On Reducts of Hrushovski's Non-Collapsed Construction

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Introduction

In this poster we show that Hrushovski's original non-collapsed construction has a proper reduct with an isomorphic geometry. This is a first step in a project aimed at finding the right setup for a reformulation of Zilber's conjecture.

Hrushovski-type Fraïssé Limits

Let $\mathcal{C}$ be a class of countably many (up to isomorphism) finite structures in a given language with $\mathcal{C}$ closed under isomorphism and taking substructures (i.e $A \subseteq B \in \mathcal{C} \implies A \in \mathcal{C}$). Let $d_0 : \mathcal{C} \to \mathbb{N}$ with

1. $d_0(\emptyset) = 0$
2. $d_0(X) \leq 1$ for any $X \in \mathcal{C}$ with $|X| = 1$
3. $d_0(A \cup B) \leq d_0(A) + d_0(B) - d_0(A \cap B)$ (Submodularity)

We call d_0 the *pre-dimension function* of $(\mathcal{C}, d_0)$.

Definition. Define $d(A, N)$, for A a finite substructure of N and N a structure whose age is a subset of $\mathcal{C}$, by

$$d(A, B) = min\{d_0(D) \mid A \subseteq D \subseteq B\}$$

We call d the *dimension function* associated with d_0.

Definition. For $A \subseteq B \in \mathcal{C}$ define $A \leqslant B$ if and only if $d(A, B) = d_0(A)$. If $A \leqslant B$ we say that A is *self-sufficient* or *strong* in B.

Definition. We say $(\mathcal{C}, d_0)$ is a Hrushovski-type amalgamation class if the following holds:

(AP) Let $A, B_1, B_2 \in \mathcal{C}$ with $B_1 \cap B_2 = A$ and $A \leqslant B_i$ for $i = 1, 2$. Then there exists some $D \in \mathcal{C}$ and embeddings $f_i : B_i \to D$ such that $f_1 \restriction A = f_2 \restriction A$ and $f_i B_i \leqslant D$ for $i = 1, 2$.

Theorem. *For $(\mathcal{C}, d_0)$ a Hrushovski-type amalgamation class, there is a single (up to isomorphism) countable structure M with age $\mathcal{C}$ that is homogeneous in the following sense:*

- *Any partial isomorphism $f : A \to B$ where $A, B \leqslant M$, extends to an automorphism of M.*

We call M the Fraïssé-Hrushovski limit of $(\mathcal{C}, d_0)$.

Definition. Let M be the Fraïssé-Hrushovski limit of a Hrushovski-type amalgamation class $(\mathcal{C}, d_0)$. Define the d-closure of A, a finite subset of M, to be

$$cl_d(A) = \{a \in M \mid d(A \cup \{a\}, M) = d(A, M)\}$$

For N an infinite subset of M, define its d-closure to be the union of the d-closures of all its finite subsets.

The d-closure operator defined above on M, in fact defines a pregeometry (M, cl_d), which in turn has an associated geometry. We call this associated geometry the geometry of M and denote it by $G(M)$.

The Original and Symmetric Constructions

Consider the language $\mathcal{L} = \{R\}$ where R is a ternary relation symbol. Assume that whenever $(a, b, c) \in R$ then a, b, c are distinct elements.

Define the following for finite $\mathcal{L}$-structures:

$$d_0(A) = |A| - |R^A|$$
$$\mathcal{C} = \{A \mid \text{A is a finite } \mathcal{L}\text{-structure with } d_0(B) \geq 0 \text{ for every } B \subseteq A\}$$
$$d_0^{\sim}(A) = |A| - |\{\{a, b, c\} \mid A \models R(a, b, c)\}|$$
$$\mathcal{C}_{\sim} = \{A \mid \text{A is a finite } \mathcal{L}\text{-structure with } d_0^{\sim}(B) \geq 0 \text{ for every } B \subseteq A\}$$

Theorem. *The pairs $(\mathcal{C}, d_0)$ and $(\mathcal{C}_{\sim}, d_0^{\sim})$ are Hrushovski-type amalgamation classes*

We denote the corresponding Fraïssé-Hrushovski limits of $(\mathcal{C}, d_0)$ and $(\mathcal{C}_{\sim}, d_0^{\sim})$ by M and $M_{\sim}$ respectively.

Theorem. *$G(M)$ and $G(M_{\sim})$ are isomorphic*

The structure M is commonly referred to as Hrushovski's non-collapsed construction. Similarly $M_{\sim}$ is Hrushovski's non-collapsed symmetric construction. Now, consider the formula

$$R_{\sim}(x_1, x_2, x_3) = \bigvee_{\sigma \in S_3} R(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})$$

Theorem. *The reduct $(M, R_{\sim})$ is a proper reduct of the structure M, and it is isomorphic to $M_{\sim}$.*

Definition of the Reduct M_s

We prove that $M_{\sim}$, too, has a proper reduct with an isomorphic geometry. Consider the formula

$$S(x_1, x_2, x_3) := \exists x_4, x_5[\bigwedge_{i \in \{1,2,3\}} R_{\sim}(x_i, x_4, x_5) \wedge \bigwedge_{i \neq j} x_i \neq x_j]$$

We denote the reduct $(M_{\sim}, S)$ by M_s. As it so happens, this will be a proper reduct of $M_{\sim}$ with an isomorphic geometry. In fact M_s is, too, the limit of a Hrushovski-type amalgamation class.

A New Kind of Dimension Function

We would like to define some d_{0s}, a pre-dimension function for the age of M_s, that will allow us to construct M_s as a Fraïssé-Hrushovski limit.

- Difficulty: A function of the sort $A \mapsto c_1|A| - c_2|S^A|$ is unbounded from below on the age of M_s, and so cannot be an appropriate pre-dimension function.

  Solution: We count the sizes of maximal cliques under the relation S (with some minor technical modifications) rather then the individual S-relations.

A New Kind of Dimension Function - Continued

- Difficulty: A small clique (of size < 5) in $A \subseteq M_s$ may extend to several distinct cliques in the structure M_s. In order to satisfy submodularity, the number of extensions that a clique may have must be reflected in the pre-dimension function.

  Solution: We introduce $\mathfrak{m}$, the *multiplicity* function for cliques, which associates each clique with the number of extensions it has. The function $\mathfrak{m}$ is definable in M_s.

- By definition of S, given an S-clique C in $M_{\sim}$, there are two elements x_C, y_C with $\models R(a, x_C, y_C)$ for all $a \in C$. Thus, adding $\{x_C, y_C\}$ to C generically lowers the dimension by $(|C| - 2)$. Accordingly, we estimate the negative contribution of a clique C, by $|C|_* = (|C| - 2)$.

We may now define d_{0s}.

Definition. For A a finite $\{S\}$-structure with set of maximal cliques $C(A)$ and multiplicity function $\mathfrak{m}$, define

$$d_{0s}(A) = |A| - \sum_{C \in C(A)} \mathfrak{m}(C) \cdot |C|_*$$

We denote the dimension function associated with d_{0s} by d_s and the associated self-sufficiency notion by $\leqslant_s$.

Main Results

Define $\mathcal{C}_s = \{A \mid A \text{ is a finite } \{S\}\text{-structure with } \emptyset \leqslant_s A\}$. Then

Theorem. *The pair $(\mathcal{C}_s, d_{0s})$ is a Hrushovski-type amalgamation class*

Theorem. *The geometry of the Fraïssé-Hrushovski limit of $(\mathcal{C}_s, d_{0s})$ is isomorphic to $G(M_{\sim})$*

Theorem. *The age of M_s is $\mathcal{C}_s$*

Theorem. *Let $A, B \leqslant_s M_s$ be finite and let $f : A \to B$ be a partial isomorphism, then f extends to an automorphism of M_s.*

Corollary. *M_s is a proper reduct of M and $G(M) \cong G(M_s)$*

References

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