Continuous model theory and the classification problem

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Outline

- Basics of C*-algebras
- Reminder about continuous model theory
- Basic model theory of operator algebras
- The Elliott classification problem
- More advanced C*-algebra basics
- A model theory conjecture

C*-algebra basics

Definition

A C*-algebra is a *-subalgebra A of the bounded linear operators $B(H)$ on a complex Hilbert space H which is closed in the operator norm topology. Alternatively, a C*-algebra is a Banach *-algebra A which satisfies the C*-identity $\|a^*a\| = \|a\|^2$ for all $a \in A$.

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The first sentence defines a concrete representation of a C*-algebra and the second gives an abstract definition.

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- C*-algebras are closed under inductive limits where the relevant morphisms are *-homomorphisms.
- C*-algebras are closed under tensor products but ...

Spectral Theorem

Definition

If A is a unital C*-algebra and $a \in A$ then $sp(a)$, the spectrum of a, is the set of $\lambda \in \mathbb{C}$ such that $a - \lambda I$ is not invertible.

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Example: If A is a C*-algebra and $p \in A$, we call p a projection if $p^2 = p$ $(= p^*)$.

Claim: For every $\epsilon > 0$ there is a $\delta > 0$ such that if a is self-adjoint and $\|a^2 - a\| < \delta$ then there is a projection p such that $\|p - a\| < \epsilon$.

Continuous model theory of C*-algebras

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- The only relation symbol is the operator norm $\| \cdot \|$.
- The basic formulas of continuous logic which are relevant here are $\|p(\bar{x})\|$ where $p(\bar{x})$ is a *-polynomial.
- Formulas are closed under composition with continuous real-valued functions; moreover, if φ is a formula then so is $\sup_{x \in B_N} \varphi$ or $\inf_{x \in B_N} \varphi$. The interpretation of these formulas in a C*-algebra is standard.

The theory of C*-algebras

- Notice the if A is a C*-algebra, $\bar{a} \in A$ and φ is a formula then $\varphi^A(\bar{a})$ is a number. In particular, if φ is a sentence then $\varphi^A \in \mathbb{R}$.
- $Th(A)$, the theory of an algebra, is the function which to every sentence φ assigns φ^A. A theory is determined by its zero set.
- We say that a class of structures K is elementary if there is a set of sentences T such that $A \in K$ iff $\varphi^A = 0$ for all $\varphi \in T$.

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Theorem

The class of C-algebras is an elementary class. In fact, in the appropriate language it is a universal class.*

Ultraproducts

- If A_i for $i \in I$ are C*- algebras and U is an ultrafilter on I, one forms the norm ultraproduct as follows:
- Let

$$\ell^\infty(\prod_{i \in I} A_i) = \{\bar{a} \in \prod_{i \in I} A_i : \text{for some } M, \|a_i\| \leq M \text{ for all } i \in I\}$$

and

$$c_U = \{\bar{a} \in \ell^\infty(\prod_{i \in I} A_i) : \lim_{i \to U} \|a_i\| = 0\}$$

- The ultraproduct is then $\prod_{i \in I} A_i / U := \ell^\infty(\prod_{i \in I} A_i)/c_U$.

Definable zero sets

Definition

Suppose that M is a metric structure and $\varphi(\bar{x})$ is a formula. We say that φ has a definable zero set if the distance function to the zero set of φ, $\{\bar{a} \in M : \varphi^M(\bar{a}) = 0\}$, is given by a definable predicate in M i.e. a uniform limit of formulas.

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Theorem

For a metric structure M and a formula φ, the following are equivalent:

- *φ has a definable zero set.*
- *The zero set of φ can be quantified over.*

Stable relations

Definition

In the language of C*-algebras, a formula $\varphi(\bar{x})$, or its zero set, is called a stable relation if for every C*-algebra A and for every $\epsilon > 0$ there is a $\delta > 0$ such that if $\bar{a} \in A$ and $|\varphi(\bar{a})| < \delta$ then there is $\bar{b} \in A$ such that $\varphi(\bar{b}) = 0$ and $\|\bar{a} - \bar{b}\| < \epsilon$.

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- the sets of self-adjoint elements, unitary elements ($u^*u = uu^* = 1$), positive elements (a^*a); in general, the range of any term.
- the sets of generators for subalgebras isomorphic to $M_n(C)$, for any $n \in \mathbb{N}$ or, in general, any finite-dimensional algebra.

The classification programme for nuclear C*-algebras

The Elliott conjecture

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- For a C*-algebra A, there is an invariant called the Elliott invariant which for the record is defined as:

$$Ell(A) = ((K_0(A), K_0^+(A), [1_A]), K_1(A), Tr(A), \rho_A)$$

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- There are other invariants which come up like KK-theory and the Cuntz semi-group but I won't focus on them.

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- The class of nuclear algebras is closed under inductive limits; UHF (uniformly hyperfinite) algebras are limits of matrix algebras; AF (approximately finite dimensional) algebras are limits of finite-dimensional algebras.

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- The class of nuclear algebras is closed under inductive limits; UHF (uniformly hyperfinite) algebras are limits of matrix algebras; AF (approximately finite dimensional) algebras are limits of finite-dimensional algebras.
- The class of nuclear algebras is not closed under ultraproducts or even ultrapowers.

Nuclear algebras, cont'd

Definition

- A element of a C*-algebra A is said to be positive if it is of the form a^*a for some $a \in A$.
- A linear map $f : A \to B$ is positive if whenever $a \in A$ is positive then so is $f(a)$.
- A linear map $f : A \to B$ is completely positive if the induced map from $M_n(A)$ to $M_n(B)$ is positive for all n.
- A map f is contractive if $\|f\| \leq 1$.

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Theorem (Stinespring)

*For any completely positive map $f : A \to B(H)$ there is a Hilbert space K, *-homomorphism $\pi : A \to B(K)$ and $V \in B(K, H)$ such that $f(a) = V\pi(a)V^*$.*

Nuclear algebras: good news and bad news

Definition

A C*-algebra A has the contractive positive approximation property (CPAP) if for every $\bar{a} \in A$ and $\epsilon > 0$ there is an n and cpc maps $\sigma : A \to M_n(\mathbb{C})$ and $\tau : M_n(\mathbb{C}) \to A$ such that $\|\bar{a} - \tau(\sigma(\bar{a}))\| < \epsilon$.

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Theorem

There are countably many partial types such that a C-algebra is nuclear iff it omits all of these types.*

The definition of K_0

Definition

For any C*-algebra A, consider the equivalence relation $\sim$ on projections in A given by $p \sim q$ iff there is some $v \in A$, $vpv^* = q$ and $v^*qv = p$.

Consider the (non-unital) *-homomorphism $\Phi_n : M_n(A) \to M_{n+1}(A)$ defined by

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$$

and let $M_\infty = \lim_n M_n(A)$. We should really complete this ...

Let $V(A) = Proj(M_\infty(A))/\sim$.

$V(A)$ has an additive structure defined as follows: if $p, q \in V(A)$ then $p \oplus q$ is

$$\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$$

The definition of K_0, cont'd

Definition

$K_0(A)$ is the Grothendieck group generated from $(V(A), \oplus)$ and $K_0^+(A)$ is the image of $V(A)$ in $K_0(A)$; if A is unital then the constant $[1_A]$ corresponds to the identity in A.

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- If H is infinite-dimensional then $K_0(B(H))$ is 0.
- Consider $A = \lim_n M_{2^n}(\mathbb{C})$ where the given morphisms are $M_{2^n}(\mathbb{C}) \hookrightarrow M_{2^{n+1}}(\mathbb{C})$ such that

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$

  Then $K_0(A)$ is the dyadic rationals with the unit associated to 1.

Examples of K_0, cont'd

- In general, if $A = \lim_k M_{n(k)}$ where $n(k)|n(k+1)$ for all k and the morphisms are given by diagonal maps

$$a \mapsto \begin{pmatrix} a & & & 0 \\ & a & & \\ & & \ddots & \\ 0 & & & a \end{pmatrix}$$

then $K_0(A) = \{m/n : m \in \mathbb{Z} \text{ and } n|n(k) \text{ for some } k\}$.

The main actors in K-theory for nuclear C*-algebras

- We have already introduced $(K_0(A), K_0^+(A), [1_A])$.
- $K_1(A) = K_0(C_0((0,1), A))$.
- $Tr(A)$ is the set of traces on A i.e. all positive linear functionals τ on A such that $\tau(1) = 1$, $\tau(x^*) = \overline{\tau(x)}$ and $\tau(xy) = \tau(yx)$.
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- The form of the Elliott conjecture which states that the Elliott invariant classifies all simple, separable, infinite-dimensional, unital nuclear algebras is false - there are counter-examples of different types with the first ones due to Toms and separately Rørdam.
- A search is on for a new invariant which might classify nuclear algebras.

Prototypical example of classification

Theorem (Elliott)

The class of AF algebras can be classified by K_0.

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Let's do a special case of this result due to Glimm.

Definition

For a UHF algebra $A = \lim_k M_{n(k)}$, let the $GI(A)$, the generalized integer of A be the function which assigns to every prime p the supremum of all n such that p^n divides $n(k)$ for some k; this can be infinite.

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Theorem (Glimm)

If A and B are separable, unital UHF algebras then $A \cong B$ iff $GI(A) = GI(B)$.

A proof of Glimm's theorem

Sketch of proof: One checks that UHF algebras have a unique trace and the values of this trace on a UHF algebra A are of the form $\{k/n : k|Gl(A)\}$.

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Now if $Gl(A) = Gl(B)$ then we can arrange in a back and forth fashion that $A = \lim_k M_{n(k)}$ and $B = \lim_k M_{m(k)}$ such that for all k, $n(k)|m(k)|n(k+1)$. It is possible then to create a sequence of maps $\varphi_k : M_{n(k)} \to M_{m(k)}$ and $\psi_k : M_{m(k)} \to M_{n(k+1)}$ which additionally have the necessary commutation to make A and B isomorphic.

A proof of Glimm's theorem

Sketch of proof: One checks that UHF algebras have a unique trace and the values of this trace on a UHF algebra A are of the form $\{k/n : k | GI(A)\}$.

Now if $GI(A) = GI(B)$ then we can arrange in a back and forth fashion that $A = \lim_k M_{n(k)}$ and $B = \lim_k M_{m(k)}$ such that for all k, $n(k)|m(k)|n(k+1)$. It is possible then to create a sequence of maps $\varphi_k : M_{n(k)} \to M_{m(k)}$ and $\psi_k : M_{m(k)} \to M_{n(k+1)}$ which additionally have the necessary commutation to make A and B isomorphic.

$$
\begin{array}{ccccccccc}
M_{n(1)}(\mathbb{C}) & \longrightarrow & M_{n(2)}(\mathbb{C}) & \longrightarrow & M_{n(3)}(\mathbb{C}) & \longrightarrow & \cdots & \quad & A \\
\phi_1 \downarrow & \overset{\psi_1}{\nearrow} & \phi_2 \downarrow & \overset{\psi_2}{\nearrow} & \phi_3 \downarrow & & & & \\
M_{m(1)}(\mathbb{C}) & \longrightarrow & M_{m(2)}(\mathbb{C}) & \longrightarrow & M_{m(3)}(\mathbb{C}) & \longrightarrow & \cdots & \quad & B
\end{array}
$$

Model theoretic version of the Elliott conjecture

Simple, separable, infinite-dimensional, unital nuclear algebras are classified by their Elliott invariant and their first order continuous theory.

$K_0(A)$ vs. $Th(A)$, round 1

- In the case of a separable, unital UHF algebra A, $K_0(A)$ is a rank 1, torsion-free abelian group where we have specified a constant. This is determined by $GI(A)$ by Glimm's theorem.

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- Round 1 - a draw.

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- Advantage K_0 (and descriptive set theory).

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- Now suppose that $A \equiv B$ where B is some simple, separable, nuclear algebra.

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- It is known that $\mathcal{R}$ satisfies property Γ and that A modulo its trace does not so $A \not\equiv B$.
- Question: what is the theory of the class of nuclear algebras? Is it the theory of C*-algebras?