Around the Canonical Base property

Zoé Chatzidakis

CNRS (UMR 7586) - Université Paris 7

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Setting

Complete theory T, which eliminates imaginaries (and hyperimaginaries). All types considered will be of finite SU-rank, and for convenience we assume that the theory is supersimple.

We work in a monster model $\mathcal{U}$.

Independence

We have a good notion of *independence*, denoted by $A \downfree_C B$: A and B do not fork, or are independent, over C. If $C =$, we omit it from the notation. It satisfies:

- stable under $\mathrm{Aut}(\mathcal{U})$,
- symmetric: $A \downfree_C B \iff B \downfree_C A$,
- transitive: If $C \subset D \subset B$, then $A \downfree_C B \iff A \downfree_C D$ and $A \downfree_D B$.
- local character: $A \downfree_C B \iff \forall$ finite $A_0 \subset A$, $A_0 \downfree_C B$.
- Extension property: for every A, B, C, there is A', $A' \equiv_C A$, such that $A' \downfree_C B$.

More properties of independence

- Independence theorem: if $M \models T$, and A, B, C_1, C_2 are such that $A \downarrow_M B$, $C_1 \downarrow_M A$, $C_2 \downarrow_M B$ and $C_1 \equiv_M C_2$, then there is $C \downarrow_M AB$ which realises $tp(C_1/MA) \cup tp(C_2/MB)$.
- $A \downarrow_C A \iff A \subset \mathrm{acl}(C)$.
- $A \downarrow_C B \iff \mathrm{acl}(CA) \downarrow_{\mathrm{acl}(C)} \mathrm{acl}(CB)$
- $A \downarrow_C B$ and $A \downarrow_B C$ imply $A \downarrow_D AB$, where $D = \mathrm{acl}(B) \cap \mathrm{acl}(C)$. (This comes from ei)

SU-rank

The SU-rank of a type $p = tp(a/A)$ is defined by induction, the important step being:
$\mathrm{SU}(p) = \mathrm{SU}(a/A) \geq \alpha$ if and only if p has a forking extension q with $\mathrm{SU}(q) \geq \alpha$.
$\mathrm{SU}(p)$ is then the smallest ordinal α such that $\mathrm{SU}(p) \not\geq \alpha + 1$, if such an ordinal exists, and ∞ otherwise.

One has:
$\mathrm{SU}(a/A) = 0$ if and only if $a \in \mathrm{acl}(A)$.
$\mathrm{SU}(a/A) = 1$ if and only if $a \notin \mathrm{acl}(A)$ and for every $B \supset A$, either $a \mathop{\smile\!\!\!\!\!\!|}_A B$ or $a \in \mathrm{acl}(B)$.

Canonical base

Definitions. (1) Let p be a type over $C = \mathrm{acl}(C)$, realised by a tuple a. Then $\overline{\mathrm{Cb}}(p) = \overline{\mathrm{Cb}}(a/C)$ is the smallest algebraically closed subset B of C such that $a \mathop{\smile\!\!\!\!\!\mid}_B C$.
(2) Let S be an ∞-definable set, defined over $C = \mathrm{acl}(C)$. Then S is *one-based* if whenever $a_1, \dots, a_n \in S$ and $B = \mathrm{acl}(B) \supset C$, then $\overline{\mathrm{Cb}}(a_1, \dots, a_n/B) \subset \mathrm{acl}(C, a_1, \dots, a_n)$. In other words,

$$a_1 \dots, a_n \mathop{\smile\!\!\!\!\!\mid}_D B$$

where $D = \mathrm{acl}(Ca_1 \dots, a_n) \cap B$.
(3) A type is *one-based* if the set of its realisations is one-based.

Some examples

1 - Infinite set with no structure.
$\mathcal{L} = \{ \}$
$A \subset \mathcal{U}$. Then $\mathrm{acl}(A) = A$.
$A \downarrow_C B \iff A \cap B \subset C$.
All types are one-based.

2 - Infinite vector spaces over a field k.
$\mathcal{L} = \{+, -, 0, \lambda\}_{\lambda \in k}$.
$A \subset \mathcal{U}$. Then $\mathrm{acl}(A)$ is the subspace generated by A.
$A \downarrow_C B \iff \langle CA \rangle \cap \langle CB \rangle = \langle C \rangle$.
All types are one-based.

Examples (ctd)

3 - ACF_0 or ACF_p (algebraically closed fields)
$\mathcal{L} = \{+, -, \cdot, 0, 1\}$
$A \subset \mathcal{U}$. Then $\mathrm{acl}(A)$ is the field-theoretic algebraic closure (inside $\mathcal{U}$) of the field generated by A.
$A \downarrow_C B$ if and only if $\mathrm{acl}(CA)$ and $\mathrm{acl}(CB)$ are free over $\mathrm{acl}(C)$. In other words, if a is a finite tuple in A, and $C \subset B$ are fields, then $trdeg(C(a)/C) = trdeg(B(a)/B)$.

Examples - ACF (ctd)

Only one non-algebraic type over a set $C = \mathrm{acl}(C)$: the type of an element transcendental over C. This type is not one-based: let $C = \mathbb{Q}$, and a, b, c algebraically independent transcendental elements over $\mathbb{Q}$, let $d = ac + b$. Then $\mathrm{acl}(a, b) \cap \mathrm{acl}(c, d) = \mathbb{Q}^{alg}$, but clearly $a, b \not\perp c, d$.
If a is a tuple, and B an algebraically closed field, then $\overline{\mathrm{Cb}}(a/B)$ is the algebraic closure of the field of definition of the algebraic locus of a over B (= smallest variety defined over B and to which a belongs).

Examples (ctd)

4 - DCF_0 Differentially closed fields of characteristic 0
$\mathcal{L} = \{+, -, \cdot, 0, 1, D\}$, D symbol for the derivation.
If $A \subset \mathcal{U}$, then $\mathrm{acl}(A)$ is the field-theoretic algebraic closure (inside $\mathcal{U}$) of the differential field generated by A.
$A \downarrow_C B \iff \mathrm{acl}(CA) \downarrow^{ACF}_{\mathrm{acl}C)} \mathrm{acl}(CB)$.
Not one-based. Culprit in rank 1: $Dx = 0$.

Examples (ctd)

5 - ACFA. Existentially closed difference fields ($=$ fields with an automorphism). $\mathcal{L} = \{+, -, \cdot, 0, 1, \sigma\}$
If $A \subset \mathcal{U}$, then $\mathrm{acl}(A)$ is the smallest algebraically closed field containing A and closed under σ, σ^{-1}.
$A \downarrow_C B \iff \mathrm{acl}(CA) \downarrow^{ACF}_{\mathrm{acl}(C)} \mathrm{acl}(CB)$.
The theory is not one-based.
Culprits in rank 1: $\mathrm{Fix}(\sigma) = \{a \in \mathcal{U} \mid \sigma(a) = a\}$; in positive characteristic also $\mathrm{Fix}(\sigma^m \mathrm{Frob}^n)$, $m \in \mathbb{N}^*$, $n \in \mathbb{Z}$.

Internality

Let $\mathcal{S}$ be a collection of types closed under $\mathrm{Aut}(\mathcal{U})$-conjugation. We say that a type q (over a set A) is $\mathcal{S}$-*internal* if whenever a realises q, then there is B independent from a over A such that $a \in \mathrm{dcl}(B, d)$ for some set d of realisations of types in $\mathcal{S}$.

Almost internality

Let $\mathcal{S}$ be a collection of types closed under $\mathrm{Aut}(\mathcal{U})$-conjugation. We say that a type q (over a set A) is *almost-$\mathcal{S}$-internal* if whenever a realises q, then there is B independent from a over A such that $a \in \mathrm{acl}(B, d)$ for some set d of realisations of types in $\mathcal{S}$.

Analysis

Fact. Let $q = tp(a/A)$ be a type of finite SU-rank, and let $\mathcal{S}$ be the set of types of SU-rank 1. Then there are $a_1, \ldots, a_n$ such that $\mathrm{acl}(Aa) = \mathrm{acl}(Aa_1, \ldots, a_n)$ and for each $i \leq n$,

$$tp(a_i/\mathrm{acl}(Aa_1, \ldots, a_{i-1})) \text{ is almost-}\mathcal{S}\text{-internal.}$$

We speak of the *levels of the analysis*.
If the analysis only involves types in some subset $\mathcal{S}_0$ of $\mathcal{S}$, then we say that $tp(a/A)$ is $\mathcal{S}_0$-*analysable*.

The Canonical Base property (CBP) - a bit of history

In 2002, Anand Pillay gives a model-theoretic translation of a property enjoyed by compact complex manifolds (and proved earlier, independently, by Campana and Fujiki). With Martin Ziegler, he then shows in 2003 that various algebraic structures enjoy this property (differentially closed fields of characteristic 0; existentially closed difference fields of characteristic 0). As with compact complex manifolds, their proof has as immediate consequence the dichotomy for types of rank 1 in these algebraic structures: if they are not one-based, they are non-orthogonal to a field (the field of constants, or the fixed field).
Several people have worked on this topic: apart from Pillay and Ziegler: Moosa, Hrushovski, Juhlin, Wagner, Palacin, and myself of course.
A recent example by Hrushovski shows that not all theories have the CBP.

Definition of the CBP

A theory T has the CBP if whenever $tp(A/B)$ has finite SU-rank, and $B = \overline{\mathrm{Cb}}(A/B)$, then $tp(B/\mathrm{acl}(A))$ is almost-$\mathcal{S}$-internal, where $\mathcal{S}$ is the family of types of SU-rank 1 with algebraically closed base.

Compare with one-based: if T is one-based, and $B = \overline{\mathrm{Cb}}(A/B)$, then $B \subset \mathrm{acl}(A)$.
Thus here, we are not saying that the canonical base of a type is contained in the algebraic closure of a realisation of the type; but that its type over this algebraic closure is almost-internal to a particular set of types of rank 1. In other words, the analysis of $tp(B/A)$ only involves one level, and as we will see later, types which are not one-based.

Example in DCF_0

(Stronger statement, proved in [PZ]) Let $S \subset X \times Y$ be definable. Viewing S as a family of definable subsets S_x of Y, assume that for $x \neq x'$ in X, S_x and $S_{x'}$ do not have the same generics, and have finite dimension. Fix some $b \in X$, a generic a of S_b. The CBP then gives strong restrictions on the set $S^a = \{x \in X \mid a \in S_x\}$: generically, it will be in bijection with $W(C)$, where W is an algebraic variety, C the field of constants. (Think of b as $\mathrm{Cb}(a/b)$, and of S^a as the set of realisations of $tp(b/a)$).

How to prove the CBP?

Theorem. Let A, B be algebraically closed with intersection C, and assume that $B = \overline{\mathrm{Cb}}(A/B)$, and $SU(B/C) < \omega$. Then there are types $p_1, \ldots, p_n$ of SU-rank 1, and $B_1, \ldots, B_n$ such that $\mathrm{acl}(B_1, \ldots, B_n) = B$, and for each i, $tp(B_i/C)$ is $\mathcal{S}_i$-analysable, where $\mathcal{S}_i$ is the set of $\mathrm{Aut}(\mathcal{U})$-conjugates of p_i. Furthermore these p_i are not-one-based, and are pairwise orthogonal.

Hence, it suffices to show the CBP for sets A, B, with $tp(B/C)$ (or $tp(A/C)$) $\mathcal{S}$-analysable [in two steps], and $\mathcal{S}$ the set of conjugates of a type of SU-rank 1.

Consequences of the CBP

From now on, theories will have the CBP, and all types considered will have finite SU-rank.

If A, B are algebraically closed, with B of finite rank over $C = A \cap B$, and $B = \overline{\mathrm{Cb}}(A/B)$, then $tp(B/C)$ is almost-$\mathcal{S}$-internal, for some family $\mathcal{S}$ of types of SU-rank 1 (and which are non-one-based).

Consequences of the CBP (ctd)

[CBP, all types considered have finite SU-rank. $\mathcal{S}$ a set of types of SU-rank 1, closed under conjugation]

(A, B, B_1, B_2 algebraically closed).
If $tp(A/B)$ is almost $\mathcal{S}$-internal, then so is $tp(A/A \cap B)$.

If $tp(A/B_1)$ and $tp(A/B_2)$ are almost-$\mathcal{S}$-internal, then so is $tp(A/B_1 \cap B_2)$.

Consequences of the CBP (ctd)

[CBP, all types considered have finite SU-rank. $\mathcal{S}$ a set of types of SU-rank 1, closed under conjugation]

There is a smallest algebraically closed B such that $tp(A/B)$ is almost-$\mathcal{S}$-internal. (And it is contained in $\mathrm{acl}(A)$)

(The UCBP). Assume $B = \overline{\mathrm{Cb}}(A/B)$ has finite SU-rank, and $tp(B/A)$ is almost-$\mathcal{S}_0$-internal for some $\mathcal{S}_0 \subset \mathcal{S}$. If D is such that $tp(A/D)$ is almost-$\mathcal{S}_0$-internal, then so is $tp(AB/D)$.

Consequences of the CBP (ctd)

[CBP, all types considered have finite SU-rank. $\mathcal{S}$ a set of types of SU-rank 1, closed under conjugation]

Given A, there is a smallest algebraically closed $B_0 \subset \mathrm{acl}(A)$ such that $tp(A/B_0)$ is almost-$\mathcal{S}$-internal. But then $B_0 = \mathrm{acl}(A \cap B_0)$.

There is a largest $B_1 \subset \mathrm{acl}(A)$ such that $tp(B_1)$ is almost-$\mathcal{S}$-internal. Then $B_1 = \mathrm{acl}(A \cap B_1)$.

Consequences of the CBP (ctd)

[CBP, all types considered have finite SU-rank. $\mathcal{S}$ a set of types of SU-rank 1, closed under conjugation]

A descent result: Let A_1, A_2, B_1, B_2 be such that $tp(B_2)$ is almost-$\mathcal{S}$-internal; $\mathrm{acl}(B_1) \cap \mathrm{acl}(B_2) = \mathrm{acl}(\emptyset)$; $A_1 \mathop{\smile\!\!\!\!\!\!\mid}_{B_1} B_2$ and $A_2 \mathop{\smile\!\!\!\!\!\!\mid}_{B_2} B_1$; $A_2 \subset \mathrm{acl}(A_1 B_1 B_2)$.
Then there is $E \subset \mathrm{dcl}(A_2 B_2)$ such that $tp(A_2/E)$ is almost-$\mathcal{S}$-internal and $E \mathop{\smile\!\!\!\!\!\!\mid} B_2$.

Application to algebraic dynamics

Let K_1, K_2 be fields intersecting in k and with algebraic closures intersecting in k^{alg}. For $i = 1, 2$ let V_i be an irreducible variety and $\phi_i : V_i \to V_i$ a dominant rational map, everything defined over K_i. Assume that K_2 is a regular extension of k, and that there are an integer $r \geq 1$, and a dominant rational map $f : V_1 \to V_2$ such that $f \circ \phi_1 = \phi_2^{(r)} \circ f$. Then there are a variety V_0 and a dominant rational map $\phi_0 : V_0 \to V_0$, all defined over k, and a rational dominant map $g : V_2 \to V_0$, such that $f \circ \phi_2 = \phi_0 \circ f$, and $\deg(\phi_0) = \deg(\phi_2)$. (In fact, one can show that the generic fiber of g is $\mathrm{Fix}(\sigma)$-internal).