(Some more) local definability theory for holomorphic functions

Gareth Jones (Manchester)

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- This is made precise on the next slide.

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- This also makes sense for real-analytic functions.

Some closure conditions

- Schwarz reflection If f is an $\mathcal{F}$-definable holomorphic germ at the origin then the Schwarz reflection of f, defined by $f^{\mathrm{SR}}(z) = \overline{f(\overline{z})}$ is also an $\mathcal{F}$-definable holomorphic germ.

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- Composition If $f, g_1, \ldots, g_m$ are $\mathcal{F}$-definable holomorphic germs such that the composition $f \circ (g_1, \ldots, g_m)$ makes sense then the germ of the composition is also $\mathcal{F}$-definable.

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$$F(0, f(0)) = 0 \neq \frac{\partial F}{\partial z_{n+1}}(0, f(0))$$

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Conjecture ($\sim$Wilkie)

Let $\tilde{\mathcal{F}}$ be the smallest collection of germs containing all germs of all functions in $\mathcal{F}$ and closed under composition and implicit definability. If f is an $\mathcal{F}$-definable holomorphic germ then f is in $\tilde{\mathcal{F}}$.

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Proposition

Suppose that f is an $\mathcal{F}$-definable holomorphic germ at 0, and takes real values at reals. Then f is in $\mathcal{F}^*$.

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- To prove the general case, we use translations and tricks with Schwarz reflection to reduce to our germs all being of the special form.
- Then we use closure under Schwarz reflection to show that the odd closure condition in the previous slide is OK.

The end