Countable structures with few reducts

Dugald Macpherson (University of Leeds)

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If $\mathcal{M}'$ and $\mathcal{M}$ have the same domain, $\mathcal{M}'$ is a **group-reduct** if $\mathrm{Aut}(\mathcal{M}') \geq \mathrm{Aut}(\mathcal{M})$, is **proper** if $\mathrm{Aut}(\mathcal{M}') > \mathrm{Aut}(\mathcal{M})$, **trivial** if $\mathrm{Aut}(\mathcal{M}) = \mathrm{Sym}(M)$.

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Basic Problem: Find examples of structures with few/no proper non-trivial reducts (in either or both senses).

Overview of Talk

- Reducts of ω-categorical structures.
- Non-ω-categorical example with no proper non-trivial reducts (both senses).
- Other possible examples (e.g. for strongly minimal sets).

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Remark. There are many (abstractly) maximal subgroups of $\mathrm{Sym}(\mathbb{N})$, e.g. stabilisers of ultrafilters on $\mathbb{N}$ (so $2^{2^{\aleph_0}}$ up to conjugacy).

Recall: A permutation group (G, X) is **primitive** if there is no proper non-trivial G-invariant equivalence relation on X.

Remarks.

- If $\mathcal{M}$ is ω-categorical, then the two notions of reduct coincide (Ryll-Nardzewski), and we just call them 'reducts'.

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So we are interested in maximal-closed subgroups of $\mathrm{Sym}(\mathbb{N})$ which act primitively on $\mathbb{N}$.

Theorem (Cameron, 1976)

The only proper non-trivial reducts of $(\mathbb{Q}, <)$ *are*

- $(\mathbb{Q}, B)$ *(ternary betweenness, '*x *is between* y *and* z*')*
- $(\mathbb{Q}, K)$ *(ternary circular order* K *induced from* $<$*)*
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BUT (Junker, Ziegler, 2008) $(\mathbb{Q}, < 0)$ has 116 reducts!

Omega-categoricity – homogeneous graphs

Theorem (Thomas, 1991)

(i) The only proper non-trivial reducts of the random graph (Γ, R) *are*

- (Γ, B) *(*B *ternary, the random graph up to anti-isomorphism)*
- $(\mathbb{Q}, K)$ *(*K *ternary, the homogeneous 'two-graph' induced from* R*, a triple satisfying* K *iff its entries are distinct and it contains an odd number of graph edges)*
- $(\mathbb{Q}, S)$ *(the above homogeneous two-graph up to anti-isomorphism)*

(ii) For $n \geq 3$*, the generic* K_n*-free graph has no proper non-trivial reducts.*

- Bennett (PhD, Rutgers, 1997): Result for random tournament like that for random graph (3 proper non-trivial reducts).
- (Pach, Pinsker, Pluhár, Pongrácz, Szabó) Similar result for generic poset (3 reducts).
- Thomas (1996): Classification of reducts of random hypergraphs.

Key tool (in treatments of such results by Bodirsky, Pinsker, Tsankov, motivated by constraint satisfaction problems (CSPs)):

Definition

Let $\mathcal{C}$ be a class of finite relational structures with a language including $<$ (interpreted by a total order). Then $\mathcal{C}$ has the **Ramsey Property** if for every $A, B \in \mathcal{C}$ and positive integer k, there is $D \in \mathcal{C}$ such that for every colouring with k colours of the copies of A in D, there is a copy of B in D all of whose substructures isomorphic to A have the same colour. In Ramsey notation,

$$D \to (B)^k_A$$

For all the above structures $\mathcal{M}$ (random graph, random hypergraph, etc.) there is a Fraïssé-homogeneous expansion $\mathcal{M}' = (\mathcal{M}, <)$ by a total order such that the $\mathrm{Age}(\mathcal{M}')$ (the class of finite structures which embed in $\mathcal{M}'$) is a Ramsey class.

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Remark. By Kechris-Pestov-Todorcevic, if $\mathcal{C}$ is as above and $\mathcal{M}$ is the Fraïssé limit, then $\mathrm{Aut}(\mathcal{M})$ is **extremely amenable**: every continuous action of $\mathrm{Aut}(\mathcal{M})$ on a compact Hausdorff space has a fixed point.

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Conjecture of Thomas (1991): If $\mathcal{M}$ is a Fraïssé-homogeneous structure over a finite relational language, then $\mathcal{M}$ has just finitely many reducts.

An example.
Let (T, R) be the unique degree 3 graph-theoretic tree. Two *rays* (infinite one-way paths) are equivalent if they have infinitely many common vertices. The equivalence classes are called *ends*. Let M^+ be the set of ends of (T, R).

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Define $D(x, y; z, w)$ to hold on M^+ iff one of

- $x = y \wedge x \neq z \wedge x \neq w$
- $z = w \wedge x \neq z \wedge y \neq z$
- x, y, z, w are distinct, and there are rays $\hat{x} \in x$, $\hat{y} \in y$ etc. such that $\hat{x} \cup \hat{y}$ and $\hat{z} \cup \hat{w}$ are disjoint two-way infinite paths.

Reducts of non-ω-categorical structures.

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Let M be a countable dense subset of M^+ (i.e. for each $a \in T$ there are $x, y, z \in M$ and rays $\hat{x} \in x$, $\hat{y} \in y$ and $\hat{z} \in z$ such that $\hat{x} \cup \hat{y}$, $\hat{y} \cup \hat{z}$, and $\hat{x} \cup \hat{z}$ are all two-way infinite paths through a).
Put $\mathcal{M} := (M, D)$.

Observations on (M, D).

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- Let K be a countable model of $Th(\mathbb{Q}_2)$ with valuation ring $\mathcal{O}$ having maximal ideal $\mathcal{M}$. On $\mathrm{PG}_1(K)$ (viewed as $K \cup \{\infty\}$), define $D(x, y; z, w)$ to hold if and only if the cross-ratio $\frac{(x-z)(y-w)}{(x-w)(y-z)} \in 1 + \mathcal{M}$. Then

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- 'D-relations' were axiomatised by Adeleke and Neumann.

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Guess: The above example (M,D) still works, starting from a higher degree tree.

Automorphisms of (M, D).

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- if U is a cone, then $\mathrm{Aut}(U, D)$ is transitive, and any automorphism of (U, D) can be extended by $\mathrm{id}_{M \setminus U}$ to an aut. of (M, D).
- if $A \subset M$ is finite, then $\mathrm{Aut}(M)_{(A)}$ (the pointwise stabiliser of A) has no finite orbits on $M \setminus A$; hence $\mathrm{acl}(A) = A$.

Definition

Let (G, X) be a permutation group (group G acting faithfully on X).

- If $A \subset X$ with $|A| > 1$, then A is a *Jordan set* if $G_{(X \setminus A)}$ is transitive on A. It is *proper* if $A \neq X$, and if $|X \setminus A| = n \in \mathbb{N}$, then (G, X) is not $(n+1)$-transitive.
- A *Jordan group* is a transitive permutation group with a proper Jordan set.

Adeleke, M, Neumann (1995, 1996) Structure theorem for primitive Jordan permutation groups. In particular,

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Steiner k-system on X: collection of 'blocks' (subsets of X all of the same size $> k$) such that any k elements of X lie on a unique block.

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Apply above classification. Rule out all but the D-relation case, and show that if H preserves a D-relation D' on M then it has the same cones as (M, D), so $D' = D$.

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Let $b_k \in M$ not lie on l, and let m be the block through $a_1, \ldots, a_{k-1}, b_k$. Choose $a_{k+1} \notin \{a_1, \ldots, a_k\}$ on l and $b_{k+1} \notin \{a_1, \ldots, a_{k-1}, b_k\}$ on m.

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Now l is the unique block containing $a_1, \ldots, a_{k-2}, a_k, a_{k+1}$, and m the unique block containing $a_1, \ldots, a_{k-2}, b_k, b_{k+1}$, and these blocks have common points $a_1, \ldots, a_{k-1}$.

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Put $A := \{a_1, \ldots, a_{k-2}, a_k, a_{k+1}, b_k, b_{k+1}\}$. Then a_{k-1} lies in a singleton orbit of $H_{(A)}$. This is impossible as $a_{k-1} \notin A$ and $H \geq G$.

To show (M, D) has no proper non-trivial definable reducts. Let $\mathcal{M}'$ be a non-trivial definable reduct of (M, D).

Claim: It suffices to show that the set of cones of (M, D) is uniformly definable in (M, D), i.e. there are formulas $\phi(x, y_1, \ldots, y_k)$ and $\psi(y_1, \ldots, y_k)$ over $\emptyset$ (in the language of $\mathcal{M}'$) such that the set of cones of (M, D) is exactly

$$\{\phi(M', \bar{a}) : \bar{a} \in M^k, \mathcal{M}' \models \psi(\bar{a})\}.$$

Proof: Easy to check that the cones determine D.

Step 1. Show that some cone of (M, D) is definable in $\mathcal{M}'$, by some formula $\phi(x, \bar{a})$. Hence, all cones are $\mathcal{M}'$-definable via ϕ (as $\mathrm{Aut}(\mathcal{M}') \geq \mathrm{Aut}(M, D)$, which is transitive on the set of cones). Now aim to modify ϕ, and find ψ.

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Step 4. Let $\psi_3(\bar{y})$ express a consequence of the Jordan property of cones. Namely,

$$\forall \bar{y}' (\bigwedge_{i=1}^{n} \neg\phi(y'_i, \bar{y}) \to ((\phi(M, \bar{y}') \supseteq \phi(M, \bar{y})) \vee \phi(M, \bar{y}') \cap \phi(M, \bar{y}) = \emptyset))).$$

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Step 5. Reduce to $l(\bar{y}) = 4$. Reduce to case when $\phi(M, \bar{a}')$ is a cone or union of two cones at adjacent nodes. Finish.

More on reducts

Partially order definable reducts of $\mathcal{M}$, putting $\mathcal{M}_1 \leq \mathcal{M}_2$ iff $\mathcal{M}_2$ is $\emptyset$-definable in $\mathcal{M}_1$ (factoring out equi-definability). Partially order group-reducts by group inclusion.

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Let $\mathcal{M}$ be saturated.
(i) The partial order of definable reducts of $\mathcal{M}$ embeds into the partial order of group-reducts of $\mathcal{M}$.
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Proof. Easy compactness and saturation argument.

Reducts of strongly minimal sets

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First try. Let (T, R) be the degree 3 tree (strongly minimal, disintegrated, i.e. $\mathrm{acl}(A) = \bigcup(\mathrm{acl}(a) : a \in A)$ for any A). The 'distance 2' graph $T^{(2)}$ is a proper non-trivial definable and group reduct. $T^{(2)}$ is the disjoint union of two graphs, each built from copies of K_3 in a treelike way, three copies of K_3 containing each vertex. There is also a (non-definable) group-reduct, the equivalence relation with 2 classes corresponding to 'even distance apart'. Any other reducts?

Second try. Let $k, l \in \mathbb{N}$ with $k \geq 2$ and $l \geq 3$. Let $\Gamma_{k,l}$ be the graph consisting of copies of K_{k+1} put together in a treelike way, with l copies of K_{k+1} containing each vertex.

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Theorem (Bodirsky, M)

$\mathcal{M} := \Gamma_{k,l}$ *has no proper non-trivial definable reducts.*

Sketch Proof.

1. Any definable reduct $\mathcal{M}'$ is strongly minimal and disintegrated, so if $\mathcal{M}'$ is non-trivial then $|\mathrm{acl}_{\mathcal{M}'}(a)| > 1$ for any a. Hence in $\mathcal{M}'$, some set $\phi(M, a)$ has finite size greater than 1. By distance transitivity, can assume there are $1 \leq n_1 < \ldots < n_t \in \mathbb{N}$ such that $\phi(x, y)$ is '$d(x, y) \in \{n_1, \ldots, n_t\}$'.

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2. Put $n := n_t$. Show that for any vertices x, y,

$$d(x, y) \leq 2n \Leftrightarrow \mathcal{M}' \models \exists z(\phi(x, z) \wedge \phi(y, z)).$$

Thus the balls $B_{2n}(x)$ are uniformly definable.

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Thus the balls $B_{2n}(x)$ are uniformly definable.

3. Show there is some $\gamma \in \mathbb{N}$ such that

$$x, y \text{ are adjacent in } \mathcal{M} \Leftrightarrow B_{2n}(x) \cap B_{2n}(y) < \gamma.$$

Problem. Describe reducts of other vertex-transitive (finite degree) connected graphs. Do maximal-closed subgroups of $\mathrm{Sym}(\mathbb{N})$ arise this way?

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Non-disintegrated locally modular strongly minimal sets. For finite fields F, vector spaces over F (and projective and affine spaces over F) are ω-categorical, and reducts can be handled by the Cherlin-Harrington-Lachlan and Zilber work.

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If F has characteristic p, there are proper reducts by viewing V as over the prime subfield.

Further questions

1. Define f on $\mathbb{Q}$ by $f(x, y, z) := x - y + z$. Does $(\mathbb{Q}, f)$ have any proper non-trivial definable reducts?

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Note: For each prime p, let v_p be the p-adic valuation on $\mathbb{Q}$ and define $C_p(x, y, z)$ to hold iff $v_p(x - y) \leq v_p(y - z)$. The structures $(\mathbb{Q}, C_p)$ are distinct group-reducts.

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4. Is every closed proper subgroup of $\mathrm{Sym}(\mathbb{N})$ contained in a maximal-closed subgroup of $\mathrm{Sym}(\mathbb{N})$?