Automorphism groups and limit laws of random nonrigid structures

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Let $V = \{R_1, \ldots, R_\rho\}$ be a vocabulary of relation symbols at least one of which has arity ≥ 2. Let $\mathbf{S}_n$ be the set of all V-structures with universe $\{1, \ldots, n\}$ and let $\mathbf{S} = \bigcup_{n=1}^{\infty} \mathbf{S}_n$. Finally, let $\mathrm{Aut}(\mathcal{M})$ denote the automorphism group of the structure $\mathcal{M} \in \mathbf{S}$.

Theorem 1. For any two finite groups G and H, each one of the following limits exists in $\mathbb{Q} \cup \{\infty\}$:

$$\lim_{n\to\infty} \frac{\big|\{\mathcal{M} \in \mathbf{S}_n : H \leq \mathrm{Aut}(\mathcal{M})\}\big|}{\big|\{\mathcal{M} \in \mathbf{S}_n : G \leq \mathrm{Aut}(\mathcal{M})\}\big|}, \quad \lim_{n\to\infty} \frac{\big|\{\mathcal{M} \in \mathbf{S}_n : H \cong \mathrm{Aut}(\mathcal{M})\}\big|}{\big|\{\mathcal{M} \in \mathbf{S}_n : G \cong \mathrm{Aut}(\mathcal{M})\}\big|} \quad \text{and}$$

$$\lim_{n\to\infty} \frac{\big|\{\mathcal{M} \in \mathbf{S}_n : G \cong \mathrm{Aut}(\mathcal{M})\}\big|}{\big|\{\mathcal{M} \in \mathbf{S}_n : G \leq \mathrm{Aut}(\mathcal{M})\}\big|}.$$

For $\mathcal{M} \in \mathbf{S}_n$ and $f \in \mathrm{Aut}(\mathcal{M})$, define

$$\mathrm{Spt}(f) = \big\{a \in M : f(a) \neq a\big\},$$
$$\mathrm{Spt}^*(\mathcal{M}) = \big\{a \in M : g(a) \neq a \text{ for some } g \in \mathrm{Aut}(\mathcal{M})\big\}.$$

Also define

$$\mathbf{S}_n(\mathrm{spt} \geq m) = \big\{\mathcal{M} \in \mathbf{S}_n : \big|\mathrm{Spt}(f)\big| \geq m \text{ for some } f \in \mathrm{Aut}(\mathcal{M})\big\},$$
$$\mathbf{S}_n(\mathrm{spt}^* \geq m) = \big\{\mathcal{M} \in \mathbf{S}_n : \big|\mathrm{Spt}^*(\mathcal{M})\big| \geq m\big\}.$$

Theorem 2. Suppose that the maximal arity among the symbols in V is 2. Let $m \geq 2$ be an integer and let $m' = m$ if m is even and $m' = m + 1$ otherwise.
(i) The proportion of $\mathcal{M} \in \mathbf{S}_n(\mathrm{spt} \geq m)$ such that $\big|\mathrm{Spt}(\mathcal{M})\big| = m'$ and $\mathrm{Aut}(\mathcal{M}) \cong \big(\mathbb{Z}_2\big)^i$ for some $i \in \{1, \ldots, m'/2\}$ converges to 1 as $n \to \infty$.
(ii) For every $i \in \{1, \ldots, m'/2\}$, there is a rational number $0 < a_i \leq 1$ (where $a_i < 1$ if $m > 2$) such that the proportion of $\mathcal{M} \in \mathbf{S}_n(\mathrm{spt} \geq m)$ such that $\mathrm{Aut}(\mathcal{M}) \cong \big(\mathbb{Z}_2\big)^i$ converges to a_i as $n \to \infty$.
(iii) Parts (i) and (ii) hold if '$\mathrm{spt} \geq m$' is replaced with '$\mathrm{spt}^* \geq m$'.

Theorem 3. Suppose that the maximal arity among the symbols in V is at least 3 and let $m \geq 2$ be an integer. Let $m' = m$ if m is even and $m' = m + 1$ otherwise. Then the proportion of $\mathcal{M} \in \mathbf{S}(\mathrm{spt} \geq m)$ such that $\big|\mathrm{Spt}^*(\mathcal{M})\big| = m'$ and $\mathrm{Aut}(\mathcal{M}) \cong \mathbb{Z}_2$ converges to 1 as $n \to \infty$. The same is true if '$\mathrm{spt} \geq m$' is replaced with '$\mathrm{spt}^* \geq m$'.

We say that $\mathbf{S}' \subseteq \mathbf{S}$ has a ***limit law*** if for every first-order sentence φ, the proportion of $\mathcal{M} \in \mathbf{S}' \cap \mathbf{S}_n$ in which φ is true converges as $n \to \infty$. If the limit is always 0 or 1, then we say that $\mathbf{S}'$ has a ***zero-one law***.

Theorem 4. Let G be a nontrivial finite group and $m \geq 2$ an integer (both fixed but arbitrary). Then each one of the following sets has a limit law, but not a zero-one law:

$$\big\{\mathcal{M} \in \mathbf{S} : G \leq \mathrm{Aut}(\mathcal{M})\big\},\ \big\{\mathcal{M} \in \mathbf{S} : G \cong \mathrm{Aut}(\mathcal{M})\big\}, \mathbf{S}_n(\mathrm{spt} \geq m), \text{ and } \mathbf{S}_n(\mathrm{spt}^* \geq m).$$

Theorem 5. Theorems 1–4 are also true if we only count structures up to isomorophism, i.e, if we count isomorphism classes of structures.

For more detail see our article, available on arXiv.