Divisible Points on Curves

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Joint work with Martin Bays (McMaster)

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$$3^2 - 2^3 = 1$$

is the only solution of

$$x - y = 1$$

with $x \in \langle 3 \rangle \subset \mathbb{C}^\times$ and $y \in \langle 2 \rangle \subset \mathbb{C}^\times$.

Lang's Conjecture and Beyond

From a *qualitative* point of view, and after work of Bombieri, Faltings, Hindry, Hrushovski, Lang, Liardet, McQuillen, Votja, ... intersections

$$X \cap \Gamma$$

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where X is a subvariety of a semi-abelian variety and Γ is a finitely generated subgroup are well-understood.
The semi-abelian varieties include $(\mathbb{C}^\times)^n$, abelian varieties, products of such, and more.

Zilber-Pink

We will restrict to the ambient group $(\mathbb{C}^\times)^n$, also known as the algebraic torus.
In this context, the Zilber-Pink Conjecture (or CIT) deals with the intersection of X with an algebraic subgroup $\mathcal{H} \subset (\mathbb{C}^\times)^n$.

Remark

Any algebraic subgroups of $(\mathbb{C}^\times)^n$ is defined by

$$\left.\begin{array}{cccc} x_1^{a_{11}} & \cdots & x_n^{a_{1n}} & = 1 \\ & \vdots & & \\ x_1^{a_{m1}} & \cdots & x_n^{a_{mn}} & = 1 \end{array}\right\} \quad \textit{with} \quad (a_{ij}) \in \mathbb{Z}.$$

Zilber-Pink cont'd

Conjecture (Variant of Zilber-Pink for the algebraic torus)

Let X be an irreducible subvariety of $(\mathbb{C}^\times)^n$. There are finitely many proper algebraic subgroups $\mathcal{H}_1, \ldots, \mathcal{H}_r \subsetneq (\mathbb{C}^\times)^n$ with

$$X \cap \bigcup_{\dim \mathcal{H} < n - \dim X} \mathcal{H} \subset \mathcal{H}_1 \cup \cdots \cup \mathcal{H}_r.$$

Here $\mathcal{H}$ runs over algebraic subgroups of $(\mathbb{C}^\times)^n$.

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Zilber expects additional uniformity for subvarieties X of fixed degree.

Some Known Cases

$$X \not\subset \text{proper algebraic subgroup} + \text{ZP}$$
$$\Downarrow$$
$$X \cap \bigcup_{\dim \mathcal{H} < n - \dim X} \mathcal{H} \quad \text{is not Zariski dense in } X.$$

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- (Bombieri-Masser-Zannier '07) X has codimension 2, and the $\mathcal{H}$ have dimension 1
- (The classical Manin-Mumford Conjecture) X has codimension 1 and the subgroups $\mathcal{H}$ are finite.

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(H. '09, generalized by Maurin '11) If

- X is defined over $\overline{\mathbb{Q}}$, and
- if X is non-degenerate, i.e.

$$\forall \text{ surjective } \varphi : (\mathbb{C}^\times)^n \twoheadrightarrow (\mathbb{C}^\times)^{\dim X} : \quad \dim \varphi(X) = \dim X$$

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These results are "pre Pila-Wilkie": there proofs rely on height-theoretic methods and (often) Ax's Theorem.

Degenerate Varieties

X is degenerate if

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Example

Take an algebraic curve $C \subset (\mathbb{C}^\times)^n$ with $n \geq 2$. The surface

$$C \times C \subset (\mathbb{C}^\times)^{2n}$$

is degenerate as $\varphi(C \times C) = C$ is a curve with

$$\varphi : (\mathbb{C}^\times)^{2n} \to (\mathbb{C}^\times)^n$$

the projection onto the first n coordinates.

Further partial results II

Theorem (Bays, Ph.D. thesis of '09)

Say X is defined over $\mathbb{R}$ and let $t \mapsto \mathcal{H}(t)$ be a 1-parameter polynomial family of algebraic subgroups. There is a finite collection of proper algebraic subgroups $\mathcal{H}_1, \ldots, \mathcal{H}_r \subsetneq (\mathbb{C}^\times)^n$ with

$$X(\mathbb{R}) \cap \bigcup_{\substack{t \in \mathbb{Z} \\ \dim \mathcal{H}(t) < n - \dim X}} \mathcal{H}(t) \subset \mathcal{H}_1 \cup \cdots \cup \mathcal{H}_r.$$

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The family $\mathcal{H}(t)$ is defined by

$$x_1^{a_{i1}(k)} \cdots x_n^{a_{in}(t)} = 1 \quad (1 \leq i \leq m)$$

with each $a_{ij} \in \mathbb{Z}[T]$.

Further partial results II cont'd

Bays also treated s-parameter polynomial families of algebraic subgroups

$$(t_1, \ldots, t_s) \mapsto \mathcal{H}(t_1, \ldots, t_s) \subset (\mathbb{C}^\times)^n.$$

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However, his result restricts to parameters

$$(t_1, \ldots, t_s) \in \mathbb{Z}^s \setminus E$$

with E definable in $\mathbb{R}_{\text{exp}}$ and $\dim E < s$.

A Question of Aaron Levin

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Suppose that C_1 and C_2 are curves in $(\mathbb{C}^\times)^n$. What can be said about the points $x \in C_1$ such that some power x^t $(t \geq 2)$ lies in C_2?

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These points produce

$$(x, x^t) \in C_1 \times C_2 \quad \text{in} \quad \mathcal{H}(t) = \{(x, y);\ y = x^t\} \subset (\mathbb{C}^\times)^{2n}.$$

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Note that $C_1 \times C_2$ is degenerate and

$$\dim \mathcal{H}(t) = n < 2n - \dim C_1 \times C_2 = 2n - 2 \quad \text{if} \quad n \geq 3.$$

As $t \to \infty$ the subgroup $\mathcal{H}_t$ "approaches" the kernel of the projection $(\mathbb{C}^\times)^{2n} \to (\mathbb{C}^\times)^n$ which is responsible for degeneracy.

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Theorem (Bays, H.)

Suppose $C \subset (\mathbb{C}^\times)^n$ is defined over $\overline{\mathbb{Q}}$ and not contained in a proper algebraic subgroup. If $n \geq 3$,

$$\mathbb{T} = S^1 \times \cdots \times S^1 \subset (\mathbb{C}^\times)^n \quad \textit{and} \quad C \cap \mathbb{T} \textit{ is finite}$$

there are only finitely many $x \in C$ with $x^t \in C$ for some $t \geq 2$.

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Finiteness $C \cap \mathbb{T}$ is not believed to be necessary.

Remarks on $C \cap \mathbb{T}$

- If C is the curve given by $x^a y^b = 1$ with $a, b \in \mathbb{Z}$, then C is a subgroup and $\# C \cap \mathbb{T} = \infty$.

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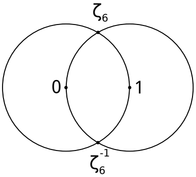
- If C is a line in general position, then $C \cap \mathbb{T}$ is finite. For example C given by $x + y = 1$:



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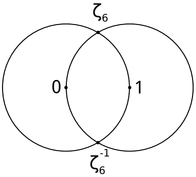

- But, if $|x| = 1$ then $\left|\frac{x-2}{2x-1}\right| = 1$, so $\left\{\left(x, \frac{x-2}{2x-1}\right)\right\} \cap \mathbb{T}$ is infinite. Many more examples can be constructed using Blaschke products.

The General Strategy

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Suppose (for simplicity) that $C \subset (\mathbb{C}^\times)^3$ is defined over $\mathbb{Q}$.
Let $x \in C$ have infinite order with $x^t \in C$ for some $t \geq 2$.

- Construct an integral point k_x in some set definable $\mathcal{Z}$ in $\mathbb{R}_{\mathrm{an,exp}}$.
  If $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ then $\sigma x \in C$ and $\sigma x^t \in C$.
  Number of integral $k_{\sigma x}$ on $\mathcal{Z}$ is $\gg [\mathbb{Q}(x) : \mathbb{Q}]$.

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- Apply Pila-Wilkie and use Ax's functional Schanuel to deduce a contradiction if $[\mathbb{Q}(x) : \mathbb{Q}] \geq t^{1/7}$ and t is large.
- Use results from transcendence theory and heights to prove $[\mathbb{Q}(x) : \mathbb{Q}] \geq t^{1/7}$.

Constructing $\mathcal{Z}$ and Integral Points

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We want to count the $k_x \in \mathbb{Z}^3$. They live on the set

$$\mathcal{Z}_t = \{k \in \mathbb{R}^3; \exists z \in \mathcal{F} \cap \exp^{-1}(C) \text{ with } tz - 2\pi i k \in \mathcal{F} \cap \exp^{-1}(C)\}$$

definable in $\mathbb{R}_{\mathrm{an,exp}}$. We think of $\mathcal{Z}_t$ as a definable family parametrized over $\mathbb{R}$ by t.

Applying Pila-Wilkie

Theorem (Pila-Wilkie '06)

Let $\mathcal{Z} \subset \mathbb{R}^n$ be definable in an o-minimal structure. For $\epsilon > 0$ there is a constant $c = c(\mathcal{Z}, \epsilon)$ such that

$$\#\left\{k \in (\mathcal{Z} \setminus \mathcal{Z}^{\mathrm{alg}}) \cap \mathbb{Q}^n;\ H(k) \leq T\right\} \leq cT^{\epsilon}$$

for all $T \geq 1$. Here $\mathcal{Z}^{\mathrm{alg}}$ is the union of all semi-algebraic, connected curves in $\mathcal{Z}$. Moreover, $c(\mathcal{Z}, \epsilon)$ is uniform over definable families.

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Uniformity in o-minimality implies $\#\{k_{\sigma x}\} \gg [\mathbb{Q}(x) : \mathbb{Q}]$.
The height $H(k)$ of $k \in \mathbb{Z}^3$ is the sup-norm of k. So $H(k_{\sigma x}) \leq t$.

Applying Pila-Wilkie cont'd

If t is large and if $[\mathbb{Q}(x):\mathbb{Q}] \geq t^{1/7}$ then Pila-Wilkie (with any fixed $\epsilon < 1/7$) implies

$$k_{\sigma x} \in \mathcal{Z}_t{}^{\mathrm{alg}}$$

for some σ.
Ax's functional Schanuel implies $\mathcal{Z}_t = \emptyset$ and thus a contraditon.
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But how?
Recall

$$\mathcal{Z}_t = \{k \in \mathbb{R}^3;\ \exists z \in \mathcal{F} \cap \exp^{-1}(C) \\ \text{with } tz - 2\pi i k \in \mathcal{F} \cap \exp^{-1}(C)\}$$

We get a semi-algebraic curve parametrized by $s \mapsto k(s)$. Then

$$\exp(2\pi i k(s)) \in \varphi(C \times C) \quad \text{with} \quad \varphi(x, y) = x^t y^{-1}.$$

Applying Ax's Theorem

$$\ldots \quad \exp(2\pi i k(s)) \text{ lies on the surface } \varphi(C \times C)$$

Now $\mathrm{trdeg}_{\mathbb{C}}\mathbb{C}(k(s)) \leq 1$ and $\mathrm{trdeg}_{\mathbb{C}}\mathbb{C}(\exp(2\pi i k(s))) \leq 2$. So

$$\mathrm{trdeg}_{\mathbb{C}}\mathbb{C}(k(s), \exp(2\pi i k(s))) \leq 3 < n + 1.$$

Ax's functional Schanuel and $n \geq 3$ imply that the components of $k(s)$ are $\mathbb{Z}$-linearly dependent.

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Ax's functional Schanuel and $n \geq 3$ imply that the components of $k(s)$ are $\mathbb{Z}$-linearly dependent.
$\Longrightarrow C$ is contained in the translate of a proper algebraic subgroup of $(\mathbb{C}^{\times})^3$. This easily contradicts our hypothesis.

Bounding the Height

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There exists B independent of x and t such that the absolute Weil height satisfies

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Recall: (Northcott) There are only finitely points of bounded height and bounded degree over $\mathbb{Q}$.

Bounded Height and Non-clustering

Theorem (Bilu '97)

Let $x_1, x_2, \ldots \in (\mathbb{Q}^\times)^n$ *be a sequence with* $\lim_{i\to\infty} h(x_i) = 0$ *such that each proper algebraic subgroup of* $(\mathbb{C}^\times)^n$ *contains at most finitely many* x_i*. Then* $\{\sigma(x_i);\ \sigma : \overline{\mathbb{Q}} \to \mathbb{C}\}$ *"become equidistributed" around* $\mathbb{T}$ *for* $i \to \infty$*.*

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We cannot apply this theorem to our points x. Instead we use:

Proposition

Let $\Sigma \subset (\mathbb{C}^\times)^n$ *be a finite set. Suppose* $x_1, x_2, \ldots \in (\overline{\mathbb{Q}}^\times)^n$ *is a sequence of distinct points with* $h(x_i) \le B$*. The conjugates of* x_i *do not "cluster around"* Σ*: there is* $\epsilon > 0$ *such that for all large* i

$$\mathrm{dist}(\sigma(x_i), \Sigma) \geq \epsilon \quad \textit{for some} \quad \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}).$$

Back to $x \in C$ and $x^t \in C$. After conjugating, x is not too close to $C \cap \mathbb{T}$:

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Figure: Black dots represent elements of $\mathbb{T} \cap C$

Without loss of generality $x = (x_1, x_2, x_3)$ satisfies $\max_i \log |x_i| \geq \epsilon$. Then x^t is close to a pole of a coordinate function. Expaning a coordinate function as a Puiseux series in another coordinate gives

$$x_1^{t\alpha} \approx c x_2^{t\beta}$$

for some $(\alpha, \beta) \in \mathbb{Z}^2 \setminus \{0\}$ and a constant $c \in \overline{\mathbb{Q}}^\times$.

Baker's Theorem

To be precisely, we obtain

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This is an exceptionally good approximating. If the absolute value is non-zero, then a theorem of Baker implies

$$\exp(-(\text{cnst.})[\mathbb{Q}(x) : \mathbb{Q}]^6 (1 + h(x))^2 \log t) \le |x_1^{t\alpha} x_2^{-t\beta} - c|.$$

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$$|x_1^{t\alpha} x_2^{-t\beta} - c| \leq \exp(-(\text{cnst.})t).$$

This is an exceptionally good approximating. If the absolute value is non-zero, then a theorem of Baker implies

$$\exp(-(\text{cnst.})[\mathbb{Q}(x) : \mathbb{Q}]^6 (1 + h(x))^2 \log t) \leq |x_1^{t\alpha} x_2^{-t\beta} - c|.$$

Since $h(x) \leq B$ and because exponential always beats polynomial we get

$$[\mathbb{Q}(x) : \mathbb{Q}] \geq t^{1/7}$$

for large t.

Knowing a height bound imposes a condition on all valuations (infinite and finite) of a point.
There's a p-adic version of Baker's Theorem due to Kunrui Yu. It allows us to drop the condition on $C \cap \mathbb{T}$ (at a price).
We obtain

Theorem (Bays+H.)

Let $C \subset (\mathbb{C}^\times)^n$ be defined over $\overline{\mathbb{Q}}$ and $n \geq 3$. Suppose C is not contained in a proper algebraic subgroup of $(\mathbb{C}^\times)^n$. There exists p_0 such that there are only finitely many

$$x \in C \quad \text{with} \quad x^t \in C \quad \text{for some } t \geq 2$$

with are integral with respect to a fixed, finite set of primes $\geq p_0$.

Open Question

Say $R \subset \mathbb{T} \subset (\mathbb{C}^\times)^3$ is a real algebraic curve not contained in a proper subgroup of $\mathbb{T}$. Are there only finitely many $x \in R$ with $x^t \in R$ for some $t \geq 2$?

Thanks for your attention!