A non-compact version of Pillay's conjecture

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Ravello 2013

Outline

1. Pillay's conjecture and related work
2. The non-compact case
3. Valued groups and model theory

An easy example

Given $\mathcal{M} = \langle M, <, +, \cdot \rangle$ a $\aleph_1$-saturated rcf, we can define a definably compact group $G = ([0,1[, \oplus)$

$$x \oplus y = \begin{cases} x+y & \text{if } x+y<1 \\ x+y-1 & \text{otherwise.} \end{cases}$$

the standard part map is a surjective homomorphism

$$st\colon [0,1[^M \to [0,1[^{\mathbb{R}}$$

$\ker st = \bigcap_{n\in\mathbb{N}} [0, \frac{1}{n}[\cup]1 - \frac{1}{n}, 1[$ is the subgroup of the infinitesimals of G.
$G/\ker st$ is isomorphic to the circle (1-dimensional torus).

Pillay's Conjecture (PC)

Pillay's conjecture

Let G be a group definable in a (sufficiently saturated) o-minimal structure $\mathcal{M}$. Then

1. $\exists$ a smallest type-definable subgroup of bounded index G^{00}.
2. With the logic topology G/G^{00} is a compact real Lie group.
3. G is definably compact $\Rightarrow$ $\dim_{\mathcal{M}} G = \dim_{\mathbb{R}} G/G^{00}$.
4. G is abelian $\Rightarrow$ G^{00} is divisible and torsion-free.

Logic topology

$X \subset G/G^{00}$ is closed $\Leftrightarrow$ $\pi^{-1}(X) \subset G$ is type-definable, where $\pi \colon G \to G/G^{00}$ is the canonical projection.

The Positive Solution

Theorem (Pillay 2004)

PC holds when $\dim G = 1$ *and when* G *is definably simple.*

Theorem (Berarducci - Otero - Peterzil - Pillay 2005)

Every group definable in an o-minimal structure has the DCC on type-definable subgroups of bounded index.

Theorem (Hrushovski - Peterzil - Pillay 2008 (fields), Eleftheriou 2008 (linear), Peterzil 2009 (groups), Edmundo - Terzo 2008 (orientable))

G *is definably compact* $\Rightarrow \dim_{\mathcal{M}} G = \dim_{\mathbb{R}} G/G^{00}$.

Related work in the compact case

Theorem (Berarducci 07, Baro 09, Berarducci - Mamino 09)

The functor $G \mapsto G/G^{00}$ *is exact and preserves the homotopy type.*

Theorem (Hrushovski - Peterzil - Pillay, 2010)

There is an elementary embedding $\sigma \colon G/G^{00} \to G$ *which is a section for the canonical projection* $\pi \colon G \to G/G^{00}$*, and therefore*

$$\langle G, \cdot \rangle \;\equiv\; \langle G/G^{00}, \cdot \rangle$$

Theorem (Eleftheriou 2009, Hrushovski - Peterzil - Pillay 2010)

G *is compactly dominated by* G/G^{00}*, i.e. the set*

$$\{c \in G/G^{00} \mid \pi^{-1}(c) \cap X \neq \emptyset \;\wedge\; \pi^{-1}(c) \cap (G \setminus X) \neq \emptyset\}$$

has Haar measure equal to 0 *for every definable* $X \subset G$.

G^{00} in the non-compact case

Theorem (Peterzil - Steinhorn 1999)

If G is not definably compact then contains a definable 1-dimensional torsion-free subgroup H.

Theorem (Pillay 2004)

H as above $\Rightarrow$ $H = H^{00}$.

Theorem (C-Pillay 2012)

H definable torsion-free $\Rightarrow$ $H = H^{00}$.

Theorem (Pillay 2004)

G non-definably compact and definably simple $\Rightarrow$ $G = G^{00}$.

Examples where $G = G^{00}$

Let $\mathcal{M}$ be a sufficiently saturated o-minimal expansion of a rcf

1. $G =$ any connected linear group of triangular matrices $= G^{00}$.
2. $G = \mathrm{SL}_n(M) = \mathrm{SL}_n(M)^{00}$.
3. (C - Pillay 2012) $G = \mathrm{SO}_2(M) \times_{\mathbb{Z}} \widetilde{\mathrm{SL}}_2(M) = G^{00}$.

G/G^{00} when G is not definably compact

Theorem (C - Pillay 2012)

Let G be a definably connected group. G contains a maximal normal definable torsion-free subgroup $\mathcal{N}(G)$, and $G/\mathcal{N}(G) = K \cdot H$, where K is definably compact and H is torsion-free.

In general G/G^{00} is a proper quotient of K/K^{00}, and $G/G^{00} = K/K^{00} \iff G/\mathcal{N}(G)$ is definably compact.

Problem

Is there a non-compact version of Pillay's conjecture?

Namely, is there a canonical way to associate a real Lie group L_G to any definable group G, so that first-order, algebraic and geometric properties (such as dimension, torsion structure, homotopy type...) are preserved?

Valued groups

Let G be a group and $\Gamma < \infty$ a totally ordered set. A valuation on G is a map

$$v\colon G \longrightarrow \Gamma \cup \{\infty\}$$

such that

- $v(x) = \infty \;\Leftrightarrow\; x = e$.
- $v(xy^{-1}) \geq \min\{v(x), v(y)\}$.

Remark
If $v(x) \neq v(y)$ then $v(xy^{-1}) = \min\{v(x), v(y)\}$

The natural valuation of ordered abelian groups

Any ordered abelian group $(G, <, +)$ has a natural valuation

$$v\colon G \longrightarrow \Gamma \cup \{\infty\}$$

$$v(x) \leq v(y) \iff \text{there is } n \in \mathbb{N} \text{ such that } n|x| > |y|.$$

For every $\gamma \in \Gamma$ we set

$$G^{\gamma} := \{a \in G : v(a) \geq \gamma\}.$$
$$G_{\gamma} := \{a \in G : v(a) > \gamma\}.$$

And

$$G^{\gamma}/G_{\gamma} = B(\gamma)$$

is called *the Archimedean component* associated to γ.

Work in progress with P. D'Aquino and S. Kuhlmann

Valued grups and model theory

Let $\mathcal{M}$ be an o-minimal expansion of an ordered group, and v its natural valuation. Then $\mathcal{M}$ is ω-saturated if and only if

1. v($\mathcal{M}$) is a dense linear ordering.
2. all Archimedean component are isomorphic to $\mathbb{R}$.
3. every pc-sequence in a substructure of finite dimension has a pseudo-limit in $\mathcal{M}$.

pc-sequences and pseudo-limits

A well ordered set $\{a_\rho\}_{\rho<\lambda}$ is a *pc-sequence* if for every $\rho < \sigma < \tau$ we have $v(a_\sigma - a_\rho) \ < \ v(a_\tau - a_\sigma)$. We say that x is a *pseudo-limit* of $\{a_\rho\}_{\rho<\lambda}$ if $\{v(x - a_\sigma)\}$ is eventually strictly increasing.

A plan (work in progress)

Let G be a definably connected group.

1. Find the "intrinsic" notion of convex hull $\mathcal{G}_x$ of $x \in G$.
2. Understand for which x there is $\mathcal{G}_x^{00}$.
3. $L_x := \mathcal{G}_x / \mathcal{G}_x^{00}$ with the logic topology is a real Lie group.
4. L_x does not depend on x (call it L_G).
5. $\dim G = \dim L_G$.
6. $G = \mathcal{G}_x$ if and only if G is definably compact.
7. $(\langle G, \cdot \rangle \equiv \langle L_G, \cdot \rangle$??)

Hopefully

- There is a valuation v on G such that

$$\mathcal{G}_x = \{g \in G : v(g) \geq v(x)\}$$

$$\mathcal{G}_x^{00} = \{g \in G : v(g) > v(x)\}$$

So a Lie group would be "the residue" of a "valuation subgroup" in a definable group.