Pfaffian functions vs Rolle leaves

Patrick Speissegger

joint work with Gareth Jones

Ravello, 13 June 2013
Let $\mathcal{R}$ be an o-minimal expansion of the real field $\overline{\mathbb{R}}$. 
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**Definition (van den Dries, Gabrielov)**

A pfaffian function over $\mathcal{R}$ is a component of some pfaffian chain over $\mathcal{R}$. 

Example 1: the exponential function is a "classical" pfaffian chain: each $g_{ij}$ is polynomial.
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**Definition (van den Dries, Gabrielov)**

$f = (f_1, \ldots, f_k) : \mathbb{R}^n \to \mathbb{R}^k$ is a **pfaffian chain** (of length $k$) over $\mathcal{R}$ if there are definable $g_{ij} : \mathbb{R}^{n+i} \to \mathbb{R}$ such that

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\[ R_{\text{pfaff}} := \text{expansion of } R \text{ by all pfaffian functions over } R \]
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**Theorem 1 (Khovanskii 1979)**

The quantifier-free definable sets in $\mathcal{R}_{\text{pfaff}}$ have finitely many connected components.
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Still open:

Conjecture 5

\( \mathcal{R}_{\text{pfaff}} \) is model complete relative to \( \mathcal{R} \).
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Line fields determined by:

$L \subseteq M$ a leaf of $d$, called a **leaf over** $\mathcal{R}$
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3. The image of every trajectory of the vector field 
\[ -y \frac{\partial}{\partial x} + (x - y) \frac{\partial}{\partial y} \text{ in } \mathbb{R}^2 \setminus \{0\} \text{ is an integral manifold of } d_s. \]
Definition (Moussu & Roche 1991, based on Khovanskii 1979)

$L$ is a **Rolle leaf** if $L$ is a closed, embedded submanifold of $M$ and, for every $C^1$ curve $\gamma : [0, 1] \to M$ with $\gamma(0), \gamma(1) \in L$, there exists $t \in [0, 1]$ such that $\gamma'(t)$ is tangent to $d(\gamma(t))$. 

Example 1

The leaves of $ds$ are not Rolle, as they spiral around the origin.

Example 2

Rolle's Theorem states that every leaf of $dh$ is a Rolle leaf.

Lemma 6 (Khovanskii 1979)

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**Theorem 7** (Lion & Rolin for $\mathcal{R} = \mathbb{R}$, S 1999) $\mathcal{P}(\mathcal{R})$ is o-minimal.

**Remarks**

1. Every Rolle leaf over $\mathcal{P}(\mathcal{R})$ is definable in $\mathcal{P}(\mathcal{R})$.

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**Definition (Lion & S 2010)**

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Example

If \( f = (f_1, \ldots, f_k) : \mathbb{R}^n \rightarrow \mathbb{R}^k \) is a pfaffian chain over \( \mathcal{R} \) with associated \( g_{ij} \), set \( d_0 := \mathbb{R}^{n+k} \) and \( L_0 := \mathbb{R}^{n+k} \) and, for \( i = 1, \ldots, k \), set \( \omega_i := g_{i1} dx_1 + \cdots + g_{in} dx_n - dx_{n+i} \) and

\[
d_i := \ker \omega_1 \cap \cdots \cap \ker \omega_i \quad \text{and} \quad L_i := \text{gr}(f_i) \times \mathbb{R}^{k-i}.
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Every nested Rolle leaf over $\mathcal{R}$ is definable in $\mathcal{P}(\mathcal{R})$.

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$\mathcal{N}(\mathcal{R})$ is model complete relative to $\mathcal{R}$ and interdefinable with $\mathcal{P}(\mathcal{R})$, provided $\mathcal{R}$ admits analytic cell decomposition.
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So, one way to approach Conjecture 5 is to consider the following:

Question

Is every nested Rolle leaf over $\mathcal{R}$ existentially definable in $\mathcal{R}_{\text{pfaff}}$?
Problem 1

While a Rolle leaf over $\mathcal{R}$ can be covered by finitely many graphs of functions satisfying pfaffian equations over $\mathcal{R}$, the domains of these functions are generally not definable in $\mathcal{R}$.
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Problem 2

Are nested Rolle leaves over $\mathcal{R}$ even locally existentially definable in $\mathcal{R}_{pfaff}$?
two problems

Problem 1
While a Rolle leaf over $\mathcal{R}$ can be covered by finitely many graphs of functions satisfying pfaffian equations over $\mathcal{R}$, the domains of these functions are generally not definable in $\mathcal{R}$. Are they at least piecewise existentially definable in $\mathcal{R}_{\text{pfaff}}$?

Problem 2
Are nested Rolle leaves over $\mathcal{R}$ even locally existentially definable in $\mathcal{R}_{\text{pfaff}}$?
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What about nested Rolle leaves over $\mathcal{R}$?

Problem 2
Are nested Rolle leaves over $\mathcal{R}$ even locally existentially definable in $\mathcal{R}_{\text{pfaff}}$?
Let \( d = (d_0, \ldots, d_k) \) be a definable nested distribution on \( \mathbb{R}^n \) and \( L = (L_0, \ldots, L_k) \) be a nested Rolle leaf of \( d \).
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**Definition**

\( L \) is a **nested pfaffian map** (over \( \mathcal{R} \)) if each \( L_i \) is the graph of a map \( f_i : \mathbb{R}^{n-i} \rightarrow \mathbb{R}^i \), for \( i > 0 \).

\( \mathcal{N}'(\mathcal{R}) := \) expansion of \( \mathcal{R} \) by all nested pfaffian maps over \( \mathcal{R} \)
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**Conjecture 9**

\( \mathcal{N}'(\mathcal{R}) \) is model complete relative to \( \mathcal{R} \) and interdefinable with \( \mathcal{N}(\mathcal{R}) \).
Example ($n = 3$, $k = 2$)

Let $g_{11}, g_{12}, g_2 : \mathbb{R}^3 \rightarrow \mathbb{R}$ be definable,

\[
\omega_1 := g_{11} \, dx_1 + g_{12} \, dx_2 - dx_3 \quad \text{and} \quad \omega_2 := g_2 \, dx_1 - dx_2
\]

and $d_0 := \mathbb{R}^3$, $d_1 := \text{ker} \, \omega_1$ and $d_2 := \text{ker} \, \omega_1 \cap \text{ker} \, \omega_2$. 
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Let $L = (L_0, L_1, L_2)$ be a nested Rolle leaf of $d = (d_0, d_1, d_2)$, and assume $L$ is a nested pfaffian map with associated $f_1 : \mathbb{R}^2 \rightarrow \mathbb{R}$ and $f_2 = (f_{21}, f_{22}) : \mathbb{R} \rightarrow \mathbb{R}^2$. 
Example \((n = 3, k = 2)\)

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\(f_1 : \mathbb{R}^2 \to \mathbb{R}\) and \(f_2 = (f_{21}, f_{22}) : \mathbb{R} \to \mathbb{R}^2\).

Then \(f_1\) is pfaffian over \(\mathcal{R}\) and \(f_{22}(x_1) = f_1(x_1, f_{21}(x_1))\), but

\[
f'_{21}(x_1) = g_2(x_1, f_{21}(x_1), f_1(x_1, f_{21}(x_1))).
\]
Let $d = (d_0, \ldots, d_k)$ be a definable nested distribution on $\mathbb{R}^n$ and $L = (L_0, \ldots, L_k)$ be a nested Rolle leaf of $d$, and assume that $L$ is a nested pfaffian map with corresponding $f_i : \mathbb{R}^{n-i} \rightarrow \mathbb{R}^i$. 

Definition

The tuple $(f_1, f_2, \ldots, f_k)$ is a nested pfaffian chain (over $\mathbb{R}$).
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**Definition**

The tuple $(f_1, f_{21}, \ldots, f_{k1})$ is a **nested pfaffian chain (over $\mathbb{R}$)**.
Let $d = (d_0, \ldots, d_k)$ be a definable nested distribution on $\mathbb{R}^n$ and $L = (L_0, \ldots, L_k)$ be a nested Rolle leaf of $d$, and assume that $L$ is a nested pfaffian map with corresponding $f_i : \mathbb{R}^{n-i} \rightarrow \mathbb{R}^i$.

**Definition**

The tuple $(f_1, f_{21}, \ldots, f_{k1})$ is a **nested pfaffian chain (over $\mathcal{R}$)**.

**Problem 2’**

Are nested pfaffian chains over $\mathcal{R}$ existentially definable in $\mathcal{R}_{pfaff}$?