebra*** *is a bounded Wajsberg hoop.*

The monoid operation of a Wajsberg algebra is usually denoted by $\odot$.

Wajsberg algebras constitute a variety generated by $\langle [0,1], \odot, \to, 0, 1\rangle$.
If $\langle A, \odot, \to, 0, 1\rangle$ is a Wajsberg algebra, then the structure $\langle A, \oplus, \neg, 0, 1\rangle$ is called MV**-algebra**. Every Wajsberg algebra is termwise equivalent to a MV-algebra ([6]). Thus we will often identify a Wajsberg algebra and the corresponding MV-algebra.

Notation 1 *We will use the following shorthand throughout the paper:*

- $\neg x$ *for* $x \to 0$
- $x \oplus y$ *for* $\neg x \to y$
- $x \ominus y$ *for* $\neg(\neg x \oplus y)$
- $x \wedge y$ *for* $x \odot (x \to y)$
- $x \vee y$ *for* $(x \to y) \to y$
- $x \leq y$ *for* $x \to y = 1$ [1]
- $\neg_\pi x$ *for* $x \to_\pi 0$
- $\Delta(x)$ *for* $\neg_\pi \neg x$,
- $u(a)$ *for* $(a \vee 0) \wedge 1$

- *We will omit the symbols* $\times$ *and* $\cdot$ *when there is no danger of confusion*
- $(n)x$ *is inductively defined by* $(0)x = 0$ *and* $(n+1)x = (n)x + x$
- x^n *is inductively defined by* $x^0 = 1$ *and* $x^{n+1} = x^n x$

In attempting to axiomatize the class of ŁΠ-algebras, in [7] the concept of PMV-algebra has been introduced.

Definition 2. *A* ***PMV-algebra*** *is an algebra* $\mathcal{A} = \langle A, \oplus, \neg, \cdot, 0, 1\rangle$ *such that:*

$\langle A, \oplus, \neg, 0, 1\rangle$ *is a MV-algebra.*
$\langle A, \cdot, 1\rangle$ *is a commutative monoid.*
For all $x, y, z \in \mathcal{A}$ *one has:* $x \cdot (y \ominus z) = (x \cdot y) \ominus (x \cdot z)$.

Definition 3. *A* ***ŁΠ-algebra*** *is an algebra* $\mathcal{A} = \langle A, \oplus, \neg, \cdot, \to_\pi, 0, 1\rangle$ *such that* $\langle A, \oplus, \neg, \cdot, 0, 1\rangle$ *is a PMV-algebra,* $\langle A, \cdot, \to_\pi, 0, 1\rangle$ *is a bounded hoop, and the following equations hold:*

- $x \to_\pi y \leq x \to y$.
- $x \wedge \neg_\pi x = 0$
- $\Delta(x) \odot \Delta(x \to y) \leq \Delta(y)$
- $\Delta(x) \leq x$
- $\Delta(\Delta(x)) = \Delta(x)$
- $\Delta(x \vee y) = \Delta(x) \vee \Delta(y)$
- $\Delta(x) \vee \neg\Delta(x) = 1$
- $\Delta(x \to y) \leq x \to_\pi y$.

A ***Ł$\Pi\frac{1}{2}$-algebra*** *is a ŁΠ-algebra with an additional constant* $\frac{1}{2}$ *satisfying* $\frac{1}{2} = \neg\frac{1}{2}$.

In [7], Lemma 2.11 and Theorem 5.1, the following is shown:

Proposition 1.
(i) Every PMV-algebra is isomorphic to a subdirect product of a family of linearly ordered PMV-algebras.
(ii) A PMV-algebra and its underlying MV-algebra have the same congruences.

Definition 4. *(Cf [2]). A* ***lattice-ordered ring*** *is a structure* $\mathcal{R} = \langle R, +, -, \times, \vee, \wedge, 0\rangle$ *such that:*

(i) $\mathcal{R} = \langle R, +, -, \times, 0\rangle$ *is a ring.*
(ii) $\mathcal{R} = \langle R, \vee, \wedge\rangle$ *is a lattice.*
(iii) Let $\leq$ *denote the partial order induced by* $\vee$ *and* $\wedge$*. Then* $x \leq y$ *implies* $x + z \leq y + z$*, and* $x, y \geq 0$ *implies* $x \times y \geq 0$.

[1] Note that $\leq$ is a distributive lattice order, and $\vee$ and $\wedge$ are the corresponding operations of join and meet

*An f-**ring** is a lattice-ordered ring which is isomorphic to a subdirect product of linearly ordered lattice-ordered rings.*

*A **strong unit** of a lattice-ordered ring $\mathcal{R}$ is an element $u \in \mathcal{R}$ such that $u^2 \leq u$, and for all $a \in \mathcal{R}$ there is $n \in \mathbf{N}$ such that $a \leq (n)u$.*

*A **commutative unitary f-ring with strong unit** (for short: a **c-s-u-f-ring**) is a commutative f-ring with a unit for product which is also a strong unit.*

In [3] the authors define a functor $\Gamma_{\mathbf{R}}$ from the category of lattice-ordered rings with strong unit into a category of algebras, called **product MV-algebras**. Here we describe the restriction of $\Gamma_{\mathbf{R}}$ to c-s-u-f-rings, which turns out to be a functor from the category of c-s-u-f-rings into the category of PMV-algebras.

Definition 5. *The functor $\Gamma_{\mathbf{R}}$ is defined as follows:*

(i) Let $\mathcal{R} = \langle R, +, -, \times, \vee, \wedge, 0\rangle$ be a c-s-u-f-ring, and let u be the unit of $\mathcal{R}$ (which by definition is also a strong unit). Then $\Gamma_{\mathbf{R}}(\mathcal{R})$ denotes the structure $\langle [0, u], \oplus, \neg, \cdot, 0, u\rangle$, where $[0, u] = \{x \in \mathcal{R} : 0 \leq x \leq u\}$, $x \oplus y = (x + y) \wedge 0$, $\neg x = u - x$, and $\cdot$ is the restriction of $\times$ to $[0, u]$.

(ii) Let $\mathcal{R}$, $\mathcal{R}'$ be lattice-ordered rings, and let u, u' be strong units of $\mathcal{R}$ and of $\mathcal{R}'$ respectively. Let h be a morphism from $(\mathcal{R}, u)$ into $(\mathcal{R}', u')$, i.e., a homomorphism from $\mathcal{R}$ into $\mathcal{R}'$ such that $h(u) = u'$. Then $\Gamma_{\mathbf{R}}(h)$ is defined as the restriction of h to $\Gamma_{\mathbf{R}}(\mathcal{R}, u)$. (Note that $\Gamma_{\mathbf{R}}(h)$ is a homomorphism from $\Gamma_{\mathbf{R}}(\mathcal{R}, u)$ into $\Gamma_{\mathbf{R}}(\mathcal{R}', u')$).

In [8], as a special case of a result contained in [3], Theorem 4.2, the following is shown:

Proposition 2. *$\Gamma_{\mathbf{R}}$ is an equivalence between the category of c-s-u-f-rings and the category of PMV-algebras.*

Definition 6. *[9]. A $Ł\Pi_q$-algebra is a structure $\mathcal{A} = \langle A, \oplus, \neg, \cdot, \rightarrow_q, q, 0, 1\rangle$ where $\langle A, \oplus, \neg, \cdot, 0, 1\rangle$ is a PMV-algebra, q is a constant, and $\rightarrow_q$ is a binary operation such that the following conditions hold:*

(A1) $q \leq \neg q$
(A2) $x \rightarrow_q y = (x \vee q) \rightarrow_q y$
(A3) $(x \vee q)(x \rightarrow_q y) = (x \vee q) \wedge y$
(A4) $q \rightarrow_q (xq) = x$
(A5) *If* $x^2 = 0$ *then* $x = 0$

In [9] it is shown that $Ł\Pi_q$-algebras constitute a quasivariety but not a variety, that every $Ł\Pi_q$-algebra is isomorphic to a subdirect product of a family of linearly ordered $Ł\Pi_q$-algebras, and that the quasivariety of $Ł\Pi_q$-algebras is generated by the class of $Ł\Pi_q$-algebras whose lattice reduct is $[0, 1]$ with the usual order.

2 f-Quasifields

Definition 7. *A* f-quasifield *is a structure* $\langle K, +, -, \times, /_q, \vee, \wedge, 0, 1, q\rangle$ *where* $\langle K, +, -, \times, \vee, \wedge, 0, 1, q\rangle$ *is a c-s-u-f-ring with strong unit 1,* q *is a constant and* $/_q$ *is a binary operation such that the following conditions are satisfied:*

(K1) $0 \leq q \leq 1 - q$
(K2) $x/_q y = u(x)/_q u(y) = u(x)/_q(u(y) \vee q)$.
(K3) $(u(x) \vee q) \times (u(y)/_q u(x)) = (u(x) \vee q) \wedge u(y)$
(K4) $(u(x) \times q)/_q q = u(x)$
(K5) If $x \times x = 0$ *then* $x = 0$.

If $\mathcal{K}$ is any f-quasifield, then for all x, y such that $q \leq x \leq 1$ and $0 \leq y \leq x$, there is z (namely $z = y/_q x$) such that $zx = y$. Thus any f-quasifield enjoys a weak form of divisibility property.

The next theorem characterizes the class of linearly ordered f-quasifields which are fields, i.e. of linearly ordered f-quasifields in which full divisibility holds. In fact, as the categorical equivalence would suggest, it is closely related to Corollary 3.12 in [9].

Theorem 2. *Let* $\mathcal{K} = \langle K, +, -, \times, /_q, \vee, \wedge, 0, 1, q\rangle$ *be a linearly ordered quasifield. The following are equivalent:*

(i) $\mathcal{K}$ *is Archimedean (i.e.* $\forall b \forall a > 0 \exists n \in \mathbf{N}(na \geq b)$*).*
(ii) $\langle K, +, -, \times, 0, 1\rangle$ *is a field.*

Proof.
(i)⇒(ii) Let $h = q/_q(q + q)$. Then $h(q + q) = q$, which immediately implies that $2h = 1$. It follows that $2hz = z$ for every $z \in \mathcal{K}$. Now let $x \in \mathcal{K} \setminus \{0\}$, and let us prove that there is a $y \in \mathcal{K}$ such that $yx = 1$. Without loss of generality we may assume that $x > 0$. Let k be minimal such that $x \leq 2^k$ (such a k exists because $\mathcal{K}$ is Archimedean). Then $h^k x \leq 1$. Moreover by the minimality of k we have $h^{k-1}x > 1$ (where we put $h^{k-1} = 2$ if $k = 0$). Hence $q \leq h < h^k x \leq 1$, and by axiom (K3) there is a $z \in \mathcal{K}$ such that $h^k xz = h$. Now let $y = h^{k-1}z$. Then $yx = h^{k-1}zx = 2h^k zx = 2h = 1$. Hence y is the desired element.
(ii)⇒(i) Let by contradiction $\mathcal{K}$ be a linearly ordered f-quasifield such that for some $a, b \in \mathcal{K}$ one has $a > 0$ and $na > b$ for every $n \in \mathbf{N}$. Then for every $n \in \mathbf{N}$ we have $n < ba^{-1}$, against the fact that 1 is a strong unit of $\mathcal{K}$.

Corollary 1. *If* $\mathcal{F}$ *is a* f*-quasifield, then the ring of rationals* $\mathbf{Q}$ *can be embedded into the ring-reduct of* $\mathcal{F}$.

Proof. By the argument used in the proof of Theorem 2 we see that for all $n \in \mathbf{N}$ there is $y \in \mathcal{K}$ such that $ny = 1$. Let us denote this y by n^{-1}. Then it is readily seen that the map Ψ from $\mathbf{Q}$ into $\mathcal{K}$ defined by $\Psi(\pm\frac{m}{n}) = \pm(n^{-1}m)$ is an embedding of $\mathbf{Q}$ into the ring-reduct of $\mathcal{K}$.

Example. Let $\mathbf{R}^\star$ be any non-trivial ultrapower of the ordered field $\mathbf{R}$ of real numbers, and let ε be any strictly positive infinitesimal. Then for all $n \in \mathbf{N}$, $n < \frac{1}{\varepsilon}$. So 1 is not a strong unit and for any choice of $q \in (0, \frac{1}{2}]$, $\langle \mathbf{R}^\star, +, -, \times, /_q, \vee, \wedge, , 0, 1, q\rangle$ (where $\times$ denotes product and $x/_q y = \frac{u(x)}{u(y) \vee q}$) is not a f-quasifield although $\langle \mathbf{R}^\star, +, -, \times, 0, 1, \rangle$ is a field.

Example. Let $\mathbf{R}^\star$ be as before, let $q = \frac{1}{2}$ and let $\mathbf{R}^\star_{fin} = \{x \in \mathbf{R}^\star : \exists n \in \mathbf{N}(\mid x \mid \leq n)\}$. It is easy to see that $\mathbf{R}^\star_{fin}$ is a c-s-u-f-ring. Now let $x, y \in [\frac{1}{2}, 1]$ be such that $x \leq y$, and let $z = \frac{x}{y}$. Then $\frac{1}{2} \leq z \leq 1$, therefore $z \in \mathbf{R}^\star_{fin}$.
It follows that, letting $a/_q b = \frac{u(a)}{(u(b) \vee q)}$, $\mathbf{R}^\star_{fin}$ is closed under $/_q$, and $/_q$ makes $\mathbf{R}^\star_{fin}$ a f-quasifield. Nevertheless $\mathbf{R}^\star_{fin}$ is not a field, because if $\varepsilon \in \mathbf{R}^\star_{fin}$ is a strictly positive infinitesimal, then $\frac{1}{\varepsilon} \notin \mathbf{R}^\star_{fin}$.

3 Categorical Equivalence

We are ready to introduce an equivalence between the category of ŁΠ_q-algebras (called $\mathbf{LP}$) and the category of f-quasifields (called $\mathbf{FQ}$).

Definition 8. *Let Π_q be the functor from* $\mathbf{FQ}$ *into* $\mathbf{LP}$ *defined as follows: for every f-quasifield $\mathcal{F}$ we define a structure $\Pi_q(\mathcal{F})$ whose domain $\Pi_q(F)$ is $[0,1] = \{x \in \mathcal{F} : 0 \leq x \leq 1\}$, whose constants 0, 1 and q are those of $\mathcal{F}$, and the other operations are: $x \oplus y = (x+y) \wedge 1$, $\neg x = 1 - x$, and $x \to_q y = y/_q x$. The operation $\cdot$ is the restriction of $\times$ to $[0,1]$. For every morphism Φ from a f-quasifield $\mathcal{F}$ into a f-quasifield $\mathcal{K}$, we define $\Pi_q(\Phi)$ to be the restriction of Φ to $\Pi_q(\mathcal{F})$.*

The following lemmas are easy to demonstrate:

Lemma 1. *$Ł\Pi_q$ is a functor from **FQ** to **LP***
□

Lemma 2. *Let $\mathcal{F}$ be any c-s-u-f-ring equipped with an additional constant q and an additional binary operation $/_q$ such that for all $x, y \in \mathcal{F}$, $x/_q y = u(x)/_q u(y)$, and $0 \leq x/_q y \leq 1$. Let $\Pi_q(\mathcal{F})$ be defined from $\mathcal{F}$ according to Definition 8, (a). The following are equivalent:*

(i) $\mathcal{F}$ is a f-quasifield.
(ii) $\Pi_q(\mathcal{F})$ is a ŁΠ_q-algebra.

□

In order to prove that Π_q is an equivalence of categories, we start from the following observation. Let $\mathbf{F}$ be the forgetful functor from $\mathbf{LP}_q$ into the category $\mathbf{PMV}$ of PMV-algebras, and let $\mathbf{S}$ be the forgetful functor from $\mathbf{FQ}$ into the category $\mathbf{FR}$ of c-s-u-f-rings. Then it follows from the definition of Π_q (and from the definition of $\Gamma_{\mathbf{R}}$, see definition 5) that the functors $\mathbf{F} \circ \Pi_q$ and $\Gamma_{\mathbf{R}} \circ \mathbf{S}$, where $\circ$ denotes the composition of functors coincide.
Now we define a functor Π_q^{-1} as follows:

- For every ŁΠ_q-algebra $\mathcal{A}$, the c-s-u-f-ring subreduct of $\Pi_q^{-1}(\mathcal{A})$ is $\Gamma_{\mathbf{R}}^{-1}(\mathbf{F}(\mathcal{A}))$. Moreover the constant q is interpreted as $q_0 = i_{\mathbf{F}(\mathcal{A})}(q^{\mathcal{A}})$, where $q^{\mathcal{A}}$ is the interpretation of q in $\mathcal{A}$. Note that the domain of $\Gamma_{\mathbf{R}}(\Gamma_{\mathbf{R}}^{-1}(\mathbf{F}(\mathcal{A})))$ is contained into the domain of $\Gamma_{\mathbf{R}}^{-1}(\mathbf{F}(\mathcal{A}))$, therefore $i_{\mathbf{F}(\mathcal{A})}(q^{\mathcal{A}}) \in \Pi_q^{-1}(\mathcal{A})$.
  Moreover we define:
$$x/_q y = i_{\mathbf{F}(\mathcal{A})}((i_{\mathbf{F}(\mathcal{A})})^{-1}(u(y)) \rightarrow_q (i_{\mathbf{F}(\mathcal{A})})^{-1}(u(x))). \tag{1}$$
  Roughly speaking, we first compute $u(x)$ and $u(y)$, two members of the ŁΠ_q-algebra $\Pi_q(\Pi_q^{-1}(\mathbf{F}(\mathcal{A}))$. Then we apply to them the inverse of $i_{\mathbf{F}(\mathcal{A})}$, thus obtaining two elements of $\mathcal{A}$. Next we apply to these elements (taken in the inverse order, because intuitively $a/_q b = u(b) \rightarrow_q u(a)$) the operation $\rightarrow_q$, thus getting an element of $\mathcal{A}$. Finally we apply the isomorphism $i_{\mathbf{F}(\mathcal{A})}$ to such an element, thus obtaining its isomorphic copy in $\Pi_q(\Pi_q^{-1}(\mathcal{A})) \subseteq \Pi_q^{-1}(\mathcal{A})$.
- If ϕ is a morphism of ŁΠ-algebras from $\mathcal{A}$ into $\mathcal{B}$, then $\Pi_q^{-1}(\phi) = \Gamma^{-1}(\mathbf{F}(\phi))$.

In order to prove that Π_q is an equivalence of categories, by Proposition 2 and Lemma 1 it is sufficient to prove the following lemma:

Lemma 3.
(i) For every ŁΠ_q-algebra $\mathcal{A}$, $i_{\mathbf{F}(\mathcal{A})}$ is an isomorphism from $\mathcal{A}$ onto $\Pi_q(\Pi_q^{-1}(\mathcal{A}))$.
(ii) For every f-quasifield $\mathcal{F}$, $j_{\mathbf{S}(\mathcal{F})}$ is an isomorphism from $\mathcal{F}$ onto $\Pi_q^{-1}(\Pi_q(\mathcal{F}))$.
(iii) For every morphism ϕ of ŁΠ_q-algebras from $\mathcal{A}$ into $\mathcal{B}$, $\Gamma_{\mathbf{R}}^{-1}(\mathbf{F}(\phi))$ is a morphism from $\Pi_q^{-1}(\mathcal{A})$ into $\Pi_q^{-1}(\mathcal{B})$.

Proof. (i). That $i_{\mathbf{F}(\mathcal{A})}$ is an isomorphism of PMV-algebras follows from Proposition 2. That $i_{\mathbf{F}(\mathcal{A})}$ preserves the constant q follows from the definition of Π_q and of Π_q^{-1}.
We prove that $i_{\mathbf{F}(\mathcal{A})}$ preserves $\rightarrow_q$. Let $\Rightarrow_q$ denote the interpretation of $\rightarrow_q$ in $\Pi_q(\Pi_q^{-1}(\mathcal{A}))$. Thus $a \Rightarrow_q b = b/_q a$, and since $u(a) = a$ and $u(b) = b$, from eq. (1) we obtain:
$$a \Rightarrow_q b = i_{\mathbf{F}(\mathcal{A})}((i_{\mathbf{F}(\mathcal{A})})^{-1}(a) \rightarrow_q (i_{\mathbf{F}(\mathcal{A})})^{-1}(b)). \tag{2}$$
Now for $x, y \in \mathcal{A}$, substituting $i_{\mathbf{F}(\mathcal{A})}(x)$ for a and $i_{\mathbf{F}(\mathcal{A})}(y)$ for b in equation (2), we obtain:
$$i_{\mathbf{F}(\mathcal{A})}(x) \Rightarrow_q i_{\mathbf{F}(\mathcal{A})}(y) = i_{\mathbf{F}(\mathcal{A})}(x \rightarrow_q y),$$
and the claim is proved.

(ii). Let us denote $\Pi_q(\mathcal{F})$ by $\mathcal{B}$. That $j_{\mathbf{S}(\mathcal{F})}$ is an isomorphism of c-s-u-f-rings follows from Proposition 2. In order to prove that $j_{\mathbf{S}(\mathcal{F})}$ preserves q, note that the interpretation of q is the same in $\mathcal{F}$ and in $\Pi_q(\mathcal{F}) = \mathcal{B}$. Moreover in $\Pi_q^{-1}(\mathcal{B})$, q is interpreted as $i_{\mathbf{F}(\mathcal{B})}(q^{\mathcal{B}})$, where $q^{\mathcal{B}}$ is the interpretation of q in both $\mathcal{B}$ and $\mathcal{F}$. Therefore we only need to prove that $i_{\mathbf{F}(\mathcal{B})}(q^{\mathcal{B}}) = j_{\mathbf{S}(\mathcal{F})}(q^{\mathcal{B}})$. Now by Proposition 2, $i_{\mathbf{F}(\mathcal{B})}(q^{\mathcal{B}}) = \Gamma_{\mathbf{R}}(j_{\mathcal{F}}(q^{\mathcal{B}})) = j_{\mathcal{F}}(q^{\mathcal{B}})$, and the claim follows.

Finally we prove that $j_{\mathbf{S}(\mathcal{F})}$ preserves $/_q$. Let $//_q$ denote the interpretation of $/_q$ in $\Pi_q^{-1}(\Pi_q(\mathcal{F}))$ (and let us identify $/_q$ with its realization in $\mathcal{F}$). Let $x, y \in \mathcal{F}$, and let us prove that $j_{\mathbf{S}(\mathcal{F})}(x/_q y) = j_{\mathbf{S}(\mathcal{F})}(x)//_q j_{\mathbf{S}(\mathcal{F})}(y)$. Since

$x/_q y = u(x)/_q u(y)$ and $j_{\mathbf{S}(\mathcal{F})}$ preserves the operation u, we may assume without loss of generality that $u(x) = x$ and $u(y) = y$. Thus for every $x, y \in \mathcal{B}$, $j_{\mathbf{S}(\mathcal{F})}(x) = \Pi_q(j_{\mathbf{S}(\mathcal{F})}(x)) = i_{\mathbf{F}(\mathcal{B})}(x)$. Similarly, $j_{\mathbf{S}(\mathcal{F})}(y) = i_{\mathbf{F}(\mathcal{B})}(y)$. Thus recalling the last claim of Proposition 2 and the definition of Π_q^{-1}, we obtain: $j_{\mathbf{S}(\mathcal{F})}(x)//_q j_{\mathbf{S}(\mathcal{F})}(y) = i_{\mathbf{F}(\mathcal{B})}(x)//_q i_{\mathbf{F}(\mathcal{B})}(y) = i_{\mathbf{F}(\mathcal{B})}(y \rightarrow_q x) = j_{\mathbf{S}(\mathcal{F})}(y \rightarrow_q x) = j_{\mathbf{S}(\mathcal{F})}(x/_q y)$, and (ii) is proved.

(iii). Set $\mathcal{F} = \Pi_q^{-1}(\mathcal{A})$, $\mathcal{K} = \Pi_q^{-1}(\mathcal{B})$, $\psi = \Gamma^{-1}(\mathbf{F}(\phi))$. That ψ is a homomorphism of c-s-u-f-rings follows from Proposition 2. We prove that ψ preserves q. The interpretation of q in $\mathcal{F}$ is $q^{\mathcal{F}} = i_{\mathbf{F}(\mathcal{A})}(q^{\mathcal{A}})$, and the interpretation of q in $\mathcal{K}$ is $q^{\mathcal{K}} = i_{\mathbf{F}(\mathcal{B})}(q^{\mathcal{B}})$. Now by Proposition 2, $\Gamma_{\mathbf{R}}(\psi) \circ i_{\mathbf{F}(\mathcal{A})} = i_{\mathbf{F}(\mathcal{B})} \circ \phi$, therefore $\psi(q^{\mathcal{F}}) = \Gamma(\psi)(q^{\mathcal{F}}) = (\Gamma(\psi) \circ i_{\mathbf{F}(\mathcal{A})})(q^{\mathcal{A}}) = (i_{\mathbf{F}(\mathcal{B})} \circ \phi)(q^{\mathcal{A}}) = i_{\mathbf{F}(\mathcal{B})}(q^{\mathcal{B}}) = q^{\mathcal{K}}$.

Finally we prove that ψ preserves $/_q$. Let $//_q$ denote the interpretation of $/_q$ in $\mathcal{K}$, and let us identify the symbol $/_q$ and its realization in $\mathcal{F}$. Let $x, y \in \mathcal{F}$. Since $x/_q y = u(x)/_q u(y)$, and since ψ preserves u, we can assume without loss of generality that $x = u(x)$ and $y = u(y)$. Then by clause (1) in the definition of Π_q^{-1} we have:

$$x/_q y = i_{\mathbf{F}(\mathcal{A})}((i_{\mathbf{F}(\mathcal{A})})^{-1}(y) \rightarrow_q (i_{\mathbf{F}(\mathcal{A})})^{-1}(y)) \tag{3}$$

$$\psi(x)//_q \psi(y) = i_{\mathbf{F}(\mathcal{B})}((i_{\mathbf{F}(\mathcal{B})})^{-1}(\psi(y)) \rightarrow_q (i_{\mathbf{F}(\mathcal{B})})^{-1}(\psi(x))). \tag{4}$$

Note that by Proposition 2, $\Gamma_{\mathbf{R}}(\Gamma_{\mathbf{R}}^{-1}(\phi))) = i_{\mathbf{F}(\mathcal{B})} \circ \phi \circ i_{\mathbf{F}(\mathcal{A})}^{-1}$. Therefore, for all $z \in \Gamma_{\mathbf{R}}(\mathcal{F})$, we have:

$$\psi(z) = (\Gamma_{\mathbf{R}}(\psi))(z) = (\Gamma_{\mathbf{R}}(\Gamma_{\mathbf{R}}^{-1}(\phi)))(z) = i_{\mathbf{F}(\mathcal{B})}(\phi(i_{\mathbf{F}(\mathcal{A})}^{-1}(z))). \tag{5}$$

In particular, $\psi(x) = i_{\mathbf{F}(\mathcal{B})}(\phi(i_{\mathbf{F}(\mathcal{A})}^{-1}(x)))$ and $\psi(y) = i_{\mathbf{F}(\mathcal{B})}(\phi(i_{\mathbf{F}(\mathcal{A})}^{-1}(y)))$, therefore

$$(i_{\mathbf{F}(\mathcal{B})})^{-1}(\psi(y)) = \phi(i_{\mathbf{F}(\mathcal{A})}^{-1}(y)) \text{ and } (i_{\mathbf{F}(\mathcal{B})})^{-1}(\psi(x)) = \phi(i_{\mathbf{F}(\mathcal{A})}^{-1}(x)). \tag{6}$$

Substituting in eq. (4), recalling that ϕ and $i_{\mathbf{F}(\mathcal{A})}^{-1}$ are homomorphisms of ŁΠ_q-algebras and using eq. (5) and eq. (3), we obtain:

$$\begin{aligned}\psi(x)//_q \psi(y) &= i_{\mathbf{F}(\mathcal{B})}(\phi(i_{\mathbf{F}(\mathcal{A})}^{-1}(y)) \rightarrow_q \phi(i_{\mathbf{F}(\mathcal{A})}^{-1}(x))) = \\ &= i_{\mathbf{F}(\mathcal{B})}(\phi(i_{\mathbf{F}(\mathcal{A})}^{-1}(y) \rightarrow_q i_{\mathbf{F}(\mathcal{A})}^{-1}(x))) = i_{\mathbf{F}(\mathcal{B})}(\phi(i_{\mathbf{F}(\mathcal{A})}^{-1}(y \rightarrow_q x))) = \\ &= \psi(y \rightarrow_q x) = \psi(x/_q y).\end{aligned}$$

This concludes the proof of the lemma.

It follows:

Theorem 3. *The categories of ŁΠ_q-algebras and of f-quasifields are equivalent via the functors Π_q and Π_q^{-1}.*

As a consequence we obtain that a number of properties of ŁΠ_q-algebras shown in [9] can be translated to f-quasifields. For example:

Corollary 2. *Every f-quasifield is isomorphic to a subdirect product of a family of linearly ordered f-quasifields.*

Proof. Let $\mathcal{F}$ be any f-quasifield. Then we may decompose $\Pi_q(\mathcal{F})$ as a subdirect product of a family of linearly ordered ŁΠ_q-algebras $\langle \mathcal{A}_i : i \in I\rangle$. Then it follows from Theorem 3 that $\mathcal{F}$ has a subdirect embedding into $\prod_{i\in I} \Pi_q^{-1}(\mathcal{A}_i)$. Moreover the c-s-u-f-subreduct of $\Pi_q^{-1}(\mathcal{A}_i)$ is $\Gamma_{\mathbf{R}}^{-1}(\mathbf{F}(\mathcal{A}_i)$, and in [8] it is shown that $\Gamma_{\mathbf{R}}^{-1}(\mathbf{F}(\mathcal{A}_i)$ is linearly ordered whenever $\mathcal{A}_i$ is linearly ordered. This concludes the proof.

Corollary 3. *f-quasifields constitute a quasivariety, but not a variety.*

Proof. f-quasifields are axiomatized by means of quasi equations, so they form a quasivariety. In order to show that they do not form a variety it is sufficient to prove that they are not closed under epimorphic images. Now in [9] it is shown that there are a ŁΠ_q-algebra $\mathcal{A}$ and a epimorphism ϕ from $\mathcal{A}$ onto a structure $\mathcal{B}$ which is not a ŁΠ_q-algebra. Let $\mathcal{F} = \Pi_q^{-1}(\mathcal{A})$. By Theorem 3, $\mathcal{F}$ is a f-quasifield. Now consider $\Gamma_{\mathbf{R}}^{-1}(\mathbf{F}(\mathcal{B}))$ with an additional constant q_0 defined by $q_0 = i_{\mathbf{F}(\mathcal{B})}(\psi(q^{\mathcal{A}}))$ (where $q^{\mathcal{A}}$ is the realization of q in $\mathcal{A}$ and $\mathbf{F}$ forgets q and $\rightarrow_q$), and with the operation $/_q$ defined by means of the formula 1 in the definition of Π_q^{-1}. Call this algebra $\mathcal{K}$. Then $\Gamma^{-1}(\mathbf{F}(\psi))$ is a epimorphism from $\mathcal{F}$ onto $\mathcal{K}$. Now let $\mathcal{E}$ be the structure $\Gamma_{\mathbf{R}}(\mathcal{S}(\mathcal{K}))$ (where $\mathbf{S}$ forgets $/_q$ and q) with a constant q interpreted as q_0 and with an operation $\rightarrow_q$ defined by $x \rightarrow_q y = y/_q x$. It is readily seen that $\mathcal{E}$ is isomorphic to $\mathcal{B}$ (under the isomorphism $i_{\mathbf{F}(\mathcal{B})}$). Now if $\mathcal{K}$ were an f-quasifield, then by Lemma 2 $\mathcal{E}$ would be a ŁΠ_q-algebra, which is impossible.

References

1. W.J. Blok , I.M.A. Ferreirim: *On the structure of hoops*, Algebra Universalis **43** 2000, 233-257.
2. A. Bigard, K. Keimel and S. Wolfenstein: *Groupes at anneaux reticulés*, Lecture Notes in Mathematics, **608**, Springer Verlag, Berlin 1977.
3. A. Di Nola, A. Dvurecenskij: *Product MV-algebras*, Multi. Val. Logic, **6**,193-215, (2001).
4. F. Esteva, L. Godo, F. Montagna: *LΠ and LΠ$\frac{1}{2}$: two fuzzy logics joining Lukasiewicz and Product logics*, Archive for Mathematical Logic **40** (2001), pp. 39-67.
5. F. Esteva, L. Godo, P. Hájek: *A complete may-valued logic with product conjunction*, Archive for Mathematical Logic **35** (1996), 191-208.
6. P. Hájek: *Metamathematics of Fuzzy Logic*, Kluwer, 1988.
7. F. Montagna: *An algebraic approach to propositional fuzzy logic*, Journal of Logic, Language and Information **9** (2000), 91-124.
8. F. Montagna: *Subreducts of MV-algebras with product and product residuation*, Preprint 2002.
9. F. Montagna, L. Spada: *Continuous approximations of MV-algebras with product and product residuation*, Preprint 2003.
10. D. Mundici: *Interpretations of AF $C^\star$ algebras in Łukasiewicz sentential calculus*, J. Funct. Analysis **65**, (1986), 15-63.