Countable structures with few reducts

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Let $\mathcal{M}$ be a countably infinite first order structure. A **definable reduct** is a structure $\mathcal{M}'$ with the same domain as $\mathcal{M}$, whose relations are definable without parameters in $\mathcal{M}$. It is an **improper reduct** if also $\mathcal{M}$ is a reduct of $\mathcal{M}'$ (so $\mathcal{M}$ and $\mathcal{M}'$ have the same 0-definable relations), and is **trivial** if $\mathcal{M}'$ is a reduct of a pure set, so has automorphism group $\text{Sym}(\mathcal{M})$.

If $\mathcal{M}'$ and $\mathcal{M}$ have the same domain, $\mathcal{M}'$ is a **group-reduct** if $\text{Aut}(\mathcal{M}') \geq \text{Aut}(\mathcal{M})$, is **proper** if $\text{Aut}(\mathcal{M}') > \text{Aut}(\mathcal{M})$, **trivial** if $\text{Aut}(\mathcal{M}) = \text{Sym}(\mathcal{M})$.

**Basic Problem:** Find examples of structures with few/no proper non-trivial reducts (in either or both senses).
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Overview of Talk

- Reducts of $\omega$-categorical structures.
- Non-$\omega$-categorical example with no proper non-trivial reducts (both senses).
- Other possible examples (e.g. for strongly minimal sets).

The first is work of Cameron, Thomas, and other authors. The last two are joint work with Manuel Bodirsky.

Group-theoretic interpretation: $M$ has no proper non-trivial group-reducts if and only if $\text{Aut}(M)$ is maximal-closed in $\text{Sym}(M)$.

Remark. There are many (abstractly) maximal subgroups of $\text{Sym}(N)$, e.g. stabilisers of ultrafilters on $N$ ($2^{2^{\aleph_0}}$ up to conjugacy).
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**Remark.** There are many (abstractly) maximal subgroups of $\text{Sym}(\mathbb{N})$, e.g. stabilisers of ultrafilters on $\mathbb{N}$ (so $2^{2^{\aleph_0}}$ up to conjugacy).
Recall: A permutation group \((G, X)\) is **primitive** if there is no proper non-trivial \(G\)-invariant equivalence relation on \(X\).

**Remarks.**

- If \(M\) is \(\omega\)-categorical, then the two notions of reduct coincide (Ryll-Nardzewski), and we just call them ‘reducts’.

- If \(E\) is a proper non-trivial equivalence relation on \(M\) with all classes of the same size then \((M, E)\) has no proper non-trivial reducts (and transitive imprimitive maximal-closed groups all arise like this.)

- If \(G < \text{Sym}(N)\) is maximal-closed and has more than one orbit on the collection of \(k\)-element sets, then \(G = \text{Aut}(\Gamma)\) for some \(k\)-uniform hypergraph \(\Gamma\) with vertex set \(N\).

So we are interested in maximal-closed subgroups of \(\text{Sym}(N)\) which act primitively on \(N\).
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So we are interested in maximal-closed subgroups of \(\text{Sym}(\mathbb{N})\) which act primitively on \(\mathbb{N}\).
Theorem (Cameron, 1976)

The only proper non-trivial reducts of \((\mathbb{Q}, <)\) are

- \((\mathbb{Q}, B)\) (ternary betweenness, ‘\(x\) is between \(y\) and \(z\)’)
- \((\mathbb{Q}, K)\) (ternary circular order \(K\) induced from \(<\) )
- \((\mathbb{Q}, S)\) (quaternary separation relation, or \(K\) considered up to reversal).
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BUT (Junker, Ziegler, 2008) \((\mathbb{Q},<0)\) has 116 reducts!
Omega-categoricity – homogeneous graphs

Theorem (Thomas, 1991)

(i) The only proper non-trivial reducts of the random graph \((\Gamma, R)\) are

- \((\Gamma, B)\) (\(B\) ternary, the random graph up to anti-isomorphism)
- \((Q, K)\) (\(K\) ternary, the homogeneous ‘two-graph’ induced from \(R\), a triple satisfying \(K\) iff its entries are distinct and it contains an odd number of graph edges)
- \((Q, S)\) (the above homogeneous two-graph up to anti-isomorphism)

(ii) For \(n \geq 3\), the generic \(K_n\)-free graph has no proper non-trivial reducts.
Bennett (PhD, Rutgers, 1997): Result for random tournament like that for random graph (3 proper non-trivial reducts).

(Pach, Pinsker, Pluhár, Pongrácz, Szabó) Similar result for generic poset (3 reducts).

Key tool (in treatments of such results by Bodirsky, Pinsker, Tsankov, motivated by constraint satisfaction problems (CSPs)):

**Definition**

Let $C$ be a class of finite relational structures with a language including $<$ (interpreted by a total order). Then $C$ has the **Ramsey Property** if for every $A, B \in C$ and positive integer $k$, there is $D \in C$ such that for every colouring with $k$ colours of the copies of $A$ in $D$, there is a copy of $B$ in $D$ all of whose substructures isomorphic to $A$ have the same colour. In Ramsey notation,

$$D \rightarrow (B)_A^k$$
For all the above structures $\mathcal{M}$ (random graph, random hypergraph, etc.) there is a Fraïssé-homogeneous expansion $\mathcal{M}' = (\mathcal{M}, <)$ by a total order such that the $\text{Age}(\mathcal{M}')$ (the class of finite structures which embed in $\mathcal{M}'$) is a Ramsey class.

Remark. By Kechris-Pestov-Todorcevic, if $\mathcal{C}$ is as above and $\mathcal{M}$ is the Fraïssé limit, then $\text{Aut}(\mathcal{M})$ is extremely amenable: every continuous action of $\text{Aut}(\mathcal{M})$ on a compact Hausdorff space has a fixed point.
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Other ω-categorical structures where reducts are describable:

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Reducts of non-$\omega$-categorical structures.

An example.
Let $(T, R)$ be the unique degree 3 graph-theoretic tree. Two rays (infinite one-way paths) are equivalent if they have infinitely many common vertices. The equivalence classes are called ends. Let $M^+$ be the set of ends of $(T, R)$. 
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Define \(D(x, y; z, w)\) to hold on \(M^+\) iff one of

- \(x = y \land x \neq z \land x \neq w\)
- \(z = w \land x \neq z \land y \neq z\)
- \(x, y, z, w\) are distinct, and there are rays \(\hat{x} \in x, \hat{y} \in y\) etc. such that \(\hat{x} \cup \hat{y}\) and \(\hat{z} \cup \hat{w}\) are disjoint two-way infinite paths.
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Let $M$ be a countable dense subset of $M^+$ (i.e. for each $a \in T$ there are $x, y, z \in M$ and rays $\hat{x} \in x, \hat{y} \in y$ and $\hat{z} \in z$ such that $\hat{x} \cup \hat{y}, \hat{y} \cup \hat{z}$, and $\hat{x} \cup \hat{z}$ are all two-way infinite paths through $a$). Put $\mathcal{M} := (M, D)$. 
Observations on $(M, D)$.

- The above determines $(M, D)$ uniquely up to isomorphism (back-and-forth).

- $(M,D)$ is interpretable in $(M,D)$: vertices are equivalence classes of triples.

- Hence, $(M,D)$ is not $\omega$-categorical.

- $\text{Aut}(M,D)$ is 3-transitive.

- $(M,D)$ has the strict order property and is NIP. Indeed, let $K$ be a countable model of $\text{Th}(Q^2)$ with valuation ring $O$ having maximal ideal $M$. On $\mathbb{P}G_1(K)$ (viewed as $K \cup \{\infty\}$), define $D(x,y;z,w)$ to hold if and only if the cross-ratio $(x-z)(y-w)(x-w)(y-z) \in 1 + M$. Then $(\mathbb{P}G_1(K),D) \cong (M,D)$.

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Let \(K\) be a countable model of \(\text{Th}(\mathbb{Q}_2)\) with valuation ring \(\mathcal{O}\) having maximal ideal \(\mathcal{M}\). On \(\text{PG}_1(K)\) (viewed as \(K \cup \{\infty\}\)), define \(D(x, y; z, w)\) to hold if and only if the cross-ratio
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Theorem (Bodirsky, M (2013))

The structure \((M, D)\) has no proper non-trivial definable reducts or group-reducts.

Remark. This says in particular that \(\text{Aut}(M, D)\) is a maximal-closed subgroup of \(\text{Sym}(M)\) which is not oligomorphic.

(A permutation group \(G\) on countably infinite \(M\) is oligomorphic if \(G\) has finitely many orbits on \(M^n\) for all \(n\).)

Problem: Find other kinds of non-\(\omega\)-categorical structures with no proper non-trivial reducts of either kind.

Guess: The above example \((M, D)\) still works, starting from a higher degree tree.
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If \(a \in T\), and \(x \in M\) there is a unique ray \(x_a \in x\) starting at \(a\). There is an equivalence relation \(E_a\) on \(M\): put \(E_a(x, y)\) iff \(x_a\) and \(y_a\) have a common edge of \(T\). An \(E_a\)-class is called a cone (at \(a\)).

Now:

- for each \(a \in T\) there are three \(E_a\)-classes, with \(\text{Aut}(M)_a\) inducing \(S_3\) on them;
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- if \(A \subset M\) is finite, then \(\text{Aut}(M)_{(A)}\) (the pointwise stabiliser of \(A\)) has no finite orbits on \(M \setminus A\); hence \(\text{acl}(A) = A\).
Permutation groups.

Definition
Let \((G, X)\) be a permutation group (group \(G\) acting faithfully on \(X\)).

- If \(A \subset X\) with \(|A| > 1\), then \(A\) is a Jordan set if \(G_{(X \setminus A)}\) is transitive on \(A\). It is proper if \(A \neq X\), and if \(|X \setminus A| = n \in \mathbb{N}\), then \((G, X)\) is not \((n + 1)\)-transitive.

- A Jordan group is a transitive permutation group with a proper Jordan set.
Adeleke, M, Neumann (1995, 1996) Structure theorem for primitive Jordan permutation groups. In particular,

**Theorem**

Let $G$ be a 3-transitive but not highly transitive Jordan permutation group on an infinite set $X$. Then $G$ preserves on $X$ one of

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Let $G$ be a 3-transitive but not highly transitive Jordan permutation group on an infinite set $X$. Then $G$ preserves on $X$ one of

- a 4-ary separation relation (from a total order on $X$)
- a $D$-relation on $X$ (in which every cone is a Jordan set)
- a Steiner $k$-system on $X$ (some $k > 1$)
- a ‘limit’ of Steiner systems on $X$. 
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**Steiner $k$-system on $X$:** collection of ‘blocks’ (subsets of $X$ all of the same size $> k$) such that any $k$ elements of $X$ lie on a unique block.
Sketch proof (that $(M, D)$ has no proper non-trivial group-reducts).
**Sketch proof** (that \((M, D)\) has no proper non-trivial group-reducts).

Any cone of \((M, D)\) is a Jordan set for \(G := \text{Aut}(M, D)\). Let \(M'\) be a non-trivial group-reduct of \((M, D)\). Then 
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\text{Aut}(M, D) \leq \text{Aut}(M') < \text{Sym}(M).
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Also, \(H := \text{Aut}(\mathcal{M}')\) is a 3-transitive but not highly transitive Jordan group.
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Apply above classification. Rule out all but the \(D\)-relation case, and show that if \(H\) preserves a \(D\)-relation \(D'\) on \(M\) then it has the same cones as \((M, D)\), so \(D' = D\).
For example, suppose \( H \) preserves a Steiner \( k \)-system on \( M \). Let \( a_1, \ldots, a_k \in M \) be distinct, and let \( l \) be the block containing \( a_1, \ldots, a_k \).
For example, suppose $H$ preserves a Steiner $k$-system on $M$. Let $a_1, \ldots, a_k \in M$ be distinct, and let $l$ be the block containing $a_1, \ldots, a_k$.

Let $b_k \in M$ not lie on $l$, and let $m$ be the block through $a_1, \ldots, a_{k-1}, b_k$. Choose $a_{k+1} \not\in \{a_1, \ldots, a_k\}$ on $l$ and $b_{k+1} \not\in \{a_1, \ldots, a_{k-1}, b_k\}$ on $m$. 
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Now $l$ is the unique block containing $a_1, \ldots, a_{k-2}, a_k, a_{k+1}$, and $m$ the unique block containing $a_1, \ldots, a_{k-2}, b_k, b_{k+1}$, and these blocks have common points $a_1, \ldots, a_{k-1}$.
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Put $A := \{a_1, \ldots, a_{k-2}, a_k, a_{k+1}, b_k, b_{k+1}\}$. Then $a_{k-1}$ lies in a singleton orbit of $H(A)$. This is impossible as $a_{k-1} \not\in A$ and $H \geq G$. 
To show \((M, D)\) has no proper non-trivial definable reducts. Let \(M'\) be a non-trivial definable reduct of \((M, D)\).

**Claim:** It suffices to show that the set of cones of \((M, D)\) is uniformly definable in \((M, D)\), i.e. there are formulas \(\phi(x, y_1, \ldots, y_k)\) and \(\psi(y_1, \ldots, y_k)\) over \(\emptyset\) (in the language of \(M'\)) such that the set of cones of \((M, D)\) is exactly

\[ \{ \phi(M', \bar{a}) : \bar{a} \in M^k, M' \models \psi(\bar{a}) \}. \]

**Proof:** Easy to check that the cones determine \(D\).
Step 1. Show that some cone of \((M, D)\) is definable in \(M'\), by some formula \(\phi(x, \bar{a})\). Hence, all cones are \(M'\)-definable via \(\phi\) (as \(\text{Aut}(M') \geq \text{Aut}(M, D)\), which is transitive on the set of cones). Now aim to modify \(\phi\), and find \(\psi\).
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Step 3. Let \(\psi_2(\bar{y})\) express also that the complement of the set \(\phi(M, \bar{y})\) has the form \(\phi(M, \bar{y}')\) for some \(\bar{y}'\). (Recall that the complement of a cone in \((M, D)\) is also a cone.)
Step 1. Show that some cone of \((M, D)\) is definable in \(\mathcal{M}'\), by some formula \(\phi(x, \bar{a})\). Hence, all cones are \(\mathcal{M}'\)-definable via \(\phi\) (as \(\text{Aut}(\mathcal{M}') \geq \text{Aut}(M, D)\), which is transitive on the set of cones). Now aim to modify \(\phi\), and find \(\psi\).

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Step 4. Let \(\psi_3(\bar{y})\) express a consequence of the Jordan property of cones. Namely,

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\forall \bar{y}' (\bigwedge_{i=1}^{n} \neg \phi(y'_i, \bar{y}) \rightarrow ((\phi(M, \bar{y}') \supseteq \phi(M, \bar{y})) \lor \phi(M, \bar{y}') \cap \phi(M, \bar{y}) = \emptyset)).
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Step 1. Show that some cone of $(M, D)$ is definable in $\mathcal{M}'$, by some formula $\phi(x, \bar{a})$. Hence, all cones are $\mathcal{M}'$-definable via $\phi$ (as $\text{Aut}(\mathcal{M}') \geq \text{Aut}(M, D)$, which is transitive on the set of cones). Now aim to modify $\phi$, and find $\psi$.

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Step 5. Reduce to $l(\bar{y}) = 4$. Reduce to case when $\phi(M, \bar{a}')$ is a cone or union of two cones at adjacent nodes. Finish.
Partially order definable reducts of $\mathcal{M}$, putting $\mathcal{M}_1 \leq \mathcal{M}_2$ iff $\mathcal{M}_2$ is $\emptyset$-definable in $\mathcal{M}_1$ (factoring out equi-definability). Partially order group-reducts by group inclusion.
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Lemma

Let $\mathcal{M}$ be saturated.

(i) The partial order of definable reducts of $\mathcal{M}$ embeds into the partial order of group-reducts of $\mathcal{M}$.

(ii) If $\text{Aut}(\mathcal{M})$ is maximal-closed, then $\mathcal{M}$ has no proper non-trivial definable reducts.

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Reducts of strongly minimal sets

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**First try.** Let \((T, R)\) be the degree 3 tree (strongly minimal, disintegrated, i.e. \(\text{acl}(A) = \bigcup(\text{acl}(a) : a \in A)\) for any \(A\)). The ‘distance 2’ graph \(T^{(2)}\) is a proper non-trivial definable and group reduct. \(T^{(2)}\) is the disjoint union of two graphs, each built from copies of \(K_3\) in a treelike way, three copies of \(K_3\) containing each vertex. There is also a (non-definable) group-reduct, the equivalence relation with 2 classes corresponding to ‘even distance apart’. Any other reducts?
**Second try.** Let $k, l \in \mathbb{N}$ with $k \geq 2$ and $l \geq 3$. Let $\Gamma_{k,l}$ be the graph consisting of copies of $K_{k+1}$ put together in a treelike way, with $l$ copies of $K_{k+1}$ containing each vertex.
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The graph $\Gamma_{k,l}$ is vertex transitive of finite degree $kl$, so strongly minimal disintegrated. In fact these are essentially the finite degree distance transitive graphs (the aut. group is transitive on the pairs of vertices at any given distance).
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Theorem (Bodirsky, M)
\[ \mathcal{M} := \Gamma_{k,l} \text{ has no proper non-trivial definable reducts.} \]
Sketch Proof.

1. Any definable reduct $\mathcal{M}'$ is strongly minimal and disintegrated, so if $\mathcal{M}'$ is non-trivial then $|\text{acl}_{\mathcal{M}'}(a)| > 1$ for any $a$. Hence in $\mathcal{M}'$, some set $\phi(M, a)$ has finite size greater than 1. By distance transitivity, can assume there are $1 \leq n_1 < \ldots < n_t \in \mathbb{N}$ such that $\phi(x, y)$ is ‘$d(x, y) \in \{n_1, \ldots, n_t\}$’.
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2. Put $n := n_t$. Show that for any vertices $x, y$,

$$d(x, y) \leq 2n \iff \mathcal{M}' \models \exists z (\phi(x, z) \land \phi(y, z)).$$

Thus the balls $B_{2n}(x)$ are uniformly definable.
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Thus the balls $B_{2n}(x)$ are uniformly definable.

3. Show there is some $\gamma \in \mathbb{N}$ such that

$$x, y \text{ are adjacent in } \mathcal{M} \iff B_{2n}(x) \cap B_{2n}(y) < \gamma.$$
**Problem.** Describe reducts of other vertex-transitive (finite degree) connected graphs. Do maximal-closed subgroups of $\text{Sym}(\mathbb{N})$ arise this way?
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Non-disintegrated locally modular strongly minimal sets. For finite fields \(F\), vector spaces over \(F\) (and projective and affine spaces over \(F\)) are \(\omega\)-categorical, and reducts can be handled by the Cherlin-Harrington-Lachlan and Zilber work.
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If $V$ is a vector space over an infinite characteristic 0 field $F$, there is a reduct (definable and group) $(V, R)$, where

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If \(F\) has characteristic \(p\), there are proper reducts by viewing \(V\) as over the prime subfield.
Further questions

1. Define $f$ on $\mathbb{Q}$ by $f(x, y, z) := x - y + z$. Does $(\mathbb{Q}, f)$ have any proper non-trivial definable reducts?

Note: For each prime $p$, let $v_p$ be the $p$-adic valuation on $\mathbb{Q}$ and define $C_p(x, y, z)$ to hold iff $v_p(x - y) \leq v_p(y - z)$. The structures $(\mathbb{Q}, C_p)$ are distinct group-reducts.

2. For $2 \leq d \leq \aleph_0$, are the groups $\text{AGL}_d(\mathbb{Q})$ and $\text{PGL}_{d+1}(\mathbb{Q})$ maximal-closed, in their natural actions? Do the corresponding strongly minimal structures have proper non-trivial definable reducts? (They will be locally modular but not disintegrated.)

3. Does $\text{Sym}(\mathbb{N})$ have any countable maximal-closed subgroups?

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