Pseudofinite Linear Groups

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Notation and Conventions

- We work in the language of groups unless stated otherwise.
- Definable means possibly with parameters.
- Ultrafilters are non-principle.
- $\prod_{i \in I} G_i/\mathcal{U}$ denotes the ultraproduct of a family of structures in a common language L over an ultrafilter $\mathcal{U}$.

Pseudofinite groups and (non)examples

Definition

A group is pseudofinite if it is elementarily equivalent to an ultraproduct of finite groups.

- $GL_n(F)$ is pseudofinite for any pseudofinite field F.
- $(\mathbb{Q}, +)$ is pseudofinite. (Later)
- $(\mathbb{Z}, +)$ is not pseudofinite.
  A sentence satisfied by all finite groups (but not $\mathbb{Z}$):

$$\forall x \forall y (x^2 = y^2 \to x = y) \leftrightarrow \forall z \exists t (z = t^2)$$

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Simple pseudofinite groups

Theorem (Wilson, 1993)

Every simple pseudofinite group is elementarily equivalent to a (twisted) Chevalley group over a pseudofinite field.

Remarks

- By a (twisted) Chevalley group over a field F we mean a simple matrix group, e.g. $PSL_2(F)$.
- Elementary equivalence can be strengthen to isomorphism (M. Ryten, 2007).
- CFSG is used in the proof.

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Definably simple pseudofinite groups

Definition

A group is called definably simple if it has no non-trivial proper definable normal subgroup.

Weaker than simplicity:

- $\prod_{i \in I} Alt(n_i)/\mathcal{U}, \; n_i \geq 5$ is definably simple but not simple unless it is finite (Felgner).
- $(\mathbb{Q}, +)$ is definably simple but not simple.
- Definable simplicity coincides with simplicity in some cases, e.g. for non-abelian groups whose theory is supersimple (Wagner).

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Definably simple pseudofinite groups of fcd

Definition

A group G has centralizer dimension $\leqslant n$ if for any $X \subseteq G$ we have $C_G(X) = C_G(X_0)$ for some $X_0 \subseteq X$ such that $|X_0| \leqslant n$. Such groups are said to have finite centralizer dimension (fcd).

The class of groups of centralizer dimension $\leqslant n$ is closed under

- taking subgroups.
- elementary equivalence.

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Abelian case

Fact (folklore)

Let A be an infinite abelian group. The following statements are equivalent.

1. *A is definably simple.*
2. *A is torsion-free divisible.*
3. *$A \equiv \prod_{p \in I} C_p / \mathcal{U} \equiv (\mathbb{Q}, +)$ where $\mathcal{U}$ is a non-principal ultrafilter on the set I of all prime numbers and C_p is the cyclic group of order p.*

Non-abelian case

Proposition (U.)

Every definably simple non-abelian pseudofinite group of finite centralizer dimension is isomorphic to a (twisted) Chevalley group over a pseudofinite field.

Proof: The ideas of Wilson + fcd property (CFSG is used).

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Where does this proposition sit?

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Simple groups of finite Morley rank

Theorem (U.)

Let G be an infinite simple group of finite Morley rank and $\alpha \in \mathrm{Aut}(G)$ such that the definable hull of $C_G(\alpha)$ is G. If $C_G(\alpha)$ is pseudofinite, then there is a definable (in $C_G(\alpha)$) normal subgroup S of $C_G(\alpha)$ such that S is isomorphic to a (twisted) Chevalley group over a pseudofinite field and $C_G(\alpha)$ embeds in $\mathrm{Aut}(S)$.

Goals:

- Identify G with a Chevalley group over an algebraically closed field.
- Eliminate the CFSG from the work.
- Weaken the pseudofiniteness assumption on the group of fixed points.

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Larsen-Pink Theorem

Theorem (Jordan, 1878)

For every n there is a constant J_n such that any finite subgroup of $GL_n(K)$ where $char(K) = 0$ has an abelian normal subgroup of index $\leqslant J_n$.

Generalization to arbitrary characteristic (special case).

Larsen-Pink Theorem (2011)

For every n there is a constant J'_n such that for any finite simple group G possessing a faithful linear representation of dimension $\leqslant n$ over a field k, we have either

- $|G| \leqslant J'_n$ *or*
- $p := char(k) > 0$ *and G is a group of Lie type (in characteristic p).*

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Implication

Below we state a special case of the proposition concerning definably simple pseudofinite groups:

Special case of the proposition

Every definably simple non-abelian pseudofinite linear group is isomorphic to a (twisted) Chevalley group over a pseudofinite field.

Special case of the proposition + Jordan's Theorem

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More special case of the proposition

Every definably simple non-abelian linear pseudofinite group over a field of characteristic $p > 2$ is isomorphic to a (twisted) Chevalley group over a pseudofinite field.

Aim: Prove this result without using the CFSG.

Idea: (Suggested by Borovik) Apply the methods from the work of Borovik about the classification of locally finite simple groups of finite Morley rank of odd type.

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Useful properties of our context

- Linearity:
  - finite centralizer dimension;
  - notion of Zariski closure;
  - induction on the Zariski dimension.
- Pseudofiniteness:
  - Definable subgroups and quotients by definable normal subgroups of pseudofinite groups are pseudofinite.
  - Brauer's Lemma: Two involutions are either conjugate or commute with a third involution.

Definable simplicity + pseudofiniteness + linearity

$\Downarrow$ (OOT + Khukro)

Existence of an involution

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Sylow theory in general

Definition

A (possibly infinite) group G is called a p-group if every element of G has order which is a power of p. As usual, Sylow p-subgroups are maximal p-subgroups.

Known results

- Sylow p-subgroups of periodic linear groups are conjugate (Platanov, 1965).
- Sylow p-subgroups of linear algebraic groups over algebraically closed fields are conjugate.
- Sylow 2-subgroups of periodic groups with DCC on centralizers are conjugate and locally finite (Wagner, 1999).
- Sylow 2-subgroups of groups of finite Morley rank are conjugate.

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Known results (continued)

For $GL_n(K)$

- Sylow p-subgroups are conjugate for arbitrary K if $p \neq 2$ (Suprunenko (1960), Volvachev (1963)).
- Sylow 2-subgroups are conjugate in positive characteristic (Volvachev, Leedham-Green and Plesken (1986)).
- Sylow 2-subgroups may not be conjugate in characteristic 0 (Volvachev, Leedham-Green and Plesken).

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Volvachev's criteria

Theorem (Volvachev, 1963)

Sylow p-subgroups of $GL_n(K)$ are conjugate except when $p = 2, char(K) \neq 2$ and the following conditions are satisfied.

1. *-1 can be written as a sum of two squares in K.*
2. *K has no element of multiplicative order 4.*
3. *In $K(i)$ where $i^2 = -1$, every 2-element $a + bi$ satisfies $(a + bi)(a - bi) = 1$.*

Example

Let $F = \prod_{p \in I} F_p/\mathcal{U}$ where $I = \{p \text{ prime} \mid p \equiv 3 \pmod 8\}$. Then F satisfies (1) and (2) but not (3). So, Sylow 2-subgroups of $GL_2(F)$ are conjugate by Volvachev's Criteria.

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Volvachev Conditions:

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- Condition (1) is a first order property which holds in all finite fields and hence holds in any pseudofinite field.
- Any field of positive characteristic satisfying condition (2) can not satisfy condition (3) (Volvachev). In particular no finite field satisfies (2) and (3) at the same time.

Volvachev's argument about positive characteristic fields works well for pseudofinite fields of characteristic 0, provided that there is a bound on the order of 2-elements of $K(i)$. This is the case for the pseudofinite field F in our example.

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Sylow 2-subgroups of pseudofinite linear groups

The following observation is the analogue of its finite Morley rank variant and a similar proof works.

Observation (U.)

Let G be a pseudofinite linear group. If one of the Sylow 2-subgroups of G is finite then all Sylow 2-subgroups of G are conjugate.

Back to the example:

$$F = \prod_{p \equiv 3 \pmod 8} F_p / \mathcal{U}.$$

Then, $GL_2(F)$ is a pseudofinite group whose Sylow 2-subgroups are finite. So we get their conjugacy again by the observation above.

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Ingredients of the proof

- Linearity is not necessary in the proof. Sylow 2-subgroups of groups with DCC on centralizers are:
  - locally finite (Wagner, 1999);
  - locally nilpotent;
  - nilpotent-by-finite (Byrant, 1979);
  - have non-trivial center and satisfy normalizer condition.
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