Weakly Projective MV-algebras

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Finitely Presented MV-algebras and Rational Polyhedra

Theorem (McNaughton (1951))

For each $n = 1, 2, \ldots$, the MV-algebra $\mathcal{M}([0, 1]^n)$ of McNaughton maps from the $n$-cube is freely generated by the projection maps $\xi_i(a_1, \ldots, a_n) = a_i$.

A McNaughton map is a continuous function $f : [0, 1]^n \rightarrow [0, 1]$ satisfying:

There are linear polynomials $p_1, \ldots, p_m$ with integer coefficients, such that for all $x \in [0, 1]^n$ there is $i \in \{1, \ldots, m\}$ with $f(x) = p_i(x)$. 
An MV-algebra $A$ is **finitely presented** if there exist $n$ and $f \in \mathcal{M}([0, 1]^n)$ such that

$$A \cong \mathcal{M}([0, 1]^n) / \text{cong}(f, 1)$$
An MV-algebra $A$ is **finitely presented** if there exist $n$ and $f \in \mathcal{M}([0, 1]^n)$ such that

$$A \cong \mathcal{M}([0, 1]^n) / \text{cong}(f, 1) \cong \mathcal{M}([0, 1]^n) \upharpoonright f^{-1}(1)$$
An MV-algebra $A$ is **finitely presented** if there exist $n$ and $f \in M([0, 1]^n)$ such that

$$A \cong M([0, 1]^n) / \text{cong}(f, 1) \cong M(f^{-1}(1))$$
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A rational polyhedron $P$ in $[0, 1]^n$ is a finite union of closed simplexxes $P = S_1 \cup \cdots \cup S_t$ in $[0, 1]^n$ such that the coordinates of the vertices of every simplex $S_i$ are rational numbers.
A rational polyhedron $P$ in $[0, 1]^n$ is a finite union of closed simplexes $P = S_1 \cup \cdots \cup S_t$ in $[0, 1]^n$ such that the coordinates of the vertices of every simplex $S_i$ are rational numbers.

**Theorem**

A subset $P \subseteq [0, 1]^n$ is a rational polyhedron if and only if it there exists $f \in \mathcal{M}([0, 1]^n)$ such that $P = f^{-1}(1)$. 
An MV-algebra $A$ is finitely presented if there exist $n$ and a rational polyhedron $P \subseteq [0, 1]^n$ such that

$$A \cong \mathcal{M}(P)$$
\[ \mathcal{M}(P \subseteq [0, 1]^n) \xrightarrow{h} \mathcal{M}(Q \subseteq [0, 1]^m) \]
\[ \xi_i \xrightarrow{\text{h}} h(\xi_i): Q \to [0, 1] \]
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Homomorphisms and $\mathbb{Z}$-maps

\[ \mathcal{M}(P \subseteq [0, 1]^n) \xrightarrow{h} \mathcal{M}(Q \subseteq [0, 1]^m) \]

\[ \xi_i \xrightarrow{\eta h} h(\xi_i) : Q \rightarrow [0, 1] \]

\[ \eta h = (h(\xi_1), \ldots, h(\xi_n)) : Q \rightarrow [0, 1]^n \]

\[ P \xleftarrow{\eta h} Q \]
Introduction

Homomorphisms and $\mathbb{Z}$-maps

\[ \mathcal{M}(P \subseteq [0, 1]^n) \xrightarrow{h} \mathcal{M}(Q \subseteq [0, 1]^m) \]

\[ \xi_i \xrightarrow{\longrightarrow} h(\xi_i) : Q \rightarrow [0, 1] \]

\[ \eta_h = (h(\xi_1), \ldots, h(\xi_n)) : Q \rightarrow [0, 1]^n \]

\[ P \xleftarrow{\eta_h} Q \]

\[ h(f) = f \circ \eta_h \]
Definition
Given rational polyhedra $P \subseteq [0, 1]^n$ and $Q \subseteq [0, 1]^m$ a continuous map $\eta: P \rightarrow Q$ is called a $\mathbb{Z}$-map if there are finite affine linear maps $q_1, \ldots, q_k$ such that for each $x \in P$, $\eta(x) = q_i(x)$ for some $i = 1, \ldots, k$. 

Lemma
$\eta$ is a $\mathbb{Z}$-map if and only if $\xi_i \circ \eta \in M(P)$ for each $i = 1, \ldots, m$. 


Definition
Given rational polyhedra $P \subseteq [0, 1]^n$ and $Q \subseteq [0, 1]^m$ a continuous map $\eta: P \to Q$ is called a $\mathbb{Z}$-map if there are finite affine linear maps $q_1, \ldots, q_k$ such that for each $x \in P$, $\eta(x) = q_i(x)$ for some $i = 1, \ldots, k$.

Lemma
$\eta$ is a $\mathbb{Z}$-map if and only if $\xi_i \circ \eta \in \mathcal{M}(P)$ for each $i = 1, \ldots, m$. 
The category of Rational Polyhedra with $\mathbb{Z}$-maps is dually equivalent to the category of finitely presented MV-algebras.
Projectives are retractions of Free algebras.

Retractions are preserved under dualities.
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Duality: Projectives

- Projectives are retractions of Free algebras.
- Retractions are preserved under dualities.

**Theorem**

A finitely generated MV-algebra is projective iff there exist a number \( n = 1, 2, \ldots \) and a \( \mathbb{Z} \)-map \( \eta : [0, 1]^n \to [0, 1]^n \) such that

(i) \( \eta \circ \eta = \eta \),

(ii) \( A \cong \mathcal{M}(\eta([0, 1]^n)) \).
Introduction

Duality: Projectives

[LMC & D. Mundici, Projective MV-algebras and rational polyhedra, Algebra Universalis Volume 62, Number 1, 63-74]
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Duality: Projectives

[LMC & D. Mundici, Projective MV-algebras and rational polyhedra, *Algebra Universalis* Volume 62, Number 1, 63-74]

[LMC & D. Mundici, Rational polyhedra and projective lattice-ordered abelian groups with order unit, *Communications in Contemporary Mathematics* (to appear)]
Can we characterize the range of $\mathbb{Z}$-maps which domain is some $n$-cube?
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Can we characterize the range of \( \mathbb{Z} \)-maps which domain is some \( n \)-cube?

In other words:

Are there intrinsic necessary and sufficient conditions for a rational polyhedron \( P \) to be equal to \( \eta([0, 1]^n) \) for some \( n \) and some \( \mathbb{Z} \)-map \( \eta \)?
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Analizing $\mathbb{Z}$-maps

$\eta$ is a $\mathbb{Z}$-map iff there is a triangulation $\Delta$ of $P$ such that over every simplex $T$ of $\Delta$, $\eta$ coincides with an (affine) linear map $\eta_T$ with integer coefficients.

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Lemma

Given rational polyhedra $P \subseteq [0, 1]^n$ and $Q \subseteq [0, 1]^m$ a map $\eta: P \to Q$ is a $\mathbb{Z}$-map iff there is a triangulation $\Delta$ of $P$ such that over every simplex $T$ of $\Delta$, $\eta$ coincides with a (affine) linear map $\eta_T$ with integer coefficients.
Combinatorics

Analizing $\mathbb{Z}$-maps

Given a $t$-simplex $S = \text{conv}(v_0, \ldots, v_t) \subseteq [0, 1]^n$ then linear maps from $S$ into $[0, 1]^m$ are in one-one correspondence with maps from $\{v_0, \ldots, v_t\}$ into $[0, 1]^m$. 
Given a $t$-simplex $S = \text{conv}(v_0, \ldots, v_t) \subseteq [0, 1]^n$ then linear maps from $S$ into $[0, 1]^m$ are in one-one correspondence with maps from $\{v_0, \ldots, v_t\}$ into $[0, 1]^m$. 
Let $f : [0, 1]^n \to \mathbb{R}^m$ be linear map with integer coefficients, i.e. there exists $M \in \mathbb{Z}^{n \times m}$ and $b \in \mathbb{Z}^m$ such that $f(w) = Mw + b$ for each $w \in [0, 1]^n$. 

Let $f : [0, 1]^n \rightarrow \mathbb{R}^m$ be linear map with integer coefficients, i.e. there exists $M \in \mathbb{Z}^{n \times m}$ and $b \in \mathbb{Z}^m$ such that $f(w) = Mw + b$ for each $w \in [0, 1]^n$.

Let $v = (x_1, \ldots, x_n) \in [0, 1]^n$ be a rational vector. We define $\text{den}(v)$ to be the least common denominator of $\{x_1, \ldots, x_n\}$, i.e. the smallest $k \in \mathbb{Z}$ such that $kv \in \mathbb{Z}^n$. 
Combinatorics

Analizing $\mathbb{Z}$-maps

Let $f : [0, 1]^n \to \mathbb{R}^m$ be linear map with integer coefficients, i.e. there exists $M \in \mathbb{Z}^{n \times m}$ and $b \in \mathbb{Z}^m$ such that $f(w) = Mw + b$ for each $w \in [0, 1]^n$.

Let $\nu = (x_1, \ldots, x_n) \in [0, 1]^n$ be a rational vector. We define $\text{den}(\nu)$ to be the least common denominator of $\{x_1, \ldots, x_n\}$, i.e. the smallest $k \in \mathbb{Z}$ such that $kv \in \mathbb{Z}^n$.

Observe that

$$\text{den}(\nu)f(\nu) = k(M\nu + b) = M(\text{den}(\nu)\nu) + \text{den}(\nu)b \in \mathbb{Z}^m.$$

Then $\text{den}(f(\nu))$ is a divisor of $\text{den}(\nu)$.
The vector $\tilde{v} = \text{den}(v)(v, 1) \in \mathbb{Z}^{n+1}$ is called the **homogeneous correspondent** of $v$. 
Combinatorics
Regular Simplexes

The vector \( \tilde{v} = \text{den}(v)(v, 1) \in \mathbb{Z}^{n+1} \) is called the **homogeneous correspondent** of \( v \).

**Definition**
A simplex \( S \subseteq [0, 1]^n \) is called **regular** if the set of homogeneous correspondents of its vertices is part of a basis of the free abelian group \( \mathbb{Z}^{n+1} \).
**Lemma**

Let $S = \text{conv}(v_0, \ldots, v_k) \subseteq [0, 1]^n$ be a regular $k$-simplex, and \{w_0, \ldots, w_k\} a set of rational points in $[0, 1]^m$. Then the following conditions are equivalent:

(i) For each $i = 1, \ldots, k$, $\text{den}(w_i)$ is a divisor of $\text{den}(v_i)$.

(ii) For some integer matrix $M \in \mathbb{Z}^{n \times m}$ and integer vector $b \in \mathbb{Z}^m$, $Mv_i + b = w_i$. 
Combinatorics
Regular Triangulations

By a **regular triangulation** of a polyhedron $P$ we understand a triangulation of $P$ consisting of regular simplexes.
Corollary

Let $P \subset [0, 1]^n$ be a polyhedron, $\Delta$ be a regular triangulation of $P$ and $f : \text{ver}(\Delta) \to ([0, 1] \cap \mathbb{Q})^m$ be such that $\text{den}(f(v))$ divides $\text{den}(v)$ for each $v \in \text{ver}(\Delta)$. Then there exists a unique $\mathbb{Z}$-map $\eta : P \to [0, 1]^m$ satisfying:

1. $\eta$ is linear on each simplex of $\Delta$,
2. $\eta \upharpoonright \text{ver}(\Delta) = f$. 
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Regular Triangulations

Corollary

Let $P \subset [0, 1]^n$ be a polyhedron, $\Delta$ be a regular triangulation of $P$ and $f : \text{ver}(\Delta) \to ([0, 1] \cap \mathbb{Q})^m$ be such that $\text{den}(f(v))$ divides $\text{den}(v)$ for each $v \in \text{ver}(\Delta)$. Then there exists a unique $\mathbb{Z}$-map $\eta : P \to [0, 1]^m$ satisfying:

1. $\eta$ is linear on each simplex of $\Delta$,
2. $\eta \upharpoonright \text{ver}(\Delta) = f$.

Lemma

Let $\eta : P \to Q$ be a $\mathbb{Z}$-map there exists a regular triangulation $\Delta$ of $P$ such that $\eta$ is linear over each simplex in $\Delta$.
Combinatorics
Weighted Abstract Simplicial Complexes

For a regular triangulation $\Delta$ of a rational polyhedron $P$, the **skeleton** of $\Delta$ is weighted abstract simplicial complex

$$W_\Delta = (\mathcal{V}, \Sigma, \omega)$$

given by the following stipulations:

1. $\mathcal{V} =$ vertices of $\Delta$.
2. For every subset $W = \{w_1, \ldots, w_k\}$ of $\mathcal{V}$, $W \in \Sigma$ iff $\text{conv}(w_1, \ldots, w_k) \in \Delta$.
3. $\omega: \mathcal{V} \to \mathbb{N}$, is defined by $\omega(v) = \text{den}(v)$
Let \( \mathcal{W} = (\mathcal{V}, \Sigma, \omega) \) be a weighted abstract simplicial complex with vertex set \( \mathcal{V} = \{v_1, \ldots, v_n\} \).
Combinatorics
Weighted Abstract Simplicial Complexes

Let $\mathcal{W} = (\mathcal{V}, \Sigma, \omega)$ be a weighted abstract simplicial complex with vertex set $\mathcal{V} = \{v_1, \ldots, v_n\}$.

Let $e_1, \ldots, e_n$ the standard basis vectors of $\mathbb{R}^n$, and $\Delta_{\mathcal{W}}$ be the complex whose vertices are

$$v'_1 = e_1/\omega(v_1), \ldots, v'_n = e_n/\omega(v_n),$$
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Weighted Abstract Simplicial Complexes

Let $\mathcal{W} = (\mathcal{V}, \Sigma, \omega)$ be a weighted abstract simplicial complex with vertex set $\mathcal{V} = \{v_1, \ldots, v_n\}$.

Let $e_1, \ldots, e_n$ the standard basis vectors of $\mathbb{R}^n$, and $\Delta_\mathcal{W}$ be the complex whose vertices are

$$v'_1 = e_1/\omega(v_1), \ldots, v'_n = e_n/\omega(v_n),$$

and whose $k$-simplexes ($k = 0, \ldots, n$) are given by

$$\text{conv}(v'_{i(0)}, \ldots, v'_{i(k)}) \in \Delta_\mathcal{W} \iff \{v_{i(0)}, \ldots, v_{i(k)}\} \in \Sigma.$$
Let $\mathcal{W} = (\mathcal{V}, \Sigma, \omega)$ be a weighted abstract simplicial complex with vertex set $\mathcal{V} = \{v_1, \ldots, v_n\}$.

Let $e_1, \ldots, e_n$ the standard basis vectors of $\mathbb{R}^n$, and $\Delta_{\mathcal{W}}$ be the complex whose vertices are

$$v'_1 = e_1/\omega(v_1), \ldots, v'_n = e_n/\omega(v_n),$$

and whose $k$-simplexes ($k = 0, \ldots, n$) are given by

$$\text{conv}(v'_{i(0)}, \ldots, v'_{i(k)}) \in \Delta_{\mathcal{W}} \quad \text{iff} \quad \{v_{i(0)}, \ldots, v_{i(k)}\} \in \Sigma.$$

Then $\Delta_{\mathcal{W}}$ is a regular triangulation of the polyhedron $|\Delta_{\mathcal{W}}| \subseteq [0, 1]^n$ called the canonical realization of $\mathcal{W}$. 
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Given a weighted abstract simplicial complex $\mathcal{W} = (V, \Sigma, \omega)$.
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Weighted Abstract Simplicial Complexes

Given a weighted abstract simplicial complex \( \mathcal{W} = (\mathcal{V}, \Sigma, \omega) \).

The maps

\[
\mathcal{V} \xrightarrow{f} [0, 1]^n
\]

such that \( \text{den}(f(v)) \) is a divisor of \( \omega(v) \)
Given a weighted abstract simplicial complex $\mathcal{W} = (\mathcal{V}, \Sigma, \omega)$.

The maps

$$\mathcal{V} \xrightarrow{f} [0, 1]^n$$

such that $\text{den}(f(\nu))$ is a divisor of $\omega(\nu)$

are in one-one correspondence with
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Weighted Abstract Simplicial Complexes

Given a weighted abstract simplicial complex \( \mathcal{W} = (V, \Sigma, \omega) \).

The maps

\[
V \xrightarrow{f} [0, 1]^n
\]

such that \( \text{den}(f(v)) \) is a divisor of \( \omega(v) \)

are in one-one correspondence with

the \( \mathbb{Z} \)-maps

\[
|\Delta_\mathcal{W}| \xrightarrow{\eta} [0, 1]^n
\]

that are linear over each simplex of \( \Delta_\mathcal{W} \).
**Z-images of cubes**

**Strongly Regular Polyhedra**

**Definition**
A rational polyhedron $P$ is said to be strongly regular if there is a regular triangulation $\Delta$ of $P$ such that the denominators of the vertices of each maximal simplex of $\Delta$ are coprime.

**Lemma**
A rational polyhedron $P$ is strongly regular if and only if every regular triangulation $\Delta$ of $P$ is such that the denominators of the vertices of each maximal simplex of $\Delta$ are coprime.
Definition

A rational polyhedron $P$ is said to be strongly regular if there is a regular triangulation $\Delta$ of $P$ such that the denominators of the vertices of each maximal simplex of $\Delta$ are coprime.
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Strongly Regular Polyhedra

Definition
A rational polyhedron \( P \) is said to be strongly regular if there is a regular triangulation \( \Delta \) of \( P \) such that the denominators of the vertices of each maximal simplex of \( \Delta \) are coprime.

Lemma
A rational polyhedron \( P \) is strongly regular if and only if every regular triangulation \( \Delta \) of \( P \) is such that the denominators of the vertices of each maximal simplex of \( \Delta \) are coprime.
Examples

- For every $n = 1, 2, \ldots$ the $n$-dimensional cube $[0, 1]^n$ is strongly regular.
- Every regular $n$-simplex $S \subset [0, 1]^n$ is strongly regular.
Theorem

Let $P$ and $Q$ be rational polyhedra and $\eta: P \to Q$ be a $\mathbb{Z}$-morphism onto $Q$. If $P$ is a strongly regular then $Q$ is strongly regular.
Theorem

Given a polyhedron $P \subseteq [0, 1]^n$ the following conditions are equivalent:

(a) There exist $m$ and a $\mathbb{Z}$-map $\eta: [0, 1]^m \to P$ onto $P$.

(b) $P$ satisfies the following conditions:

1. $P$ is connected,
2. $P \cap \{0, 1\}^n \neq \emptyset$, and
3. $P$ is strongly regular.
Z-images of cubes
First Step: Strongly Regular Collapsible Polyhedra

Given an abstract simplicial complex $\langle V, \Sigma \rangle$ a simplex $T \in \Sigma$ is said to have a free face $F$ if $\emptyset \neq F \subseteq T$ is a facet of $T$, and if $F \subseteq S \in \Sigma$ then $S = F$ or $S = T$.

The transition from $\langle V, \Sigma \rangle$ to the subcomplex $\langle V', \Sigma' = \Sigma \setminus \{T, F\} \rangle$ of $\langle V, \Sigma \rangle$, where $V' = V \setminus F$ if $F$ is a singleton and otherwise $V' = V$ is called an (abstract) elementary collapse.

We say that $\langle V, \Sigma \rangle$ is collapsible if it collapses to the abstract simplicial complex consisting of one of its vertices (equivalently any of its vertices).
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First Step: Strongly Regular Collapsible Polyhedra

Given an abstract simplicial complex \( \langle V, \Sigma \rangle \) a simplex \( T \in \Sigma \) is said to have a free face \( F \) if \( \emptyset \neq F \subseteq T \) is a facet of \( T \), and if \( F \subseteq S \in \Sigma \) then \( S = F \) or \( S = T \).
Given an abstract simplicial complex \( \langle V, \Sigma \rangle \) a simplex \( T \in \Sigma \) is said to have a free face \( F \) if \( \emptyset \neq F \subseteq T \) is a facet of \( T \), and if \( F \subseteq S \in \Sigma \) then \( S = F \) or \( S = T \).

The transition from \( \langle V, \Sigma \rangle \) to the subcomplex \( \langle V', \Sigma' = \Sigma \setminus \{T, F\} \rangle \) of \( \langle V, \Sigma \rangle \), where \( V' = V \setminus F \) if \( F \) is a singleton and otherwise \( V' = V \) is called an (abstract) elementary collapse.
Given an abstract simplicial complex $\langle V, \Sigma \rangle$ a simplex $T \in \Sigma$ is said to have a \textit{free face} $F$ if $\emptyset \neq F \subseteq T$ is a facet of $T$, and if $F \subseteq S \in \Sigma$ then $S = F$ or $S = T$.

The transition from $\langle V, \Sigma \rangle$ to the subcomplex $\langle V', \Sigma' \rangle = \Sigma \setminus \{T, F\}$ of $\langle V, \Sigma \rangle$, where $V' = V \setminus F$ if $F$ is a singleton and otherwise $V' = V$ is called an (abstract) \textit{elementary collapse}.

We say that $\langle V, \Sigma \rangle$ is \textbf{collapsible} if it collapses to the abstract simplicial complex consisting of one of its vertices (equivalently any of its vertices).
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First Step: Strongly Regular Collapsible Polyhedra
Theorem (Mundici & LMC)

Let $P \subseteq [0, 1]^n$ be a polyhedron. Suppose

(i) $P$ has a collapsible triangulation $\nabla$;

(ii) $P$ contains a vertex $v$ of $[0, 1]^n$;

(iii) $P$ is strongly regular.

Then there is a $\mathbb{Z}$-map $\eta : [0, 1]^n \rightarrow P$ onto $P$. 
Theorem (Mundici & LMC)

Let \( P \subseteq [0,1]^n \) be a polyhedron. Suppose

(i) \( P \) has a collapsible triangulation \( \nabla \);

(ii) \( P \) contains a vertex \( v \) of \([0,1]^n\);

(iii) \( P \) is strongly regular.

Then \( P \) is a \( \mathbb{Z} \)-retract of \([0,1]^n\).
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Second Step: Collapsible Wrapping

Let $P \subseteq [0, 1]^n$ be a rational polyhedron satisfying:

1. $P$ is connected,
2. $P \cap \{0, 1\}^n \neq \emptyset$, and
3. $P$ is strongly regular.

Let $\Delta$ be a regular triangulation of $P$. 
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Weakly Projective MV-algebras

Definition

An MV-algebra $A$ is said to be (finitely generated) **weakly projective** if there exist positive integers $m$ and $n$, and homomorphisms $f: \mathcal{M}([0, 1]^m) \to A$ and $g: A \to \mathcal{M}([0, 1]^n)$, such that $f$ is onto $A$ and $g$ is one-one.
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Weakly Projective MV-algebras

Theorem
An MV-algebra is weakly projective if and only if there exist $n$ and a rational polyhedron $P \subseteq [0, 1]^n$ satisfying the following conditions:

1. $A \cong \mathcal{M}(P)$,
2. $P \cap \{0, 1\}^n \neq \emptyset$,
3. $P$ is connected, and
4. $P$ is a strongly regular.
Admissible Rules

An MV-algebra \( A \) is weakly projective if and only if it is finitely presented and...
An MV-algebra $A$ is weakly projective if and only if it is finitely presented and $A \in \mathbb{I}(\mathcal{M}([0, 1]^n))$ for some $n$. 
An MV-algebra $A$ is weakly projective if and only if it is finitely presented and $A \in \mathbb{IS}(\text{Free}_{\text{MV}}(\omega))$. 
Admissible Rules

If a pair of formulas $\varphi, \psi$ with variables $\{x_1, \ldots, x_n\}$ are such that $\mathcal{M}([0, 1]^n) / \text{cong}(f_\varphi, f_\psi) \in \mathbb{ISP}_U(\text{Free}_{MV}(\omega)),$
Admissible Rules

If a pair of formulas $\varphi, \psi$ with variables $\{x_1, \ldots, x_n\}$ are such that $\mathcal{M}([0, 1]^n) / \text{cong}(f\varphi, f\psi) \in \mathbb{ISP}_U(\text{Free}_{MV}(\omega))$, then:

$$
\varphi \equiv \psi \models_{\text{Free}_{MV}(\omega)} \{ \alpha_i \equiv \beta_i \mid i = 1, \ldots, m \}
$$

if and only if there exist $j \in \{1, \ldots, m\}$ such that

$$
\varphi \equiv \psi \models_{\text{MV}} \alpha_j \equiv \beta_j
$$
Admissible Rules

If a pair of formulas \( \varphi, \psi \) with variables \( \{x_1, \ldots, x_n\} \) are such that \( M([0,1]^n)/\text{cong}(f_\varphi, f_\psi) \in \text{ISP}_U(\text{Free}_{MV}(\omega)) \), then:

\[
\varphi \simeq \psi \models_{\text{Free}_{MV}(\omega)} \{ \alpha_i \simeq \beta_i \mid i = 1, \ldots, m \}
\]

if and only if there exist \( j \in \{1, \ldots, m\} \) such that

\[
\varphi \simeq \psi \models_{MV} \alpha_j \simeq \beta_j
\]

if and only if

\[
\varphi \leftrightarrow \psi \models_\mathbb{L}_\infty \alpha_j \leftrightarrow \beta_j
\]
Admissible Rules

If a formula \( \varphi \) with variables \( \{x_1, \ldots, x_n\} \) is such that 
\[ M([0, 1]^n) / \text{cong}(f_\varphi, 1) \in \text{ISP}_U(\text{Free}_{MV}(\omega)) \], then:

\[ \varphi \equiv \varphi \rightarrow \varphi \models_{\text{Free}_{MV}(\omega)} \{ \alpha_i \approx \alpha_j \rightarrow \alpha_j \mid i = 1, \ldots, m \} \]

if and only if there exist \( j \in \{1, \ldots, m\} \) such that

\[ \varphi \equiv \varphi \rightarrow \varphi \models_{MV} \alpha_j \approx \alpha_j \rightarrow \alpha_j \]

if and only if

\[ \varphi \models_{L_\infty} \alpha_j \]
Admissible Rules
Admissible Saturated Formulas

Admissible Rules
Admissible Saturated Formulas


**Definition**
A formula $\varphi$ is admissibly saturated in a logic $L$ if for every finite set $\Delta$ of formulas, $\varphi \vdash_L \Delta$ implies $\varphi \vdash_L \psi$ for some $\psi \in \Delta$. 
If $\varphi$ is a formula such that $\mathcal{M}([0, 1]^n) / \text{cong}(f_{\varphi}, 1)$ is weakly projective, then $\varphi$ is admissible saturated (in $\mathcal{L}_\infty$).
If \( \varphi \) is a formula such that \( \mathcal{M}([0, 1]^n) / \text{cong}(f_\varphi, 1) \) is weakly projective, then \( \varphi \) is admissible saturated (in \( \mathcal{L}_\infty \)).

**Theorem (E. Jeřábek)**

A formula \( \varphi \) with variables in \( \{x_1, \ldots, x_n\} \) is admissibly saturated in \( \mathcal{L}_\infty \) if and only if

1. \( f_{\varphi}^{-1}(1) \cap \{0, 1\}^n \neq \emptyset \),
2. \( f_{\varphi}^{-1}(1) \) is connected, and
3. \( f_{\varphi}^{-1}(1) \) is a finite union of anchored polytopes.
Theorem

And MV-algebra is weakly projective if and only if there exist \( n \) and a rational polyhedron \( P \subseteq [0, 1]^n \) satisfying the following conditions:

1. \( A \cong \mathcal{M}(P) \),
2. \( P \cap \{0, 1\}^n \neq \emptyset \),
3. \( P \) is connected, and
4. \( P \) is a strongly regular.
A formula \( \varphi \) with variables in \( \{x_1, \ldots, x_n\} \) is admissibly saturated in \( \mathcal{L}_\infty \) if and only if

\[
\mathcal{M}([0, 1]^n)/\text{cong}(f_\varphi, 1) \text{ is weakly projective.}
\]
Summarizing

Images of $\mathbb{Z}$-maps.
Introduction

Finitely Presented MV-algebras and Rational Polyhedra
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Duality
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Strongly Regular Polyhedra
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Summarizing

► Images of $\mathbb{Z}$-maps.
► Weakly projective MV-Algebras.
Summarizing

- Images of $\mathbb{Z}$-maps.
- Weakly projective MV-Algebras.
- Relation with admissible saturated formulas.
Thank you!

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