The Zilber-Pink Conjecture
Diophantine geometry, functional transcendence, differential algebra and o-minimality

Thomas Scanlon

UC Berkeley

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The plan

- Origins of Zilber's conjecture in the theory of $\mathbb{C}_{\exp}$
- Ax's functional Schanuel conjecture
- Differential algebraic generalizations, especially in the work of Bertrand and Pillay
- Pink's formulation with mixed Shimura varieties
- O-minimal proofs and uses of Ax-Lindemann-Weierstrass

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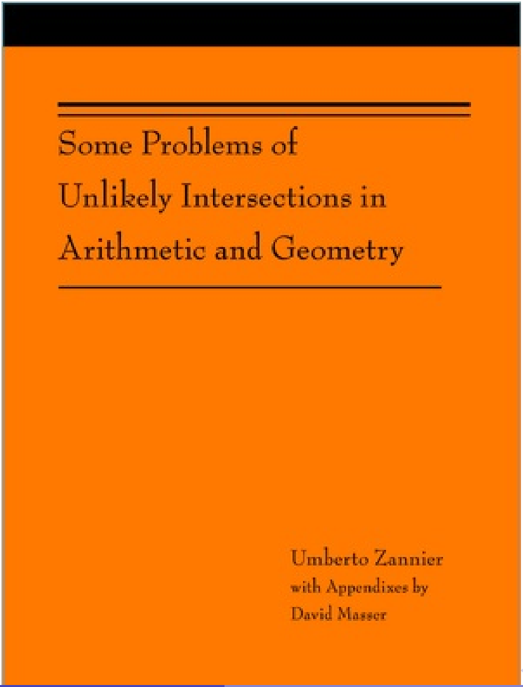
Some Problems of
Unlikely Intersections in
Arithmetic and Geometry
Umberto Zannier
with Appendixes by
David Masser

Planned omissions

- Bombieri-Masser-Zannier theory of anomalous intersections
- More generally, the strictly Diophantine geometric approaches to the conjecture
- Pila-Wilkie o-minimal counting and (most of) its applications

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General form of Pink-Zilber conjectures

We are given an algebraic variety X and a collection $\mathscr{S}$ of irreducible special subvarieties of X.

For each positive integer e, we define

$$\mathscr{S}^{[e]} := \bigcup_{Y \in \mathscr{S}, \dim(Y) = e} Y .$$

More generally, we write

$$\mathscr{S}^{[<e]} := \bigcup_{n<e} \mathscr{S}^{[n]} \text{ and } \mathscr{S}^{[\leq e]} := \bigcup_{n \leq e} \mathscr{S}^{[n]} .$$

The Zilber-Pink conjecture for this context asserts that if $Z \subseteq X$ is a subvariety which is not contained in a proper special subvariety of X, then $Z \cap \mathscr{S}^{[<\operatorname{codim}_X(Z)]}$ is not Zariski dense in Z.

At this level of generality, the Pink-Zilber conjecture is either meaningless or just plain false. To find cogent specifications of the classes of special varieties is an important aspect of this research program.

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Caveat on attribution

It is not correct to attribute the formulation of this conjecture, even in some of the more precise versions which go under the name of the Pink-Zilber conjecture, solely or even primarily to Pink and Zilber.

- The idea of considering anomalous intersections is due to Bombieri-Masser-Zannier and Zilber.
- Pink's enunciation of an amalgam of the Mordell-Lang, Manin-Mumford and André-Oort conjectures using the language of mixed Shimura varieties was influenced by earlier suggestions of André.
- Refinements and further generalizations both conjectural and in very few cases proven are due to many other people including Bertrand, Ghioca, Habegger, Maurin, Tucker, Ullmo, and Yafaev amongst others.

Zilber-Pink and heights

In some cases, the Zilber-Pink conjecture includes an assertion about the set

$$Z \cap \mathscr{S}^{[\leq \mathrm{codim}_X(Z)]}$$

(namely, under the stronger hypothesis that Z is not contained in a weakly special variety that this intersection is a set of bounded height).

However, there are many cases in which the Zilber-Pink conjecture should be true, but this refined statement about complementary dimensional intersections is known to be false.

Zilber-Pink examples

- $X := \mathbb{G}_m^g$ ($X(\mathbb{C}) = (\mathbb{C}^\times)^g$) and $\mathscr{S}$ is the collection of all components of algebraic subgroups of X (Originally considered by Bombieri-Masser-Zannier and independently by Zilber under the name of the Conjecture on Intersection with Tori; details to follow)
- X an abelian variety ($X(\mathbb{C}) \cong \mathbb{C}^g/\Lambda$ (with Λ a lattice) $\cong (S_1)^{2g}$) and $\mathscr{S}$ the collection of all components of algebraic subgroups.
- $k \subseteq K$ an extension of algebraically closed fields, X a variety over k and $\mathscr{S}$ the collection of all k-subvarieties of X. (Chatzidakis-Ghioca-Masser-Maurin)
- X a mixed Shimura variety and $\mathscr{S}$ the collection of components of images of mixed Shimura varieties under generalized Hecke correspondences (Pink's generalization)
- Not an example: $X = \mathbb{A}^n$ and $\mathscr{S}$ the set of all affine subspaces defined over $\mathbb{Q}$.

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Recall that the Manin-Mumford conjecture asserts that if $Y \subseteq X$ is a closed irreducible subvariety which is not a translate by a torsion point of an algebraic subgroup of X, then the set of torsion points of X lying on Y is not Zariski dense in Y.

Replacing X by the smallest algebraic group containing Y (and then taking components and translating by a torsion point if need be), we may assume that Y is not contained in any proper special subvariety if X.

The set

$$\mathscr{S}^{[0]} = \bigcup \{Z : \dim(Z) = 0, Z \text{ irreducible component of algebraic subgroup of } X\}$$

is the set of torsion points. Zilber-Pink asserts that $Y \cap \mathscr{S}^{[<\mathrm{codim}_X(Y)]} \supseteq Y \cap \mathscr{S}^{[0]}$ is not Zariski dense in Y.

Pink-Zilber implies Manin-Mumford

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EXPONENTIAL SUMS EQUATIONS AND THE SCHANUEL CONJECTURE

BORIS ZILBER

Abstract

A uniform version of the Schanuel conjecture is discussed that has some model-theoretical motivation. This conjecture is assumed, and it is proved that any 'non-obviously-contradictory' system of equations in the form of exponential sums with real exponents has a solution.

1. *Introduction*

In [**8**] we started a model-theoretical study of the formal theory of exponentiation (pseudo-exponentiation). The crux of the analysis is the observation that the Schanuel conjecture on the degree of algebraic independence between complex numbers and their exponentials (see [**5**]) is responsible for very basic geometric properties of fields allowing a function ex satisfying $\mathrm{ex}(x+y) = \mathrm{ex}(x) \cdot \mathrm{ex}(y)$. It turns out that in the class of fields with pseudo-exponentiation, given an uncountable cardinal, there is a unique 'right' one of the given cardinality. This model we call *canonical*. The mere fact of the canonicity tempts us to conjecture that the classical exponentiation on the complex numbers is formally equivalent to the pseudo-exponentiation in a canonical model, or, even more concretely, the structure of complex numbers in the language $(+, \cdot, \exp)$ is the canonical model of the field with pseudo-exponentiation of cardinality continuum. This is a very strong conjecture, which presumes among other properties the Schanuel conjecture.

As a matter of fact, the analysis in [**8**] shows that the 'other properties' could be reduced to a unique and natural one: the property of *exponential-algebraic closedness*, meaning that any 'non-obviously-inconsistent' system of equations has a solution in the field. In this paper this condition is explained in precise technical terms (*normal* and *free* system of equations).

On the other hand, the further logical analysis [**9**] of pseudo-exponentiation shows that the Schanuel conjecture is much better motivated under an extra conjecture of Diophantine type (the conjecture on intersections with tori). We show that this conjecture holds in a 'function field case', using a result of J. Ax.

- We must show that if X is an abelian variety over $\mathbb{C}$, $\Gamma < X(\mathbb{C})$ is a finite dimensional subgroup, and $Y \subseteq X$ is an irreducible subvariety which is not a translate of an algebraic subgroup of X, then $Y \cap \Gamma$ is not Zariski dense.
- Translating and replacing X if need be, we may assume that Y is not contained in a translate of a proper algebraic subgroup of X.
- Assuming that we have a counterexample to Mordell-Lang, we find a generic sequence $(a_i)_{i=0}^{\infty}$ from $Y \cap \Gamma$ by which we mean that $\mathrm{tp}(a_i)$ converges to the generic type of Y. Equivalently, for every proper subvariety $Z \subsetneq Y$ the set $\{i \in \omega \ : \ a_i \in Z\}$ is finite.

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- Set $d := \dim(X)$, $e := \dim(Y)$, and $r := \dim_{\mathbb{Q}}(\Gamma \otimes \mathbb{Q})$.
- Noting that $e < d$, we may find k so that $rd < k(d - e)$.
- If $(\gamma_1, \ldots, \gamma_k) \in \Gamma^k$, then because $\dim_{\mathbb{Q}} \Gamma \otimes \mathbb{Q} = r$, there are $k - r$ independent $\mathbb{Z}$-linear forms vanishing on $(\gamma_1, \ldots, \gamma_k)$. These together define an algebraic subgroup $T < X^k$ of dimension rd. Thus, $\Gamma^k \cap U^k \subseteq U^k \cap \mathscr{S}^{[rd]}$.
- Because $rd < kd - ke = \operatorname{codim}_{X^k}(Y^k)$, by ZP, there is a finite list $T_1, \ldots, T_n$ of algebraic groups of dimension at most $k(d - e) - 1$ and points $c_1, \ldots, c_n$ so that $\Gamma^k \cap U^k \subseteq \bigcup c_i T_i$.

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- By Ramsey's Theorem, we may assume that $(a_{i_1}, \ldots, a_{i_k}) \in c_1 T_1$ for all increasing k-tuples $i_1 < \ldots < i_k$.
- Replacing k with a smaller number if need be, we may assume that $T \cap ((0, \ldots, 0) \times X) = (0, \ldots, 0) \times T'$ where $T' < X$ is a proper subgroup of X.
- Looking at fibres, we see that all a_i (for $i > k - 1$) belong to the same coset of T'. This violates genericity and our reduction.

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- ker(exp) is cycle
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- ACF First-order
- $\exp(x + y) = \exp(x)\exp(y)$ First order
- exp is onto the multiplicative group First order
- ker(exp) is cycle Omitting a partial type
- Exponential-algebraic closedness First order by "function field" CIT
- Schanuel's conjecture Requires CIT for first-order formulation

Schanuel's conjecture

Conjecture

If $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ *are linearly independent over* $\mathbb{Q}$*, then* $\operatorname{tr.deg} \mathbb{Q}(\alpha_1, \ldots, \alpha_n, \exp(\alpha_1), \ldots, \exp(\alpha_n)) \geq n$*.*

Reformulated geometrically, if $W \subseteq \mathbb{A}^n \times \mathbb{G}_m^n$ is an irreducible affine variety over $\mathbb{Q}^{\text{alg}}$ of dimension $< n$ for which there is a point $(a_1, \ldots, a_n) \in \mathbb{C}^n$ with $(a_1, \ldots, a_n; \exp(a_1), \ldots, \exp(a_n)) \in W(\mathbb{C})$, then a belongs to some proper subspace defined over $\mathbb{Q}$.

Equivalently: The set of points of the form $(a_1, \ldots, a_n; \exp(a_1), \ldots, \exp(a_n))$ is contained in

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In Zilber's conjecture, a torus is an algebraic subgroup $T \leq (\mathbb{C}^\times)^g$ of some Cartesian power of the multiplicative group of complex numbers.

More concretely, T is defined by a system of equations of the form

$$\prod_{j=1}^{g} x_j^{\alpha_{i,j}} = 1$$

where each $\alpha_{i,j}$ is an integer.

For most authors, a torus must be connected which from the monomial equations defining the group means that the matrix

$$\begin{pmatrix} \alpha_{1,1} & \cdots & \alpha_{1,g} \\ \vdots & \ldots & \vdots \\ \alpha_{m,1} & \cdots & \alpha_{m,g} \end{pmatrix} \in M_{m \times g}(\mathbb{Z})$$

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Conjecture

For each (not necessarily irreducible) algebraic variety $W \subset \mathbb{A}^g_{\mathbb{Q}}$ defined over the rational numbers, there is finite set $\{T_1, \ldots, T_n\}$ of proper tori so that for every torus $T \leq (\mathbb{C}^\times)^g$ if S is a component of $W \cap T$ with $\dim(S) > \dim(W) + \dim(T) - g$, then $S \subseteq T_j$ for some $j \leq n$.

- Since W need not be irreducible, it is easy to deduce the corresponding conjecture for W defined over $\mathbb{Q}^{\mathrm{alg}}$ by replacing W by the union of its $\mathrm{Gal}(\mathbb{Q}^{\mathrm{alg}}/\mathbb{Q})$-conjugates.
- A uniform version over $\mathbb{C}$ may be deduced via a compactness argument, but the correct statement requires *weakly special* varieties: Let B be an algebraic variety and $W \subseteq \mathbb{A}^g_B = \mathbb{A}^g \times B$ a family of affine varieties parametrized by B. Then there is a finite set $\{T_1, \ldots, T_n\}$ of proper tori so that for every every parameter $b \in B$ there are points $c_1, \ldots, c_n \in (\mathbb{C}^\times)^g$ so that for every torus $T \leq (\mathbb{C}^\times)^g$ if S is anomalous in the sense that it is a component of $W_b \cap T$ with $\dim(S) > \dim(W) + \dim(T) - g$, then $S \subseteq c_j T_j$ for some $j \leq n$.

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Using CIT to write Schanuel's conjecture

Assuming CIT, we may express Schanuel's conjecture as a first-order theory. The reduction is routine but not completely trivial.

The key is to show that (under the assumptions of CIT and SC) if $W \subseteq \mathbb{G}_a^n \times \mathbb{G}_m^n$ is a variety of dimension $< n$, then there is a finite set $S(W)$ of proper tori $T < \mathbb{G}_m^n$ so that if

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Some details of the implication from CIT and SC to first-order expressibility: first steps

- Let $\pi : W \to W' \subseteq \mathbb{G}_m^n$ be the projection to the last n coordinates.
- Working by Noetherian induction on W, we may assume that $\dim(W_{\exp(a)}) = \dim(W) - \dim(W') =: d$ is the generic fibre dimension.
- By SC, (because $\operatorname{tr.deg}(a, \exp(a)) < n$) there is a proper torus $T < \mathbb{G}_m^n$ with $\exp(a) \in T$.
- Minimizing T, by SC again (because the multiplicative rank of $\exp(a)$ would be $\dim(T)$), we have $\dim(T) \leq \operatorname{tr.deg}(a, \exp(a))$.

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Details, continued

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$$\begin{aligned}\dim(T) &\leq \operatorname{tr.deg}(a, \exp(a)) \\ &\leq d + \dim(T \cap W')\end{aligned}$$

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Hence, $\exp(a)$ must lie on an atypical component of $W' \cap T$. By CIT, we may cover the atypical intersections by finitely many proper tori.

Unlike the Schanuel condition, exponential closedness can be expressed unconditionally due to the truth of a weakened form of CIT.

Schanuel proposed a converse to the conjecture we have been calling Schanuel's Conjecture: if K is a countable field with an exponential $E : K \to K^\times$ having $\ker E \cong \mathbb{Z}$, then $(K, +, \cdot, E)$ embeds into $\mathbb{C}_{\exp}$.

Zilber's axiomatization of exponential closedness captures the finitary content of the reverse Schanuel Conjecture: for each irreducible algebraic variety $V \subseteq \mathbb{G}_a^g \times \mathbb{G}_m^g$, there should be a point of the form $(a_1, \ldots, a_g, \exp(a_1), \ldots, \exp(a_g)) \in V(\mathbb{C})$ provided that this does not produce an obvious counterexample to the Schanuel conjecture.

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Exponential closedness, more precisely

Each matrix $M \in M_{\ell \times g}(\mathbb{Z})$ gives a map $[M] : \mathbb{G}_a^g \times \mathbb{G}_m^g \to \mathbb{G}_a^\ell \times \mathbb{G}_m^\ell$ via the rule $(x_1, \ldots, x_g; y_1, \ldots, y_g) \mapsto$
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We say that an irreducible variety $Y \subseteq \mathbb{G}_a^g \times \mathbb{G}_m^g$ is exponentially normal if for every $M \in M_{\ell \times g}(\mathbb{Z})$ of rank ℓ one has $\dim([M]Y) \geq \ell$. The variety is free if its projection to $\mathbb{G}_a^g$ is not contained in any translate of a proper subspace defined over $\mathbb{Q}$ and its projection to $\mathbb{G}_m^g$ is not contained in any translate of a proper subtorus.

Exponential closedness is the assertion that for every such exponentially normal and free variety Y there are points of the form $(a_1, \ldots, a_g; \exp(a_1), \ldots, \exp(a_g)) \in Y(\mathbb{C})$.

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First-order expressibility of exponential closedness

Several parts of the expression of exponential closedness are not obviously first-order. For example, that one may express that Y is irreducible as a first-order condition on the coefficients of some set of defining polynomials is true but nontrivial.

Issues specific to this case include saying that

- the projection to $\mathbb{G}_a^g$ is not contained in a translate of a proper $\mathbb{Q}$-vector space,
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- the projection to $\mathbb{G}_m^g$ is not contained in a translate of a proper subtorus, and weak CIT
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Very unlikely intersections

Given a variety X, a collection $\mathscr{S}$ of special subvarieties, and $Y \subseteq X$ an irreducible subvariety not contained in any proper special subvariety, we say that $a \in Y$ belongs to very unlikely intersection if there is some $T \in \mathscr{S}^{[<\operatorname{codim}_X Y]}$ for which $\dim_a(Y \cap T) > 0$.

By the weak Pink-Zilber conjecture (for X, $\mathscr{S}$, and Y) we mean the assertion that the set of very unlikely intersection points is not Zariski dense in Y. When specialized to $X = \mathbb{G}_m^g$ and $\mathscr{S}$ the collection of components of subtori, we will call this statement the weak CIT.

Using the weak CIT, the exponential closedness axioms may be shown to be first-order expressible.

Unlike the CIT, the weak CIT is a theorem whose proof using differential algebra and then generalized using model theoretic differential algebra generalizes to other cases of the weak Zilber-Pink conjecture.

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On Schanuel's conjectures

By JAMES AX*

In this paper proofs are given of conjectures of Schanuel on the algebraic relations satisfied by exponentiation in a differential-algebraic setting. The methods and results are then used to give new proofs and generalizations of the theorems of Chabauty, Kolchin, and Skolem.

1. Introduction

(i) *Statement of the conjectures and our main results.* S. Schanuel has made a conjecture [1, p. 30–31] concerning the exponential function which embodies all its known transcendentality properties such as the theorems of Lindemann [2, p. 225 or 1, Ch. VII, § 2, Th. 1], Baker [3, Cor. 1, 2, and 4, Th. 1, 2], and other results (e.g. [1, Ch. II, Th. 1; Ch. V, Th. 1]) and implies a whole collection of special conjectures (e.g. [1, p. 11, Remark], [5, p. 138, Problems 1, 7, 8] and the algebraic independence of π and e over $\mathbf{Q}$).

The conjecture runs as follows:

(S) Let $y_1, \cdots, y_n \in \mathbf{C}$ be $\mathbf{Q}$-linearly independent. Then

$$\dim_{\mathbf{Q}} \mathbf{Q}(y_1, \cdots, y_n, e^{y_1}, \cdots, e^{y_n}) \geqq n .$$

Here $\dim_E F$, for any extension of fields F/E, denotes the cardinality of

Ax's differential Schanuel conjecture

Theorem

Let $\alpha_1, \ldots, \alpha_n \in t\mathbb{C}[[t]]$ *be a sequence of* $\mathbb{Q}$*-linearly independent formal power series over the complex numbers having no constant terms. Then* $\operatorname{tr.deg}_{\mathbb{C}} \mathbb{C}(\alpha_1, \ldots, \alpha_n, \exp(\alpha_1), \ldots, \exp(\alpha_n)) \geq n + 1$.

Theorem

Let (K, ∂) *be a differential field of characteristic zero with field of constants* $C := \{x \in K \ : \ \partial(x) = 0\}$ *and elements* $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in K$ *satisfying*

- $\partial(\alpha_i) = \frac{\partial(\beta_i)}{\beta_i}$ *for all* $i \leq n$ *and*
- $\partial(\alpha_1), \ldots, \partial(\alpha_n)$ *are linearly independent over* $\mathbb{Q}$.

Then $\operatorname{tr.deg}_C C(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n) \geq n + 1$

The formal theorem is a consequence of the differential version taking $K = \mathbb{C}((t))$, $\partial = \frac{d}{dt}$ and $\beta_i := \exp(\alpha_i)$.

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The formal theorem is a consequence of the differential version taking $K = \mathbb{C}((t))$, $\partial = \frac{d}{dt}$ and $\beta_i := \exp(\alpha_i)$.

Uniformity from Ax-Schanuel

Theorem

If $Z \subseteq (\mathbb{G}_a^g \times \mathbb{G}_m^g) \times B$ is a family of irreducible subvarieties of dimension at most g, then there is finite collection of nontrivial integer vectors $M^{[1]} = (M_1^{[1]}, \ldots, M_g^{[1]}), \ldots, M^{[n]} = (M_1^{[n]}, \ldots, M_g^{[n]})$ so that for any parameter $b \in B$ and any infinite component of $Z_b \cap \mathit{Graph}(\exp)$ equations of the form $\sum M_j^{[i]} x_j = c$ and $\prod y_j^{M_j^{[i]}} = c$ hold.

- If this were false, we would deduce that it would be consistent to have a differential field K extending $\mathbb{C}$, a point $b \in B(K)$, and $(\alpha_1, \ldots, \alpha_g; \beta_1, \ldots, \beta_g) \in Z_b(K)$ for which $\partial \alpha_i = \frac{\partial \beta_i}{\beta_i}$ (for $i \leq g$) and $\partial \alpha_1, \ldots, \partial \alpha_g$ are $\mathbb{Q}$-linearly independent contradicting Ax's theorem.
- How? If the theorem were false, then for each finite list of nontrivial integer vectors we could find some b and a component S of $Z_b \cap \mathrm{Graph}(\exp)$ on which none of the stated equations held. Taking a nonconstant analytic curve $z \mapsto (\alpha_1(z), \ldots, \alpha_g(z); \beta_1(z), \ldots, \beta_g(z)) \in S(\mathbb{C})$ we find a point in the differential field of meromorphic functions satisfying this finite fragment of the type.

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The (uniform) weak CIT follows from the uniform Ax-Schanuel theorem:

- Consider a family $Z \subseteq \mathbb{G}_m^g \times B$ of irreducible subvarieties of $\mathbb{G}_m^g$ of dimension $d < g$.
- By uniform Ax-Schanuel applied to the family of varieties $H \times Z_b$ (where H ranges through the family of affine spaces of dimension $< g - d$) the projections to the multiplicative group of the infinite components of $(H \times Z_b) \cap \mathrm{Graph}(\exp)$ are contained in translates of one of a finite list of tori.
- Taking H to be a component of $\log T$ where T is a translate of a torus with $\dim(T) < g - d$ we see that all of the points of very unlikely intersection are contained in the intersections with translates from a finite list of tori independent of parameters.

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Recall that for any commutative ring k and k-algebra R there is a universal k-derivation $d : R \to \Omega_{R/k}$ there $\Omega_{R/k}$ is an R-module.

The map $d\log : R^\times \to \Omega_{R/k}$ given by $a \mapsto \frac{1}{a}da$ is a homomorphism from the multiplicative group of R to the module of differentials.

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If $C \subseteq F$ is an inclusion of fields of characteristic zero, then then the natural map $F \otimes_{\mathbb{Z}} d\log(F) \to \Omega_{F/C}/dF$ is injective.

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- More concretely, the proposition asserts that whenever elements $\alpha_1, \ldots, \alpha_n$, and β of F satisfy an equation of the form $\sum c_i \frac{d\beta_i}{\beta_i} = d\alpha$ with $c_1, \ldots, c_n \in C$, then $d\log(\beta_1), \ldots, d\log(\beta_n)$ are $\mathbb{Q}$-linearly dependent.
- Arguing by induction on n and on $\mathrm{tr.deg}_C(F)$, we may assume that F is a function field of a smooth curve X over C and that the c_i's are $\mathbb{Q}$-linearly independent.
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- Assuming that $\operatorname{tr.deg}_C C(\alpha, \beta) \leq n$, $\dim \Omega_{C(\alpha,\beta)/C} \leq n$ so that there must be a nontrivial linear relation of the form $\sum f_j(d\alpha_j - \frac{d\beta_j}{\beta_j}) = gd\beta_1$.
- We may extend the derivation to a map $D : \Omega \to \Omega$ via the rule $D(xdy) = \partial(x)dy + xd(\partial(y))$,
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Kolchin's logarithmic derivatives

In general, if G is a commutative algebraic group over the constants of a differential field K, there is a differential algebraic homomorphism

$$\partial \log_G : G(K) \to T_0 G(K) \cong K^g$$

where $T_0 G$ is the tangent space to G at the identity and $g = \dim G$ coming from the trivialization of the tangent bundle of G.

$$G(K) \xrightarrow{\nabla} TG(K) = G(K) \times T_0 G(K) \longrightarrow T_0 G(K)$$

Associated to the commutative algebraic group G over the constants, we have the logarithmic differential equation $\partial \log_G(x) = \partial y$ where we identify $T_0 G$ with its own tangent space.

Note that when G is defined over the complex numbers and $x = \exp_G(y)$ where $\exp_G$ is the analytic exponential map coming from Lie theory, then (x, y) satisfies the logarithmic differential equation. The logarithm $\log_G : G(\mathbb{C}) \to T_0 G(\mathbb{C})$ is analytic but not well-defined in that the exponential map (usually) has a kernel. In differentiating we eliminate the ambiguity.

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Ax's differential Schanuel theorem says that if $(x, y) \in \mathbb{G}_m^g \times T_0\mathbb{G}_m^g$ satisfies the logarithmic differential equation, then either $\mathrm{tr.deg}_C\, C(x, y) \geq g + 1$ (where C is the field of constants) or $\partial(y)$ belongs to a proper subspace defined over $\mathbb{Q}$.

In generalizing, to other algebraic groups, it does not suffice to ask merely that the coordinates of $\partial(y)$ be $\mathbb{Q}$-linearly independent. For example, if E is an elliptic curve with complex multiplication by some quadratic imaginary α, and t is nonconstant, then the point $(t, \alpha t; \exp_E(t), \exp_E(\alpha t))$ has transcendence degree at most two over C.

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The theory of the exponential differential equations of semiabelian varieties

Jonathan Kirby

Abstract. The complete first-order theories of the exponential differential equations of semiabelian varieties are given. It is shown that these theories also arise from an amalgamation-with-predimension construction in the style of Hrushovski. The theories include necessary and sufficient conditions for a system of equations to have a solution. The necessary conditions generalize Ax's differential fields version of Schanuel's conjecture to semiabelian varieties. There is a purely algebraic corollary, the "Weak CIT" for semiabelian varieties, which concerns the intersections of algebraic subgroups with algebraic varieties.

Mathematics Subject Classification (2000). Primary 12H20; Secondary 03C60, 14L10.

1. Introduction

1.1. The exponential differential equation

Let $\langle F; +, \cdot, D\rangle$ be a differential field of characteristic zero, and consider the

Theorem (Kirby)

If G is a semiabelian variety over the constants C of a differential field K of characteristic zero and $(x, y) \in (G \times T_0(G))(K)$ satisfies the logarithmic differential equation, then either $\mathrm{tr.deg}_C\, C(x, y) \geq \dim(G) + 1$ *or there is a proper algebraic subgroup $H < G$ and a constant point $c \in TG(C)$ so that $(x, y) \in c + TH$.*

As with the deduction of weak CIT from Ax's differential-Schanuel theorem, Kirby deduces a weak form of the Zilber-Pink conjecture (namely for very unlikely intersections) for semiabelian varieties.

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A LINDEMANN-WEIERSTRASS THEOREM FOR SEMI-ABELIAN VARIETIES OVER FUNCTION FIELDS

DANIEL BERTRAND AND ANAND PILLAY

Contents

Bertrand-Pilay generalized Ax-Lindemann-Weierstraß

Bertrand and Pillay have generalized Kirby's theorem to the case that G is not necessarily defined over the constants. There are some highly nontrivial subtleties.

- The logarithmic differential equation involves the theory of Manin homomorphisms.
- In some ways, the conclusion is stronger because there are fewer constant points by which to translate, but one must make sense of the part which descends to the constants.
- Kirby's proof is a careful extension of Ax's differential algebraic argument. This part appears in the Bertrand-Pillay proof, but so does the socle theorem, Manin's theorem of the kernel, and generalized differential Galois theory.
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- Kirby's proof is a careful extension of Ax's differential algebraic argument. This part appears in the Bertrand-Pillay proof, but so does the socle theorem, Manin's theorem of the kernel, and generalized differential Galois theory.
- In another sense, the conclusion is weaker in that it is an Ax-Lindemann-Weierstraß theorem: assuming sufficient independence from algebraic subgroups, if $(x, y) \in G(K) \times T_0 G(K)$ satisfies the generalized logarithmic differential equation and $\mathrm{tr.deg}_C(C(y)) = 1$, then $\mathrm{tr.deg}_C(C(x)) = \dim G$.

Looking ahead to o-minimal approaches

With the Pila-Zannier o-minimal method to prove theorems of André-Oort/Manin-Mumford-type, there three crucial ingredients:

- a definability theorem showing that the relevant analytic covering maps are o-minimally definable,
- a theorem on the size of Galois orbits of special points showing, and
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Special varieties from analytic coverings: example of $\mathbb{G}_m^g$

Take $\mathfrak{X} := \mathbb{C}^g$, $X = \mathbb{G}_m^g$, and $E : \mathfrak{X} \to X(\mathbb{C})$ to be the map

$$(z_1, \ldots, z_g) \mapsto (\exp(2\pi i z_1), \ldots, \exp(2\pi i z_g))$$

Then,

- Relative to the usual real coordinates, the restriction of E to the semialgebraic set $\mathfrak{D} := \{(z_1, \ldots, z_g) \ : \ 0 \leq \mathrm{Re}(z_j) < 1 \text{ for } j \leq g\}$ is o-minimally definable and surjective.
- The special points are the g-tuples of roots of unity and have large Galois orbits, of size greater than $C_\epsilon n^{1-\epsilon}$ for $0 < \epsilon < 1$ where n is the order of the torsion point.
- The weakly special subvarieties $Y \subseteq \mathbb{G}_m^g$ are the translates of subtori and these are precisely the varieties for which $E^{-1}Y(\mathbb{C})$ is a translate of a vector space defined over $\mathbb{Q}$.

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- That the class of special subvarieties of $\mathbb{G}_m^g$ which on its own does not form a definable family may be seen to come from the rational points of a definable family of algebraic subvarieties of $\mathbb{C}^g$ is the starting point of many applications of o-minimal to Zilber-Pink problems. For example, Kirby and Zilber showed as an easy corollary of this observation that the real Schanuel conjecture already implies its own uniform version.
- This presentation is essential for the Pila-Zannier strategy.
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Uses of the definable presentation of the analytic covering

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In general, for those complex algebraic varieties X for which the Zilber-Pink conjecture should hold, we should have an analytic covering $E : \mathfrak{X} \to X(\mathbb{C})$ for which

- $\mathfrak{X}$ is an open semialgebraic subset of some ambient complex algebraic variety $\widetilde{\mathfrak{X}}$,
- $\mathfrak{X}$ is a homogeneous space for some open subgroup $G(\mathbb{R})^+$ of a real algebraic group,
- there is a semi-algebraic subset $\mathfrak{D} \subseteq \mathfrak{X}$ for which the restriction of E to $\mathfrak{D}$ is surjective and o-minimally definable, and
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Special varieties from analytic coverings: general picture

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- Pink works with mixed Shimura varieties and then defines the class of special varieties in terms of generalized Hecke correspondences.
- A theorem of Ullmo and Yafaev suggests an alternate definition of the special varieties: An irreducible variety $Y \subseteq X$ should be (weakly) special just in case each component of $E^{-1}Y(\mathbb{C})$ is (the intersection with $\mathfrak{D}$ of) an algebraic subvariety of $\widetilde{\mathfrak{X}}$.

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A Combination of the Conjectures of Mordell–Lang and André–Oort

Richard Pink

Dept. of Mathematics, ETH-Zentrum, CH-8092 Zürich, Switzerland
pink@math.ethz.ch

Summary. We propose a conjecture combining the Mordell–Lang conjecture with an important special case of the André–Oort conjecture, and explain how existing results imply evidence for it.

1 Introduction

We begin with some remarks on the history of related conjectures; the reader wishing to skip them may turn directly to Conjecture 1.6. Let us start (arbitrarily) with the following theorem.

Theorem 1.1 (Mordell–Weil). *For any abelian variety A over a number field K, the group of rational points $A(K)$ is finitely generated.*

(Mixed) Shimura varieties

A (connected) mixed Shimura variety is given by group theoretic $(G, \mathfrak{X}, \Gamma)$ where

- G is an algebraic group over $\mathbb{Q}$,
- $\mathfrak{X}$ is a complex manifold on which $G(\mathbb{R})$ acts transitively,
- Γ is an arithmetic group,
- $X(\mathbb{C}) = \Gamma \backslash \mathfrak{X}^+$ is an algebraic variety defined over a number field, and
- several other necessary and nontrivial, but technical, conditions hold.

The additional conditions include

- a presentation of $\mathfrak{X}^+$ as a conjugacy class of cocharacters,
- requirements on the Hodge decomposition of the adjoint representation of G, and
- a reductivity hypothesis for pure Shimura varieties.

The quintessential mixed Shimura varieties are the universal abelian schemes over the moduli spaces of abelian varieties with fixed polarization and level structure, $\mathscr{X}_g \to \mathscr{A}_g$.

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The j-function

A complex elliptic curve may be presented as $E(\mathbb{C}) = \mathbb{C}/\Lambda$ where Λ is a lattice. After an appropriate change of basis, one may take Λ to have the form $\Lambda_\tau := \mathbb{Z} + \mathbb{Z}\tau$ where $\mathrm{Im}(\tau) > 0$. A simple computation shows that $\mathbb{Z} + \mathbb{Z}\tau = \mathbb{Z} + \mathbb{Z}\tau'$ if and only if there is some $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ with $\tau' = \frac{a\tau+b}{c\tau+d}$.

The analytic j-function is a map $j : \mathfrak{h} \to \mathbb{C}$ (where $\mathfrak{h} := \{z \in \mathbb{C} \ : \ \mathrm{Im}(z) > 0\}$) having the property that $j(\tau) = j(\tau')$ if and only if $\Lambda_\tau = \Lambda_{\tau'}$.

Here $\mathfrak{X} = \mathfrak{H}$, $G = \mathrm{PSL}_2$ acting by fractional linear transformations, and $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$.

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Universal elliptic curve

Using the Weierstraß $\wp$-function we may represent the universal family of elliptic curves as an analytic quotient.

$$\begin{array}{ccc} \mathfrak{h} \times \mathbb{C} & \longrightarrow & \mathscr{E}(\mathbb{C}) \\ \widetilde{\pi} \downarrow & & \downarrow \pi \\ \mathfrak{h} & \xrightarrow{j} & \mathbb{C} \end{array}$$

where we realize $\mathscr{E}(\mathbb{C})_{j(\tau)}$ as $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$.
Here $\mathfrak{X} = \mathfrak{h} \times \mathbb{C}$, $G = \mathrm{PSL}_2 \ltimes \mathbb{G}_a^2$, and $\Gamma = \mathrm{PSL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$.

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Morphisms of mixed Shimura varieties and special subvarieties

Given two mixed Shimura data $(G_1, \mathfrak{X}_1, \Gamma_1)$ and $(G_2, \mathfrak{X}_2, \Gamma_2)$ with corresponding mixed Shimura varieties $X_1 = \Gamma_1 \backslash \mathfrak{X}_1$ and $X_2 = \Gamma_2 \backslash \mathfrak{X}_2$, a map $\phi : X_1 \to X_2$ of varieties is a morphism mixed Shimura varieties if it lifts to an equivariant map $(\widetilde{\phi}, \bar{\phi}) : (G_1, \mathfrak{X}_1) \to (G_2, \mathfrak{X}_2)$.

Note in particular, if $\Gamma_1 \leq \Gamma_2$ has finite index, then there is a morphism $\pi_{\Gamma_1, \Gamma_2} : X_1 \to X_2$ corresponding to the natural quotient $\Gamma_1 \backslash \mathfrak{X}_1 \to \Gamma_2 \backslash \mathfrak{X}_2$.

The image of a mixed Shimura variety is called special. Given two morphisms of Shimura varieties $\alpha : X \to Y$ and $\beta : X \to Z$, and a point $y \in Y$, each component of $\beta(\alpha^{-1}\{y\})$ is called weakly special.

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Special subvarieties of $\mathscr{X}_g$

- Morally, the special subvarieties of $\mathscr{A}_g$, the moduli space of abelian varieties of dimension g (with polarization and level structure notationally suppressed) are the submoduli varieties. More precisely, they correspond to moduli varieties for certain restrictions on the Hodge structure and this is generally true for pure Shimura varieties.
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Warning: a special subvarieties of relative abelian varieties need not be (components of) group schemes

Special points and Poincaré bi-extensions

by Daniel Bertrand, with an *Appendix* by Bas Edixhoven

April 2011

The context is the following :

i) in a joint project with D. Masser, A. Pillay and U. Zannier [7], we aim at extending to semi-abelian schemes the Masser-Zannier approach [11] to Conjecture 6.2 of R. Pink's preprint [13]; this conjecture also goes under the name "*Relative Manin-Mumford*". Inspired by Anand Pillay's suggestion that the semi-constant extensions of [6] may bring trouble, I found a counter-example, which is described in Section 1 below.

ii) at a meeting in Pisa end of March, Bas Edixhoven found a more concrete way of presenting the counter-example, with the additional advantage that the order of the involved torsion points can be controlled in a precise way : this is the topic of the Appendix.

iii) finally, I realized that when rephrased in the context of mixed Shimura varieties, the construction, far from providing a counter-example, actually *supports* Pink's general Conjecture 1.3 of [13]; a sketch of this view-point is given in Section 2.

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A CHARACTERIZATION OF SPECIAL SUBVARIETIES

EMMANUEL ULLMO AND ANDREI YAFAEV

Abstract. We prove that an algebraic subvariety of a Shimura variety is weakly special if and only if analytic components of its preimage in the symmetric space are algebraic. We also prove an analogous result in the case of abelian varieties.

§1. *Introduction.* The aim of this note is to obtain a new characterization of special subvarieties of Shimura varieties. For generalities on Shimura varieties we refer to [3, 5] or [12].

Let (G, X) be a Shimura datum and X^+ a connected component of X. We let K be a compact open subgroup of $G(\mathbb{A}_f)$ and $\Gamma := G(\mathbb{Q})_+ \cap K$ where $G(\mathbb{Q})_+$ denotes the stabilizer in $G(\mathbb{Q})$ of X^+. We let $S := \Gamma\backslash X^+$, a connected component of $\mathrm{Sh}_K(G, X)$.

A special subvariety of S is a subvariety of Hodge type in the sense of [15]. In §2 we give a description of slightly more general notion of weakly special subvarieties in terms of sub-Shimura data of (G, X). In [15] Moonen proves that a subvariety of S is weakly special if and only if it is a totally geodesic submanifold of S. A special point is a special subvariety of dimension zero and a weakly special subvariety containing a special point is special.

Special subvarieties are interesting for many reasons, one of which is the

Dual algebraicity

Given a Shimura variety presented via a Shimura datum $(G, \mathfrak{X}, \Gamma)$, there are various ways to realize $\mathfrak{X}$ as an open semialgbraic subset of some complex algebraic variety $\widetilde{\mathfrak{X}}$. For the purposes of their characterization theorem, Ullmo and Yafaev work with the Borel embedding.

Theorem

We are given

- X *is a Shimura variety defined over* $\mathbb{Q}^{alg}$
- *given by a Shimura datum* $(G, \mathfrak{X}, \Gamma)$ *and*
- *analytic covering map* $\pi : \mathfrak{X} \to \Gamma \backslash \mathfrak{X} = X(\mathbb{C})$ *with*
- $\mathfrak{X} \hookrightarrow \widetilde{\mathfrak{X}}$ *its the Borel embedding, and*
- $Y \subseteq X$ *an irreducible subvariety.*

Then Y *is a weakly special subvariety if and only if each component* $\mathfrak{Y}$ *of* $E^{-1}Y$ *is algebraic in the sense that there is an algebraic variety* $Z \subseteq \widetilde{\mathfrak{Y}}$ *with* $Z \cap \mathfrak{X} = \mathfrak{Y}$*. Moreover,* Y *is special if and only if both* Y *and* $\mathfrak{Y}$ *are defined over* $\mathbb{Q}^{alg}$*.*

THE RATIONAL POINTS OF A DEFINABLE SET

J. PILA and A. J. WILKIE

Abstract
Let $X \subset \mathbb{R}^n$ be a set that is definable in an o-minimal structure over $\mathbb{R}$. This article shows that in a suitable sense, there are very few rational points of X which do not lie on some connected semialgebraic subset of X of positive dimension.

1. Introduction

This article is concerned with the distribution of rational and integer points on certain nonalgebraic sets in $\mathbb{R}^n$. To contextualize the kind of results sought and, in particular, to motivate the present setting of definable sets in o-minimal structures over $\mathbb{R}$ (see Definition 1.7), we begin by describing earlier results.

The ideas pursued here grew from the article [4] of Bombieri and Pila, where a technique using elementary real-variable methods and elementary algebraic geometry was used to establish upper bounds for the number of integer points on the graphs of functions $y = f(x)$ under various natural smoothness and convexity hypotheses. Results were obtained for f variously assumed to be (sufficiently) smooth, algebraic, or real analytic. Several results concerned the *homothetic dilation* of a fixed graph $X : y = f(x)$.

Definition 1.1
Let $X \subset \mathbb{R}^n$. For a real number $t \geq 1$ (which is always tacitly assumed), the *homothetic dilation* of X by t is the set $tX = \{\langle tx_1, \ldots, tx_n\rangle : \langle x_1, \ldots, x_n\rangle \in X\}$. By $X(\mathbb{Z})$ we denote the subset of X comprising the points with integer coordinates.

Suppose now that X is the graph of a function $f : [0, 1] \to \mathbb{R}$. Trivially, one has $\#(tX)(\mathbb{Z}) \leq t + 1$ (with equality, e.g., for $f(x) = x$ and positive integral t). According to Jarník [15], a strictly convex arc $\Gamma : y = g(x)$ of length ℓ contains at most

Pila-Wilkie counting theorem: set-up

- For $q = \frac{a}{b} \in \mathbb{Q}^\times$ written in lowest terms, we define $H(q) := \max\{|a|, |b|\}$ (and $H(0) := 0$).
- For $q = (q_1, \ldots, q_n) \in \mathbb{Q}^n$ we define $H(q) := \max\{h(q_i) \ : \ i \leq n\}$
- For $X \subseteq \mathbb{R}^n$ and $t > 0$ we define $X(\mathbb{Q}, t) := \{q \in X \cap \mathbb{Q}^n \ : \ H(q) \leq t\}$ and $N(X, t) := \# X(\mathbb{Q}, t)$.
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Pila-Wilkie counting theorem

Theorem

Let $X \subseteq \mathbb{R}^n$ be definable in some o-minimal expansion of the real field. Then for every $\epsilon > 0$ there is a constant $C = C_\epsilon$ so that $N(X^{tr}, t) \leq Ct^\epsilon$.

The counting theorem admits a natural generalization to counting algebraic points of bounded degree. Specifically, for each $d \in \mathbb{Z}_+$ and $X \subseteq \mathbb{R}^n$, we define $N_d(X, t) := \#\{(a_1, \ldots, a_n) \in X \ : \ [K(a_i) : K] \leq d \text{ and } H(a_i) \leq t \text{ for } i \leq n\}$.

Theorem

Let $X \subseteq \mathbb{R}^n$ be definable in some o-minimal expansion of the real field. Then for every $d \in \mathbb{Z}_+$ and $\epsilon > 0$ there is a constant $C = C_{\epsilon,d}$ so that $N_d(X^{tr}, t) \leq Ct^\epsilon$.

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Number theory. — *Rational points in periodic analytic sets and the Manin–Mumford conjecture*, by JONATHAN PILA and UMBERTO ZANNIER, communicated by U. Zannier on 18 April 2008.

ABSTRACT. — We present a new proof of the Manin–Mumford conjecture about torsion points on algebraic subvarieties of abelian varieties. Our principle, which admits other applications, is to view torsion points as rational points on a complex torus and then compare (i) upper bounds for the number of rational points on a transcendental analytic variety (Bombieri–Pila–Wilkie) and (ii) lower bounds for the degree of a torsion point (Masser), after taking conjugates. In order to be able to deal with (i), we discuss (Thm. 2.1) the semialgebraic curves contained in an analytic variety supposed invariant under translations by a full lattice, which is a topic with some independent motivation.

KEY WORDS: Torsion points on algebraic varieties; rational points on analytic varieties; conjecture of Manin–Mumford.

MATHEMATICS SUBJECT CLASSIFICATION (2000): 11J95, 14K20, 11D45.

1. INTRODUCTION

The so-called *Manin–Mumford conjecture* was raised independently by Manin and Mumford and first proved by Raynaud [R1] in 1983; its original form stated that *a curve C (over $\mathbb{C}$) of genus ≥ 2, embedded in its Jacobian J, can contain only finitely many torsion points* (relative of course to the Jacobian group structure). Raynaud actually considered the more general case when C is embedded in any abelian variety. Soon afterwards, Raynaud [R2] produced a further significant generalization, replacing C and J respectively by a subvariety X in an abelian variety A; in this situation he proved that *if X contains a Zariski dense set of torsion points, then X is a translate of an abelian subvariety of A by a torsion point*. Other proofs (sometimes only for the case of curves) appeared later, due to Serre, Coleman, Hindry, Buium, Hrushovski (see [Py1]), Pink & Roessler [PR], and M. Baker & Ribet [BR]. We also remark that a less deep precedent of this problem was an analogous question for multiplicative algebraic groups, raised by Lang already in the '60s. (See [L];

Pila-Zannier method

Working with a mixed Shimura variety X with analytic covering $E : \mathfrak{X} \to X(\mathbb{C})$ and o-minimally definable restriction to a fundamental domain $\widetilde{E} := E \upharpoonright \mathfrak{D} : \mathfrak{D} \to X(\mathbb{C})$, we consider some irreducible variety $Y \subseteq X$ which contains a Zariski dense set of special points.

Through various steps, one reduces to considering the situation that Y is defined over some number field K and that there is a generic sequence (ξ_i) of special points on Y so that $[K(\xi_i) : K] \to \infty$ and $E^{-1}(\xi_i) \subseteq \widetilde{E}^{-1} Y$ is a set of algebraic points of bounded degree but of cardinality greater than would be allowed by the counting theorem.

To finish, under the assumption that Y is not special, one must show that it is possible to take $E^{-1}(\xi_i) \subseteq \widetilde{E}^{-1} Y$.

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By definition, $\widetilde{E}^{-1}Y(\mathbb{C})^{\mathrm{alg}}$ is the union of all infinite, connected semialgebraic subsets, but because E is complex analytic, one may compute the algebraic part as the union of the images of nonconstant algebraic functions into $\mathfrak{Y}$.

In the case the $\mathfrak{X} = \mathbb{C}^g$ and $X = \mathbb{G}_m^g$, then Ax's theorem implies that if $f = (f_1, \ldots, f_g) : \Delta := \{z \in \mathbb{C} \ : \ |z| < 1\} \to \mathbb{C}^g$ is an algebraic function for which no nontrivial $\mathbb{Q}$-linear combination is constant, then the image of $(\exp(2\pi i f_1(t)), \ldots, \exp(2\pi i f_g(t))$ is Zariski dense.

In general, the Ax-Lindemann-Weierstraß statement (in the sense of Pila) for $E : \mathfrak{X} \to X$ should be that if $f : \Delta \to \mathfrak{X}$ is an algebraic function for which the image of $E \circ f$ is not contained in any (weakly) special subvariety of X, then this image is Zariski dense.

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Ax-Lindemann-Weierstraß and the algebraic part

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Differential algebraic proof of ALW

In the case that X is a semiabelian variety and $E : \mathfrak{X} \to X(\mathbb{C})$ is the corresponding exponential map, then for any algebraic function $f : \Delta \to \mathfrak{X}$, then point $(f, E \circ f)$ satisfies the logarithmic exponential equation for X and hence must have transcendence degree at least $1 + \dim X$ unless the image of $E \circ f$ is contained in a constant translate of a proper algebraic subgroup.

In general, there is a natural differential equation corresponding to a covering $E : \mathfrak{X} \to X$: the inverse $E^{-1} : X(\mathbb{C}) \to \mathfrak{X}$ is locally analytic (as long as Γ is small enough) and is well defined up to translation by $\Gamma < G(\mathbb{R}) < G(\mathbb{C})$. By elimination of imaginaries in the theory of differentially closed fields, there is a differential rational function χ on $\widetilde{X}$ so that for K-points in any differential field with field of constant $\mathbb{C}$, $\chi(x) = \chi(y)$ if and only if there is some $\gamma \in G(\mathbb{C})$ with $x = \gamma \cdot y$. The differential-analytic function $\ell := \chi \circ E^{-1}$ is well defined X. If $f : \Delta \to \mathfrak{X}$ is a holomorphic function, then $(x, y) := (f, E \circ f)$ satisfies the differential equation $\chi(x) = \ell(y)$.

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Sketch of proof of Modular Ax-Lindemann-Weierstraß

- We are given $E : \mathfrak{X} \to X(\mathbb{C})$, the analytic covering map expressing $X(\mathbb{C}) = \Gamma \backslash \mathfrak{X}$ where $(G, \mathfrak{X}, \Gamma)$ is a mixed Shimura datum and an irreducible subvariety $Y \subseteq X$ together with a semialgebraic fundamental domain $\mathfrak{D}$ for which the restriction of E to $\mathfrak{D}$ is o-minimally definable. Our charge is to describe $(\mathfrak{D} \cap E^{-1} Y(\mathbb{C}))^{\mathrm{alg}}$.
- Let $\mathfrak{Y}$ be a component of $E^{-1} Y$ having $\dim(\mathfrak{Y} \cap \mathfrak{D}) = \dim Y$. Using the fact that E is complex analytic, one shows that if $Z \subseteq (\mathfrak{Y} \cap \mathfrak{D})$ is a connected, semialgebraic set, then $\overline{Z}^{\mathrm{Zar}} \subseteq \mathfrak{Y}$. Thus, the problem is reduced to describing those complex algebraic varieties contained in $\mathfrak{Y}$.
- Let $Z \subseteq \mathfrak{Y}$ be a maximal, irreducible, positive dimensional algebraic subvariety with $\dim(Z \cap \mathfrak{D}) = \dim Z$.
- Towards showing that Z is stabilized by a large subgroup of $G(\mathbb{R})$, consider $S := \{g \in G(\mathbb{R}) \ : \ \dim(gZ \cap \mathfrak{Y} \cap \mathfrak{D}) = \dim Z\}$, a definable set.

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- Let $N := \{\gamma \in \Gamma \ : \ \overline{\mathfrak{D}} \cap \overline{\gamma\mathfrak{D}} \neq \varnothing\}$ which is a finite generating set for Γ.
- Following $\mathfrak{Y}$ out of $\mathfrak{D}$, it is easy to see that if $\gamma \in N$ with $\mathfrak{Y} \cap \gamma\mathfrak{D} \neq \varnothing$, then $\gamma \in S$. Continuing in this way, one finds that $S \cap \Gamma$ is fairly large.
- What is more difficult to establish (and has been done so in print only under additional hypotheses) is that $N(S \cap \Gamma, t)$ grows faster than allowed by the Pila-Wilkie bounds. In the case that $X(\mathbb{C})$ is compact, Ullmo and Yafaev achieve this by comparing the word metric on Γ to a G-invariant metric on $\mathfrak{X}$.
- It then follows that there is a semi-algebraic curve $C \subseteq S$ which we may take to contain the identity. The set $C \cdot Z$ is a connected semi-algebraic set contained in $\mathfrak{Y}$ and containing Z. Hence, by maximality, $C \cdot Z = Z$.
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- A further application of o-minimality, namely that a countable definable set must be finite, is used to complete the proof: the maximal algebraic components of $(\mathfrak{Y} \cap \mathfrak{D})^{\mathrm{alg}}$ may be seen as instance of a single family of varieties defined using rational parameters.
- It is relatively easy to complete the above sketch of ALW for semiabelian varieties.
- The case of $E : \mathfrak{H}^n \to \mathbb{C}^n$ given by $(\tau_1, \ldots, \tau_n) \to (j(\tau_1), \ldots, j(\tau_n))$ appears in Pila's proof of the André-Oort conjecture.
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Conclusion

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Thank you