Pfaffian functions vs Rolle leaves

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joint work with Gareth Jones

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Example

1. the exponential function
2. "classical" pfaffian chain: each g_{ij} is polynomial

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Still open:

Conjecture 5

$\mathcal{R}_{\text{pfaff}}$ is model complete relative to $\mathcal{R}$.

coordinate-independent formulation

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Example

1. The horizontal lines are the leaves of d_h.
2. The graph of exp is a leaf of d_e.
3. The image of every trajectory of the vector field $-y\frac{\partial}{\partial x} + (x - y)\frac{\partial}{\partial y}$ in $\mathbb{R}^2 \setminus \{0\}$ is an integral manifold of d_s.

rolle leaves

Definition (Moussu & Roche 1991, based on Khovanskii 1979)

L is a **Rolle leaf** if L is a closed, embedded submanifold of M and, for every C^1 curve $\gamma : [0,1] \longrightarrow M$ with $\gamma(0), \gamma(1) \in L$, there exists $t \in [0,1]$ such that $\gamma'(t)$ is tangent to $d(\gamma(t))$.

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Lemma 6 (Khovanskii 1979)

Let $f = (f_1, \dots, f_k) : \mathbb{R}^n \longrightarrow \mathbb{R}^k$ be a pfaffian chain over $\mathcal{R}$. Then the graph of each f_i is a Rolle leaf over $(\mathcal{R}, f_1, \dots, f_{i-1})$.

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2. $\mathcal{R}_{\text{pfaff}}$ is a reduct of $\mathcal{P}(\mathcal{R})$.
3. $\mathcal{P}(\mathcal{R})$ admits quantifier elimination.

nested rolle leaves

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Example

If $f = (f_1, \dots, f_k) : \mathbb{R}^n \longrightarrow \mathbb{R}^k$ is a pfaffian chain over $\mathcal{R}$ with associated g_{ij}, set $d_0 := \mathbb{R}^{n+k}$ and $L_0 := \mathbb{R}^{n+k}$ and, for $i = 1, \dots, k$, set $\omega_i := g_{i1}dx_1 + \cdots + g_{in}dx_n - dx_{n+i}$ and

$$d_i := \ker\omega_1 \cap \cdots \cap \ker\omega_i \quad \text{and} \quad L_i := \mathrm{gr}(f_i) \times \mathbb{R}^{k-i}.$$

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$\mathcal{N}(\mathcal{R})$ is model complete relative to $\mathcal{R}$ and interdefinable with $\mathcal{P}(\mathcal{R})$, provided $\mathcal{R}$ admits analytic cell decomposition.

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So, one way to approach Conjecture 5 is to consider the following:

Question

Is every nested Rolle leaf over $\mathcal{R}$ existentially definable in $\mathcal{R}_{\text{pfaff}}$?

two problems

Problem 1

While a Rolle leaf over $\mathcal{R}$ can be covered by finitely many graphs of functions satisfying pfaffian equations over $\mathcal{R}$, the domains of these functions are generally not definable in $\mathcal{R}$.

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Problem 2

Are nested Rolle leaves over $\mathcal{R}$ even *locally* existentially definable in $\mathcal{R}_{\text{pfaff}}$?

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Definition

L is a **nested pfaffian map (over $\mathcal{R}$)** if each L_i is the graph of a map $f_i : \mathbb{R}^{n-i} \longrightarrow \mathbb{R}^i$, for $i > 0$.

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Conjecture 9

$\mathcal{N}'(\mathcal{R})$ is model complete relative to $\mathcal{R}$ and interdefinable with $\mathcal{N}(\mathcal{R})$.

Example ($n = 3$, $k = 2$)

Let $g_{11}, g_{12}, g_2 : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be definable,

$$\omega_1 := g_{11} dx_1 + g_{12} dx_2 - dx_3 \quad \text{and} \quad \omega_2 := g_2 dx_1 - dx_2$$

and $d_0 := \mathbb{R}^3$, $d_1 := \ker \omega_1$ and $d_2 := \ker \omega_1 \cap \ker \omega_2$.

Example ($n = 3$, $k = 2$)

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Let $L = (L_0, L_1, L_2)$ be a nested Rolle leaf of $d = (d_0, d_1, d_2)$, and assume L is a nested pfaffian map with associated $f_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}$ and $f_2 = (f_{21}, f_{22}) : \mathbb{R} \longrightarrow \mathbb{R}^2$.

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Then f_1 is pfaffian over $\mathcal{R}$ and $f_{22}(x_1) = f_1(x_1, f_{21}(x_1))$, but

$$f_{21}'(x_1) = g_2(x_1, f_{21}(x_1), f_1(x_1, f_{21}(x_1))).$$

nested pfaffian chains

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Definition

The tuple $(f_1, f_{21}, \ldots, f_{k1})$ is a **nested pfaffian chain (over $\mathcal{R}$).**

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Problem 2'

Are nested pfaffian chains over $\mathcal{R}$ existentially definable in $\mathcal{R}_{\text{pfaff}}$?