A fundamental dichotomy for definably complete expansions of ordered fields

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Introduction

Joint work with Philipp Hieronymi.

A definably complete structure is either a model of second order Peano Arithmetic, or it is “tame”.

This dichotomy can be used to transfer several theorems from $\mathbb{R}$ to DC structures.

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Hieronymi’s dichotomy

Let $\mathfrak{R}$ be an expansion of the ordered field $\mathbb{R}$.

Theorem (Hieronymi ’10)

Only 2 cases are possible:

Restrained case: *either, for every definable discrete set $D \subset \mathbb{R}^n$ and every definable function $f : D \to \mathbb{R}$, $f(D)$ is nowhere dense in $\mathbb{R}$;*

Unrestrained case: *or $\mathbb{Z}$ is definable in $\mathfrak{R}$.*

Proposition

In the restrained case, if $f : \mathbb{R}^n \to \mathbb{R}$ is a definable continuous function, then f is differentiable outside a nowhere dense set (or, equivalently, outside a closed set of measure 0).

Definably complete structures

Definition

Let $\mathbb{K} = \langle K, <, \dots \rangle$ be a (linearly) ordered structure.
$\mathbb{K}$ is **definably complete** (DC) if every definable subset $X \subseteq K$ has a least upper bound in $K \cup \{\pm\infty\}$.

Examples

- Any expansion of $\mathbb{R}$ is DC.
- Any o-minimal structure is DC.
- "Being definably complete" is a first-order property: hence, any structure elementary equivalent to an expansion of $\mathbb{R}$ is also DC.
- The ordered field of rational numbers is not DC.

Dichotomy in DC fields

Let $\mathbb{K} = \langle K, <, +, \cdot, \dots \rangle$ be a DC expansion of an ordered field.

Theorem

Only 2 cases are possible:

Restrained case: *either, for every definable discrete set $D \subset K^n$ and every definable function $f : D \to K$, $f(D)$ is nowhere dense in K;*

Unrestrained case: *or a (unique) discrete subring Z is definable in $\mathbb{K}$.*

Proposition

In the restrained case, if $f : K^n \to K$ is a definable continuous function, then f is differentiable outside a nowhere dense set.

Examples

$\overline{\mathbb{R}} := \langle \mathbb{R}, +, \cdot, <, 0, 1 \rangle$.

Restrained structures:

- O-minimal structures;
- Locally o-minimal structures; e.g., ultraproducts of o-minimal structures;
- D-minimal structures; e.g., $\langle \overline{\mathbb{R}}, 2^{\mathbb{Z}} \rangle$ (Dries '85)
- Dense elementary pairs of o-minimal (more generally, d-minimal) structures; e.g., $\langle \overline{\mathbb{R}}, 2^{\mathbb{Z}}, \mathbb{R}^{alg} \rangle$.

Unrestrained structures:

- $\langle \overline{\mathbb{R}}, \mathbb{Z} \rangle$;
- $\langle \overline{\mathbb{R}}, 2^{\mathbb{Z}}, 3^{\mathbb{Z}} \rangle$ (Hieronymi '10);
- $\langle \overline{\mathbb{R}}, C \rangle$, where $C \subseteq \mathbb{R}^n$ is any closed set with non-integer Hausdorff dimension.

Models of arithmetic

Assume that $\mathbb{K}$ defines a discrete subring Z; let $N := Z_{\geq 0}$.
Since every definable subset of N has a minimum, N is a model of Peano arithmetic.
We can encode definable functions from N to N as elements of K (like in the real case, where we can use the continuous fraction expansion of a real number to encode sequences of natural numbers).
Therefore, N is a model of **second-order Peano arithmetic**.
We can also encode definable functions from N to K as elements of K (like in the real case, where we can encode a sequence of real numbers as a single real number).

Lemma

*The family of definable functions from N to K is a **definable** family.*

Application 1: Category Theory

Definition

$X \subseteq \mathbb{K}$ is **definably meager** if

$$X = \bigcup_{t \in K} X_t,$$

where $\left(X_t : t \in K\right)$ is a definable increasing family of nowhere dense subsets of $\mathbb{K}$.

Theorem (Hieronymi '13)

$\mathbb{K}$ is not definably meager.

Proof.

If $\mathbb{K}$ is unrestrained, we can transfer the proof of the usual Baire Category Theorem.
If $\mathbb{K}$ is restrained, every definably meager subset of K is nowhere dense. □

Application 2: Measure Theory

Assume that $\mathbb{K}$ is unrestrained and $N := Z_{\geq 0}$.
Given $a < b \in K$, let $|(a, b)| := b - a$.
Let $\mathfrak{A} := \left(A_i : i \in N\right)$ be a definable family of intervals: define

$$M(\mathfrak{A}) := \sum_{i \in N} |A_i|.$$

(I.e., there exists a unique definable function $f : N \to K_{\geq 0}$ such that $f(0) = 0$ and $f(n+1) = f(n) + |A_n|$.
Define $M(\mathfrak{A}) := \sup_n f(n)$).

Given $X \subseteq K$ definable set, let the definable measure of X be

$$\mu^*(X) := \inf\left\{ M(\mathfrak{A}) : \mathfrak{A} \text{ definable family of intervals covering } X \right\}.$$

μ^* is the definable analogue of the (outer) Lebesgue measure.

Lebesgue's Differentiation Theorem

Theorem (Miller's Conjecture)

Let $f : K \to K$ be a definable monotonic function.
Then, f is differentiable on a dense subset of K.

Proof.

If $\mathbb{K}$ is restrained, then f is continuous on an open dense set.
For every definable continuous function f, f is differentiable on an open dense set.

If $\mathbb{K}$ is unrestrained, then f is differentiable outside a set of definable measure zero. □

Sketch of proof of the Dichotomy

Theorem

Let $D \subset K_{\geq 0}$ be a definable, closed, unbounded, and discrete set and $f : D \to K$ be a definable function.
If $f(D)$ is dense in K, then $\mathbb{K}$ defines a discrete subring.

Definition

Let $D \subset K_{\geq 0}$ be a definable, closed, discrete set.
For every $a \in D$, define $s_D(a)$ to be the **successor** of a in D.
We say that D has **step 1** if, for every $a \in D$ not the maximum, $s_D(a) = a + 1$.
D is a **natural fragment** if it has step 1 and $\min(D) = 0$.

Remark

Assume that $N \subset K$ is an unbounded natural fragment.
Then, N is the positive part of a discrete subring of K.

Asymptotic extraction

Let $\mathfrak{A} := \left(A_i : i \in I\right)$ be a definable family of closed discrete subset of $K_{\geq 0}$.

Definition

$a \in K$ is in the **natural fragment extracted from** $\mathfrak{A}$ if:
for every $\varepsilon > 0$ there exists $i \in I$ such that:

$$\begin{aligned} d(A_i, 0) &< \varepsilon; \\ d(A_i, a) &< \varepsilon; \\ \forall d \in A_i \quad |s_{A_i}(d) - d - 1| &< \varepsilon. \end{aligned}$$

The natural fragment extracted from $\mathfrak{A}$ is indeed a natural fragment.

Proof

...continued

Let $f : D \to K$ be definable, with $D \subset K_{>0}$ closed, discrete, and unbounded, and $f(D)$ dense in K.
Then, there exists a definable family

$$\mathfrak{Y} = \left(Y_{s,t,d} : s, t \in K, d \in D\right)$$

of closed, discrete, and bounded subsets of K, such that the natural fragment extracted from $\mathfrak{Y}$ is unbounded.

More generally, for every $\varepsilon > 0$ and $X \subset K_{\geq 0}$ definable, closed, discrete, and bounded, there exist (s, t, d), such that $d(Y_{s,t,d}, X) < \varepsilon$.

Open problems

Brouwer's Fixed Point Theorem Let $f : [0,1]^2 \to [0,1]^2$ be a continuous definable function.
Does f have a fixed point?

Pigeon Hole Principle Let $D \subset K$ be definable, closed, discrete, and bounded. Let $f : D \to D$ be definable and injective.
Is f surjective?

In both case, answer is "YES" if $\mathbb{K}$ is either unrestrained or o-minimal.

A. Fornasiero and P. Hieronymi.
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