Galois theories, transcendence, Manin maps, and the model theory of differentially closed fields

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- Equivalently, assuming that $x \notin L(H) + B_0(\mathbb{C})$ for any proper algebraic subgroup H of B, then $tr.deg(K^\sharp(y)/K^\sharp) = dim(B)$.
- Here $K^\sharp$ is the differential field generated over K by solutions of $\mu_B(-) = 0$ in K^{diff} where μ_B is the Manin map, and replacing K by $K^\sharp$ is the strengthening.

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- The "theorem of the kernel" implies that $ker(\mu_A)(K^{alg})$ is precisely the torsion points.

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- Note that the $n = 1$ case is the differential algebraic theorem of the kernel.
- This general statement is true in many cases (such as when A is an elliptic curve), but an example of Yves André yields a counterexample (with some work).
- However the following statement, also generalizing the theorem of the kernel does hold: If $y_1, .., y_n \in A(K^{alg})$ are linearly independent over $End(A)$ then $\mu_A(y_1), .., \mu_A(y_n)$ are linearly independent over $\mathbb{C}$, also with a differential Galois-theoretic proof.

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- $Aut_\partial(L/K)$ has naturally the form $G(\mathcal{C}(K))$ for an algebraic subgroup G of GL_n over $\mathcal{C}(K)$.

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- A ∂-group structure on G is an extension of the derivation ∂ (on the differential field over which G is defined) to the structure sheaf of G which respects co-multiplication, equivalently a rational homomorphic section $s : G \to T_\partial G$ from G to a certain shifted tangent bundle of G.

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- Then ∂ln_G is the (definable) map $y \to \partial(y) \cdot s(y)^{-1}$ ($\cdot$ computed in the algebraic group $T_\partial(G)$).

Algebraic ∂-groups and logarithmic derivatives III

- If B is a semiabelian variety then its “universal vectorial extension” $\tilde{B}$ (where we have $0 \to W_B \to \tilde{B} \to B \to 0$) has a unique ∂-group structure, as does the quotient of $\tilde{B}$ by an algebraic ∂-subgroup contained in W_B.

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- Then $\partial ln_{\tilde{B}}$ induces a (definable) map from B to $L\tilde{B}/\partial ln_{\tilde{B}}(W_B)$ (a $\mathcal{C}$-vector space) which is the differential algebraic/model-theoretic Manin map μ_B.

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- Differentiating $\partial ln_{\tilde{B}}$ at the identity yields a definable homomorphism $\partial_{L\tilde{B}}$ from $L\tilde{B}$ to itself (the Gauss-Manin connection).

Algebraic ∂-groups and logarithmic derivatives IV

- The Ax-Lindemann theorem mentioned in the introduction is proved by passing to $\tilde{B}$, lifting x to $\tilde{x} \in L\tilde{B}(K^{alg})$, and y to $\tilde{y} \in \tilde{B}$ such that $exp(\tilde{y}) = \tilde{x}$.

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- Then noting that $\partial ln_{\tilde{B}}(\tilde{y}) = \partial_{L\tilde{B}}(\tilde{x})$, and giving a differential-Galois-theoretic proof (using also Manin-Coleman-Chai..) that $tr.deg(K^\sharp(\tilde{y})/K^\sharp) = dim(\tilde{B})$.

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- Now $K^{\sharp}$ is the (algebraic closure if one wishes of the) differential field generated by $K = \mathbb{C}(t)$ and the solutions of $\partial ln_{\tilde{B}}(-) = 0$ in the differential closure K^{diff} of K.

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- So in so far as transcendence/Galois theory issues are concerned it is natural to work over $K^\sharp$ the differential field generated by K and $G^\partial(K^{diff})$.

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- Let (G, s) be an algebraic ∂-group defined over the algebraically closed differential field K and let $\partial ln_G(y) = a$ be a logarithmic differential equation over K, y any solution in K^{diff} and $L = K^\sharp(y)$. Then $Aut_\partial(L/K^\sharp)$ has naturally the structure of $H^\partial(K^{diff}) = H^\partial(K^\sharp)$ for some algebraic ∂-subgroup H of (G, s) defined over $K^\sharp$. Moreover $tr.deg(K^\sharp(y)/K^\sharp) = dim(H)$ and in fact a generator y_1 for L over $K^\sharp$ can be chosen such that $y \in H$ and $\partial ln_H(y_1) = b$ for some $b \in LH(K^\sharp)$.

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- On of our main Galois theoretic results is that in suitable contexts the Galois data are all defined over K. It is a "Galois descent" argument, as Bertrand puts it.

Galois theory III

So the first Galois descent result, possibly of independent interest, is:

Theorem 0.1

Let B be a semi-abelian variety over an algebraically closed differential field K. Let $G = \tilde{B}$ equipped with its unique ∂-group structure, and let $\partial ln_{\tilde{B}}(-) = a$ be a log-differential equation over K. Let y be a solution in $G(K^{diff})$ Then $tr.deg(K^\sharp(y)/K^\sharp)$ is the dimension of the smallest algebraic ∂-subgroup H of G, defined over K, *such that $a \in LH + \partial ln_G(G(K))$. Equivalently the smallest such H such that $y \in H + G(K) + G^\partial(K^{diff})$. Moreover $H^\partial(K^{diff})$ is the Galois group of $K^\sharp(y)$ over $K^\sharp$.*

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- So A is an abelian variety over an algebraically closed differential field K, $\tilde{A}$ its universal vectorial extension. We fix $y \in \tilde{A}(K)$, and let x be a solution of $\partial_{L\tilde{A}}(x) = y$ in K^{diff}.

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- We now let $K^\sharp$ denote the differential field over K generated by $(L\tilde{A})^\partial$ the solution set of $\partial_{L\tilde{A}}(-) = 0$ in K^{diff}. Let A_0 be the $\mathcal{C}(\mathcal{K})$-trace of A and $\bar{y}$ the image of y in A.

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Theorem 0.2

Let B be the smallest abelian subvariety of A such that some multiple of $\bar{y}$ is in $B + A_0(\mathcal{C}(K))$. Then the (differential) Galois group of $K^{\sharp}(x)$ over $K^{\sharp}$ is $L\tilde{B}^{\partial}(K^{diff})$.

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- As before we have $1 \to W_A \to \tilde{A} \to A \to 1$. Let G be any quotient of $\tilde{A}$ by a unipotent (contained in W_A) algebraic ∂ subgroup and let W_G be the unipotent part of G.
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Theorem 0.3

Suppose $y \in G(K)$, $x \in LG(K)$ be such that $\partial ln_G(y) = \partial_{LG}(x)$. Then the projection of y to A is a torsion point, and in particular $x \in LW_G$ (which can be identified with W_G).

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- We let y, x be K^{diff}-rational points of $\tilde{G}$, $L\tilde{G}$ respectively such that $\partial ln_G(y) = \partial_{L\tilde{G}}(x)$, and let $K^\sharp$ be generated by K and the solutions of $\partial ln_{\tilde{G}} = 0$ in K^{diff}

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- Let $y_2 = y_1 + g$. Then $\partial ln_G(y_2) = a$ too.

Conclusion III

- Then, by the Galois theory, $y_2/H \in (G/H)(K)$, $x/LH \in L(G/H)(K)$, we still have that $\partial ln(y_2/H) = \partial(x/LH)$ (with relevant subscripts), and using $(HG)_0$, the HK hypothesis is preserved in the quotient.

Conclusion III

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- I now touch on the result concerning the Manin map with no details:
- Let A be simple traceless over $K = \mathbb{C}(t)^{alg}$. Then the statement: if $y_1, .., y_n \in A(K^{alg})$ are linearly independent over $End(A)$ then $\mu_A(y_1), .., \mu_A(y_n)$ are linearly independent over $\mathbb{C}$,
  follows fairly quickly from Theorem 0.2.

Conclusion III

- When W_A contains no algebraic ∂ subgroup of $\tilde{A}$, then we obtain with additional work the stronger statement:
  if $y_1, .., y_n \in A(K^{alg})$ are linearly independent over $\mathbb{Z}$ then $\mu_A(y_1), .., \mu_A(y_n)$ are linearly independent over $\mathbb{C}$.

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- And an example (mentioned in our Lindemann-Weierstrass paper) due to Yves André of a simple $\mathbb{C}$-trace 0 abelian variety over K such that $\tilde{A}$ DOES have have a nontrivial unipotent ∂ subgroup, yields a counterexample to the stronger statement.