Merzlyakov-type theorems after Sela

Part I

Goals & Motivation

Question (Tarski, 1946)

Let $\mathbb{F}_n$ be the free group of rank n. Is $\bigcap_{n=2}^{\omega} Th(\mathbb{F}_n)$ complete?

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The following chain of groups is elementary

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Theorem (Sela)

The common theory of non abelian free groups T_{fg} is stable.

Questions & Problems

- *Understand* $Def(T_{fg})$*, e.g. definable/interpretable groups, fields;*
- *Identify regular types;*
- *Characterize the superstable part;*
- *Understand forking independence;*
- *Does* T_{fg} *has nfcp?*
- *What does a saturated model of* T_{fg} *look like?*

$\mathbb{F}$ is a finitely generated non abelian free group.
$\Sigma(\bar{x}, \bar{y}) \subset_{\text{finite}} \langle \bar{x}, \bar{y} \rangle$.

Theorem (Merzlyakov)

Let $\mathbb{F} \models \forall \bar{x} \exists \bar{y} (\Sigma(\bar{x}, \bar{y}) = 1)$. *Then there exists a retract* $r : G_\Sigma \to \langle \bar{x} \rangle$, *where* $G_\Sigma := \langle \bar{x}, \bar{y} \mid \Sigma(\bar{x}, \bar{y}) \rangle$.

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Theorem (Extended Merzlyakov)

Let $(\bar{b}_n)_{n<\omega}$ *be a "test sequence" in* $\mathbb{F}$. *Suppose for each* n *there is* $\bar{c}_n$ *such that* $\mathbb{F} \models \Sigma(\bar{b}_n, \bar{c}_n) = 1$. *Then there exists a retract* $r : G_\Sigma \to \langle \bar{x} \rangle$.

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Combine "Extended Merzlyakov" with Sela's "Diophantine envelopes" technique.

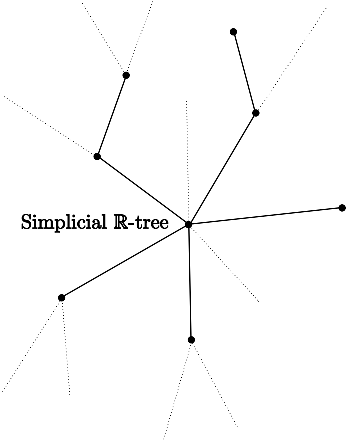
Main tools and notions

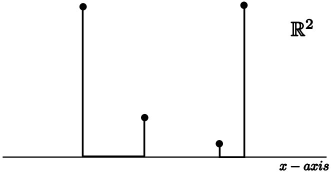
- Real trees ($\mathbb{R}$-trees), group actions on $\mathbb{R}$-trees, Rips' Machine;
- Limit groups, Sela's shortening argument;
- Towers, Hyperbolic Towers, Diophantine Envelopes.

Real trees

Definition

An $\mathbb{R}$-tree T is a geodesic metric space such that for any two points $x, y \in T$ there is a unique arc joining them.





Non-simplicial $\mathbb{R}$-tree

$$d((x_1, y_1), (x_2, y_2)) = \begin{cases} |y_2 - y_1| & \text{if } x_1 = x_2 \\ |y_1| + |x_2 - x_1| + |y_2| & \text{if } x_1 \neq x_2 \end{cases}$$

Alternatively: an $\mathbb{R}$-tree is a geodesic 0-hyperbolic metric space.

Definition (Rips)

A geodesic metric space is called δ-hyperbolic if every geodesic triangle is δ-thin.

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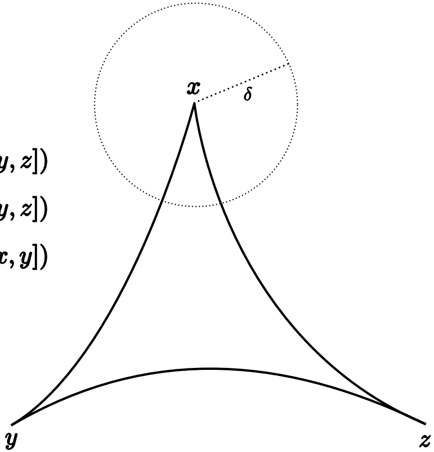

Hyperbolic metric spaces

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Definition (Gromov)

A based metric space $(X, *, d)$ is called (δ)-hyperbolic if it satisfies Gromov's three point condition:

- for $x, y \in X$, let $(x \cdot y)_* = \frac{1}{2}(d(*, x) + d(*, y) - d(x, y))$;
- then for any $x, y, z \in X$, $(x \cdot y)_* \geq min((x \cdot z)_*, (y \cdot z)_*) - \delta$.

Group actions on $\mathbb{R}$-trees

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Motivation: Bass-Serre Theory

Theorem

Suppose G acts freely on a simplicial tree T (a contractible $1 \geq$-dimensional CW-complex). Then G is a free group.

Proof.

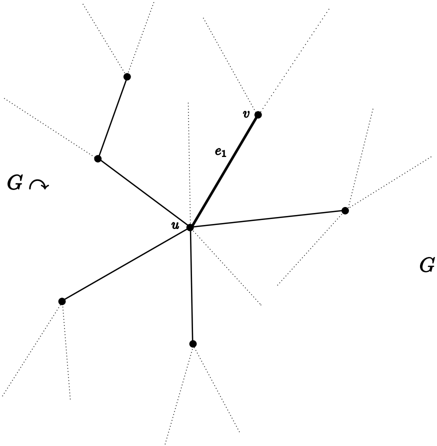
G acts freely $\Rightarrow$ G acts by a covering space action. Thus, $G \cong \pi_1(T/G)$, but T/G is a graph. □

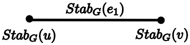
Corollary (Nielsen-Schreier Theorem)

Let $G < \mathbb{F}$. Then G is a free group.

Theorem (Main Theorem)

Suppose G acts (without inversions) on a simplicial tree T. Then G splits as a graph of groups, where the underlying graph is T/G, vertex groups come from vertex stabilizers and edge groups from edge stabilizers.





$$G \cong Stab_G(u) *_{Stab_G(e_1)} Stab_G(v)$$

Some types of actions on $\mathbb{R}$-trees

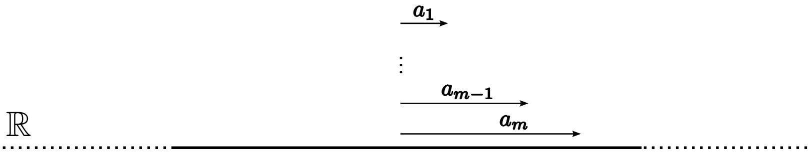
Suppose G acts on a real tree T (by isometries). Then the action is of:

- *Discrete type*, if branching points in T form a dicrete closed set and the orbit of each point is discrete. It is essentially an action on a simplicial tree as in the Bass-Serre theory;
- *Axial type*, if T is isometric to $\mathbb{R}$, and the orbit of each point is dense in T;
- *Surface type*, if $ker\lambda \hookrightarrow G \twoheadrightarrow L$ where $ker\lambda$ is the kernel of the action and $L := \pi_1(\Sigma)$, where Σ is a surface with (possibly empty) boundary carrying an arational measured foliation and T is dual to $\tilde{\Sigma}$, i.e. T is the lifted leaf space in $\tilde{\Sigma}$ after identifying leaves of distance 0 (with respect to the pseudometric induced by the measure);

Axial type

- Let $a \in \mathbb{R}$ and $f_a \in Isom^+(\mathbb{R})$ be defined by $f_a(x) = a + x$;
- Let $\mathbb{Z}^m := \langle z_1, \dots, z_m \rangle$ and $h : \mathbb{Z}^m \to Isom^+(\mathbb{R})$ defined as follows:

$$h(z_i) = f_{a_i} \text{ and } \{a_1, \dots, a_m\} \text{ is linearly independent}$$



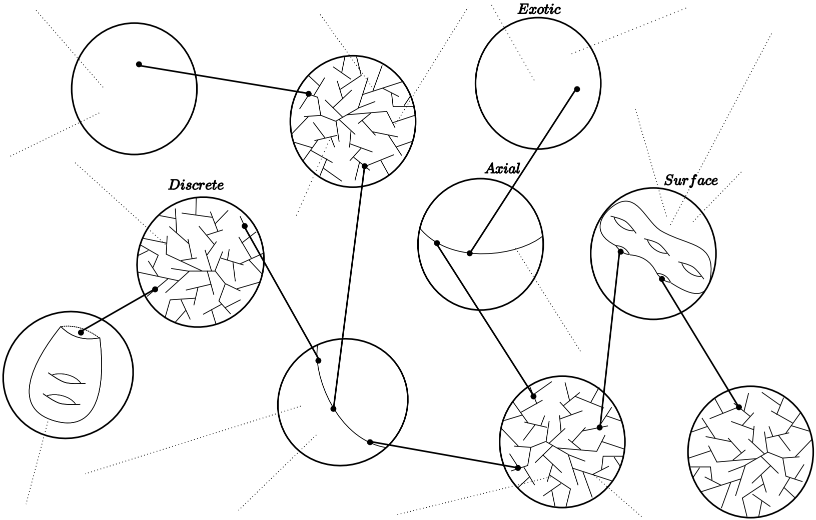
Rips' Machine

Suppose G acts on a real tree T. Then the action is:

- *minimal*, if there is no G-invariant proper subtree;
- *non-trivial*, if there is no globally fixed point;
- *super-stable*, if for any arc I and subarc $J \subset I$ we have that $Stab(J) \neq Stab(I) \Rightarrow Stab(I) = \{1\}$.

Theorem (Rips' Machine)

Let G be a finitely generated group. Suppose G acts non-trivially and minimally on an $\mathbb{R}$-tree T. Moreover, assume that the action is super-stable and tripod stabilizers are trivial. Then the action can be understood in terms of simpler components which are of discrete, axial, surface or exotic type


Exotic
Discrete
Axial
Surface

Bestvina-Paulin Method

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Let G be a (discrete) group and consider the space of G-equivariant pseudo-metrics equipped with the compact-open topology.
Recall that:

$$(d_i)_{i<\omega} \to d \text{ if and only if } d_i(1,g) \to d(1,g)\ \forall g \in G$$

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- An action $G \curvearrowright (X, *, d_X)$ on a based metric space induces a G-equivariant pseudo-metric:

$$d(g,h) = d_X(g \cdot *, h \cdot *)$$

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- A morphism $h : G \to H$ (H finitely generated) induces an action $G \curvearrowright X_H$:

$$(g,x) \mapsto h(g)x$$

Theorem A

Let Γ be a δ-hyperbolic group. Let $(h_n)_{n<\omega} : G \to \Gamma$ be a sequence of pairwise non-conjugate morphisms. Then there exists a sequence of base points $(*_n)_{n<\omega}$ in X_Γ and a sequence of rescaling constants $(r_n)_{n<\omega} \in \mathbb{R}^+$ such that a subsequence of the induced pseudo-metrics $(d_n/r_n)_{n<\omega}$ converges to a pseudo-metric d which is induced by a non-trivial action of G on a real tree $(T, *)$.

Proof.

Fix a finite generating set S for G. Consider the function $f_n : X_\Gamma \to [0, \infty)$, given by:

$$f_n(x) = max_{s\in S}\{d_{X_\Gamma}(x, h_n(s) \cdot x)\}$$

- Take $*_n$ to be a point minimizing the value of f_n; and
- $r_n = f_n(*_n)$.

□

- $f_n(x) = max_{s \in S}\{d_{X_\Gamma}(x, h_n(s) \cdot x)\}$;
- let $*_n$ be a point minimizing the value of f_n and $r_n = f_n(*_n)$;
- $r_n \to \infty$ (Hint: h_n are pairwise non-conjugate);
  - Suppose not, and $r_n < M$;
  - then for each h_n there is $\gamma_n \in \Gamma$ such that $|\gamma_n^{-1} h_n(s) \gamma_n|_\Gamma < M$ (assume that $*_n$ is a vertex in X_Γ and take $\gamma_n = *_n$);
  - so $d_{X_\Gamma}(\gamma_n, h_n(s) \cdot \gamma_n) \leq max_{s \in S}\{d_{X_\Gamma}(\gamma_n, h_n(s) \cdot \gamma_n)\} < M$;
  - but only finitely many morphisms exist that send the generating set of G in the ball of radious M, a contradiction.
- let $\hat{d}_n := \frac{d_n}{r_n}$. Then $\hat{d}_n(1, s) \leq 1$, $\forall s \in S$;
  - $d_n(1, s) = d_{X_\Gamma}(*_n, h_n(s) \cdot *_n)$; and
  - $r_n = max_{s \in S}\{d_{X_\Gamma}(*_n, h_n(s) \cdot *_n)\}$.
- a subsequence of $(\hat{d}_n)_{n<\omega}$ converges to d (Exercise);
- $(G, 1, \hat{d}_n)$ is $\frac{\delta}{r_n}$-hyperbolic, thus $(G, 1, d)$ is 0-hyperbolic.

Lemma (Connecting the dots)

Let $(X, *, d)$ be a 0-hyperbolic metric space. Then there exists an $\mathbb{R}$-tree (T, d_T) and an isometric embedding $i : X \to T$ such that:

- *no proper subtree of T contains $i(X)$;*
- *furthermore, if a group G acts by isometries on X, then the action extends to an isometric action on T.*

- $f_n(x) = max_{s\in S}\{d_{X_\Gamma}(x, h_n(s)\cdot x)\}$;
- let $*_n$ be a point minimizing the value of f_n and $r_n = f_n(*_n)$;
- $r_n \to \infty$ (Hint: h_n are pairwise non-conjugate);
- let $\hat{d}_n := \frac{d_n}{r_n}$. Then $\hat{d}_n(1, s) \leq 1$, $\forall s \in S$;
- a subsequence of $(\hat{d}_n)_{n<\omega}$ converges to d;
- $(G, 1, \hat{d}_n)$ is $\frac{\delta}{r_n}$-hyperbolic, thus $(G, 1, d)$ is 0-hyperbolic;
- d is induced by an action of G on the real tree $(T, *, d_T)$;
  - $d(1, g) = d_T(*, g \cdot *)$, for any $g \in G$.

Lemma (Approximating Sequences)

Assume $(X_\Gamma, *_n, d_{X_\Gamma})_{n<\omega}$ *"converges" to* $(T, *, d_T)$ *as in Theorem A. Then for any* $x \in T$*, the following hold:*

- *there exists a sequence* $(x_n)_{n<\omega} \in X_\Gamma$ *such that* $\frac{d_{X_\Gamma}}{r_n}(x_n, g \cdot *_n) \to d_T(x, g \cdot *)$ *for any* $g \in G$*, we call such a sequence an approximating sequence;*
- *if* $(x_n)_{n<\omega}$ *is an approximating sequence for* x*, then* $(g \cdot x_n)_{n<\omega}$ *is an approximating sequence for* $g \cdot x$*;*
- *if* $(x_n)_{n<\omega}, (y_n)_{n<\omega}$ *are approximating sequences for* x, y *respectively, then* $\frac{d_{X_\Gamma}}{r_n}(x_n, y_n) \to d_T(x, y)$*.*

- $f_n(x) = max_{s \in S}\{d_{X_\Gamma}(x, h_n(s) \cdot x)\}$;
- let $*_n$ be a point minimizing the value of f_n and $r_n = f_n(*_n)$;
- $r_n \to \infty$ (Hint: h_n are pairwise non-conjugate);
- let $\hat{d}_n := \frac{d_n}{r_n}$. Then $\hat{d}_n(1, s) \leq 1$, $\forall s \in S$;
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- d is induced by an action of G on the real tree $(T, *, d_T)$;
- let $x \in T$ be a globally fixed point and $(x_n)_{n<\omega}$ be an approximating sequence for x. Then $\frac{d_{X_\Gamma}}{r_n}(x_n, s \cdot x_n) \to 0$ for any $s \in S$, a contradiction to the choice of $*_n$.

Theorem A

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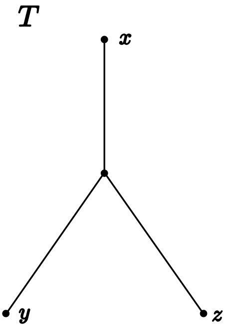
Theorem B

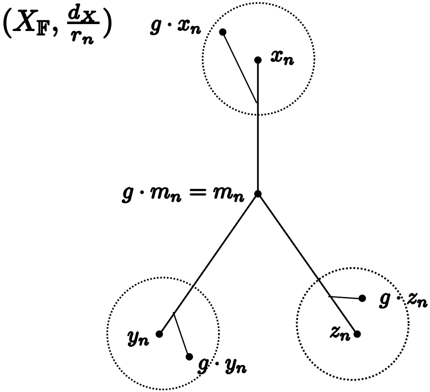
Assume the hypothesis of Theorem A and moreover that Γ is torsion-free. Let $G \curvearrowright^\lambda T$ be the action in the limit. Then $G/\ker\lambda$ acts on T as follows:

- *tripod stabilizers are trivial;*
- *arc stabilizers are abelian;*
- *the action is super-stable.*

Proof of Theorem B (case $\Gamma := \mathbb{F}$)

Suppose g gixes a tripod in T. Then $h_n(g)$ fixes m_n for all large enough n, but then $h_n(g) = 1$ for all large enough n. Thus $g \in ker\lambda$.

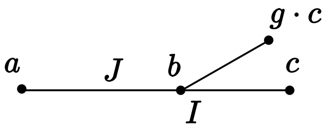




Claim I: Arc stabilizers are abelian.
Proof: Exercise.

Claim II: The action is super-stable.
Proof:

- Let J be a subarc of I, with $Stab(J) \neq Stab(I)$ and $Stab(I) \neq \{1\}$;
- Suppose $g \in Stab(J) \setminus Stab(I)$;



- Let $\gamma \in Stab(I)$;
- Since γ commutes with g (Claim I), we have that γ fixes a tripod, thus γ is trivial, a contradiction.

Definition

The quotient $G/ker\lambda$ obtained by a sequence $(h_n)_{n<\omega} : G \to \mathbb{F}$ is called a limit group.

Theorem (Sela)

Let L be a limit group. Then:

- *L is finitely presented;*
- *every abelian subgroup is finitely generated;*
- *either L is abelian or it admits a non-trivial cyclic splitting.*

Fact

Let L be a finitely generated group. The following are equivalent:

- *L is a limit group;*
- *L is ω-residually free;*
- *$L \models Th_\forall(\mathbb{F})$ for some free group $\mathbb{F}$ (including $\mathbb{Z}$);*