Height gaps versus Spectral gaps

E. Breuillard

Université Paris-Sud, Orsay

Ravello, June 11 2013

Growth of groups

Let Γ be a group generated by a finite symmetric set S. The growth type of Γ is the asymptotics of $|S^n|$ as $n \to \infty$.

Growth of groups

Let Γ be a group generated by a finite symmetric set S. The growth type of Γ is the asymptotics of $|S^n|$ as $n \to \infty$.

• For example Gromov's theorem says that if $|S^n|$ grows at most polynomially in n, then Γ is virtually nilpotent (= has a nilpotent subgroup of finite index).

Growth of groups

Let Γ be a group generated by a finite symmetric set S. The growth type of Γ is the asymptotics of $|S^n|$ as $n \to \infty$.

• For example Gromov's theorem says that if $|S^n|$ grows at most polynomially in n, then Γ is virtually nilpotent ($=$ has a nilpotent subgroup of finite index).

• new proofs by Hrushovski (2009), Breuillard-Green-Tao (2011) related to new developments on approximate groups involving model-theoretic ideas...

• If Γ has exponential growth, we may want to consider the rate of growth:

$$\rho_S := \lim_{n\to\infty} \frac{1}{n} \log |S^n|$$

Growth of groups

When Γ is a linear group we have the following conjecture:

Conjecture (Growth Gap Conjecture)

There is $\varepsilon = \varepsilon(d) > 0$ such that for every field K and finite set $S \subset \mathrm{GL}_d(K)$, either $\rho_S = 0$ and $\langle S \rangle$ is virtually nilpotent, or

$$\rho_S := \lim_{n\to\infty} \frac{1}{n} \log |S^n| > \varepsilon.$$

Growth of groups

When Γ is a linear group we have the following conjecture:

Conjecture (Growth Gap Conjecture)

There is $\varepsilon = \varepsilon(d) > 0$ such that for every field K and finite set $S \subset \mathrm{GL}_d(K)$, either $\rho_S = 0$ and $\langle S \rangle$ is virtually nilpotent, or

$$\rho_S := \lim_{n \to \infty} \frac{1}{n} \log |S^n| > \varepsilon.$$

We know the following:

Theorem (B '08)

This holds unless $\langle S \rangle$ is virtually solvable.

When Γ is a linear group we have the following conjecture:

Conjecture (Growth Gap Conjecture)

There is $\varepsilon = \varepsilon(d) > 0$ such that for every field K and finite set $S \subset \mathrm{GL}_d(K)$, either $\rho_S = 0$ and $\langle S \rangle$ is virtually nilpotent, or

$$\rho_S := \lim_{n\to\infty} \frac{1}{n} \log |S^n| > \varepsilon.$$

We know the following:

Theorem (B '08)

This holds unless $\langle S \rangle$ is virtually solvable.

- It also holds if K is of positive characteristic, so the key case is $K = \overline{\mathbb{Q}}$.

When Γ is a linear group we have the following conjecture:

Conjecture (Growth Gap Conjecture)

There is $\varepsilon = \varepsilon(d) > 0$ such that for every field K and finite set $S \subset \mathrm{GL}_d(K)$, either $\rho_S = 0$ and $\langle S \rangle$ is virtually nilpotent, or

$$\rho_S := \lim_{n \to \infty} \frac{1}{n} \log |S^n| > \varepsilon.$$

• The conjecture reduces to the case when $d = 2$ and $S = S(x)$ is as follows:

$$S(x) := \{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \}$$

where $x \in \overline{\mathbb{Q}}$ is an algebraic unit.

$$S(x) := \{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}\}$$

where $x \in \overline{\mathbb{Q}}$ is an algebraic unit.

In fact we have the following inequality:

$$[\mathbb{Q}(x) : \mathbb{Q}] \cdot h(x) \geqslant \rho_{S(x)}$$

where $h(x)$ is the absolute Weil height of x.

$$h(x) := \frac{1}{[\mathbb{Q}(x) : \mathbb{Q}]} \sum_v n_v \log^+ |x|_v$$

In particular, we see that:

The growth gap conjecture implies the Lehmer conjecture.

Non amenability

Definition (Amenable group)

A discrete group Γ is said to be amenable if there is a sequence $f_n \in \ell^2(\Gamma)$ of unit vectors such that $||\gamma \cdot f_n - f_n||_{\ell^2(\Gamma)}$ tends to 0 as $n \to +\infty$.

Definition (Amenable group)

A discrete group Γ is said to be amenable if there is a sequence $f_n \in \ell^2(\Gamma)$ of unit vectors such that $||\gamma \cdot f_n - f_n||_{\ell^2(\Gamma)}$ tends to 0 as $n \to +\infty$.

Examples of

• amenable groups: $\mathbb{Z}^d$, solvable groups, finite groups, groups with sub-exponential growth, etc.

• non amenable groups: free groups (von Neumann), large Burnside groups (Adian), non virtually solvable subgroups of GL_n (Tits alternative), arithmetic groups in semisimple Lie groups, etc.

Non amenability

Definition (Amenable group)

A discrete group Γ is said to be amenable if there is a sequence $f_n \in \ell^2(\Gamma)$ of unit vectors such that $||\gamma \cdot f_n - f_n||_{\ell^2(\Gamma)}$ tends to 0 as $n \to +\infty$.

• Equivalently, a group Γ is non-amenable if for some (every) generating set S the associated *Kazhdan constant* $\kappa_S := \kappa(S, \ell^2(\Gamma))$ is positive.

$$\kappa_S := \inf_{f \in \ell^2(\Gamma) \setminus \{0\}} \{\max_{s \in S} \frac{||\pi(s)f - f||}{||f||}\} > 0$$

Amenability vs. Growth

Non amenable groups have exponential growth, indeed they satisfy the following *linear isoperimetric inequality*:
For every finite subset $A \subset \Gamma$ and finite generating set S,

$$|AS| \geqslant (1 + \frac{\kappa_S^2}{2})|A|$$

[just take $f = \mathbf{1}_A$ the indicator function of A.] In particular:

Amenability vs. Growth

Non amenable groups have exponential growth, indeed they satisfy the following *linear isoperimetric inequality*:
For every finite subset $A \subset \Gamma$ and finite generating set S,

$$|AS| \geqslant (1 + \frac{\kappa_S^2}{2})|A|$$

[just take $f = \mathbf{1}_A$ the indicator function of A.] In particular:

$$\rho_S := \lim_{n\to\infty} \frac{1}{n} \log|S^n| \geqslant 1 + \frac{\kappa_S^2}{2}$$

So non-amenability implies exponential growth.

Uniform non-amenability

Theorem (B '08)

There is $\varepsilon = \varepsilon(d) > 0$ *such that for every field* K *and every finite subset* $S \subset \mathrm{GL}_d(K)$ *such that* $\langle S \rangle$ *is non amenable,*

$$\kappa_S > \varepsilon$$

Remark: this builds on earlier work with T. Gelander (partial uniformity: in S only, not K).

Question: Does this relate to a height lower bound as in the solvable case ?

Theorem (B '08)

There is $\varepsilon = \varepsilon(d) > 0$ *such that for every field* K *and every finite subset* $S \subset \mathrm{GL}_d(K)$ *such that* $\langle S \rangle$ *is non amenable,*

$$\kappa_S > \varepsilon$$

Remark: this builds on earlier work with T. Gelander (partial uniformity: in S only, not K).

Question: Does this relate to a height lower bound as in the solvable case ?

Answer: Yes

Theorem (B '08)

There is $\varepsilon = \varepsilon(d) > 0$ such that for every field K and every finite subset $S \subset \mathrm{GL}_d(K)$ such that $\langle S \rangle$ is non amenable,

$$\kappa_S > \varepsilon$$

Let $h(g)$ be a height function on $M_{d,d}(\overline{\mathbb{Q}})$, e.g. if $g \in M_{d,d}(K)$ and K is a number field with local degrees $n_v := [K_v : \mathbb{Q}_v]$,

$$h(g) := \frac{1}{[K:\mathbb{Q}]} \sum_{v \in V_K} n_v \log^+ ||g||_v$$

Let $\mathbb{A}_K$ be the ring of adèles of K.

Spectral gap vs. Height gap

Lemma

The function $g \mapsto e^{-C[K:\mathbb{Q}]h(g)}$ *is in* $L^2(\mathrm{SL}_d(\mathbb{A}_K))$ *if* $C > d^2$.

But $\langle S \rangle$ is discrete in $\mathrm{GL}_d(\mathbb{A}_K)$, so in restriction to $\langle S \rangle$ this function is in $\ell^2(\langle S \rangle)$.

Corollary (Lehmer-type bound)

For some $c = c(d) > 0$, *if* $S \subset \mathrm{GL}_d(K)$ *for some number field* K,

$$\widehat{h}(S) := \lim_{m \to +\infty} \frac{1}{m} h(S^m) > \frac{c(d)}{[K:\mathbb{Q}]} \kappa_S^2$$

- Here for $A \subset \mathrm{GL}_d(K)$, $h(A) := \max_{g \in A} h(g)$.
- Proof of Corollary: Jensen's inequality.

The normalized height $\widehat{h}(S)$

$$\widehat{h}(S) := \lim_{m\to+\infty} \frac{1}{m} h(S^m)$$

Basic properties:

- $\widehat{h}(S^n) = n\widehat{h}(S)$,
- $\widehat{h}(S) = 0 \Leftrightarrow \langle S \rangle$ quasi unipotent
- $\widehat{h}(gSg^{-1}) = \widehat{h}(S)$ if $g \in \mathrm{GL}_d(\overline{\mathbb{Q}})$.

The normalized height $\widehat{h}(S)$

$$\widehat{h}(S) := \lim_{m \to +\infty} \frac{1}{m} h(S^m)$$

Basic properties:

- $\widehat{h}(S^n) = n\widehat{h}(S)$,
- $\widehat{h}(S) = 0 \Leftrightarrow \langle S \rangle$ quasi unipotent
- $\widehat{h}(gSg^{-1}) = \widehat{h}(S)$ if $g \in \mathrm{GL}_d(\overline{\mathbb{Q}})$.

$\Rightarrow$ $\widehat{h}(S)$ is really defined on the character variety $(\mathrm{GL}_d)^k // \mathrm{GL}_d$, where $k = |S|$.

The normalized height $\widehat{h}(S)$

$$\widehat{h}(S) := \lim_{m \to +\infty} \frac{1}{m} h(S^m)$$

Basic properties:

- $\widehat{h}(S^n) = n\widehat{h}(S)$,
- $\widehat{h}(S) = 0 \Leftrightarrow \langle S \rangle$ quasi unipotent
- $\widehat{h}(gSg^{-1}) = \widehat{h}(S)$ if $g \in \mathrm{GL}_d(\overline{\mathbb{Q}})$.

$\Rightarrow$ $\widehat{h}(S)$ is really defined on the character variety $(\mathrm{GL}_d)^k // \mathrm{GL}_d$, where $k = |S|$.

On this variety it is comparable (up to multiplicative and additive constants) to an ordinary height function (e.g. Procesi (1970's) gave generators for the ring of invariants: $\mathrm{tr}(w(s_1, ..., s_k))$, $w \in F_k$, $|w| \leqslant C(d,k)$.)

Height gap theorem

Theorem (Bogomolov-type theorem, B. 08)

There is $\varepsilon = \varepsilon(d) > 0$ *such that if* $S \subset \mathrm{GL}_d(\overline{\mathbb{Q}})$ *is a finite subset generating a non virtually solvable subgroup, then*

$$\widehat{h}(S) > \varepsilon$$

The previous theorem that $\kappa_S > c(d) > 0$ is a consequence of this.

Theorem (Bogomolov-type theorem, B. 08)

There is $\varepsilon = \varepsilon(d) > 0$ such that if $S \subset \mathrm{GL}_d(\overline{\mathbb{Q}})$ is a finite subset generating a non virtually solvable subgroup, then

$$\widehat{h}(S) > \varepsilon$$

The previous theorem that $\kappa_S > c(d) > 0$ is a consequence of this.

• The proof makes use of local estimates for the joint displacement of S on the symmetric spaces and Bruhat-Tits building associated to reductive groups at each place. Bilu's equidistribution theorem and Zhang's Bogomolov-type theorem on tori are also ingredients.

• The proof that $\kappa_S > c(d) > 0$ goes by finding two words of bounded length with letters in S, which are generators of a free subgroup.

Borel density and co-amenability

Using similar ideas, one can give the following generalisation of the uniform non-amenability result:

Theorem (B. 2013)

Let $\mathbf{G}$ *be a semisimple* $\mathbb{Q}$*-group. There is* $\varepsilon = \varepsilon(\mathbf{G}) > 0$ *such that if* $S \subset \mathbf{G}(\overline{\mathbb{Q}})$ *is a finite set generating a Zariski dense subgroup* Γ *and* $\Gamma' \leqslant \Gamma = \langle S \rangle$ *a subgroup which is not Zariski-dense (e.g.* $\Gamma' = \{1\}$*), then*

$$\kappa_S(\ell^2(\Gamma/\Gamma')) > \varepsilon$$

Borel density and co-amenability

Using similar ideas, one can give the following generalisation of the uniform non-amenability result:

Theorem (B. 2013)

Let $\mathbf{G}$ *be a semisimple* $\mathbb{Q}$*-group. There is* $\varepsilon = \varepsilon(\mathbf{G}) > 0$ *such that if* $S \subset \mathbf{G}(\overline{\mathbb{Q}})$ *is a finite set generating a Zariski dense subgroup* Γ *and* $\Gamma' \leqslant \Gamma = \langle S \rangle$ *a subgroup which is not Zariski-dense (e.g.* $\Gamma' = \{1\}$*), then*

$$\kappa_S(\ell^2(\Gamma/\Gamma')) > \varepsilon$$

Remark: this can be seen as a uniform relative version of Borel's density theorem ($=$ if Γ is co-amenable, then it is Zariski dense).

Borel density and co-amenability

Using similar ideas, one can give the following generalisation of the uniform non-amenability result:

Theorem (B. 2013)

Let $\mathbf{G}$ *be a semisimple* $\mathbb{Q}$*-group. There is* $\varepsilon = \varepsilon(\mathbf{G}) > 0$ *such that if* $S \subset \mathbf{G}(\overline{\mathbb{Q}})$ *is a finite set generating a Zariski dense subgroup* Γ *and* $\Gamma' \leqslant \Gamma = \langle S \rangle$ *a subgroup which is not Zariski-dense (e.g.* $\Gamma' = \{1\}$*), then*

$$\kappa_S(\ell^2(\Gamma/\Gamma')) > \varepsilon$$

Remark: this can be seen as a uniform relative version of Borel's density theorem (= if Γ is co-amenable, then it is Zariski dense).

The uniformity here allows to prove new spectral gaps, e.g. for groups of toral automorphisms (Bekka-Guivarch).

Spectral gaps for finite groups and expanders

If G is a finite group with generating set S, define:

$$\kappa(G,S) := \inf_{f \in \ell^2_0(G)} \max_{s \in S} \frac{||sf - f||_2}{||f||_2}$$

The only difference with the previously defined κ_S is that we work in the orthogonal of constants in $\ell^2(G)$.

This quantity is closely related to the first non zero eigenvalue $\lambda_1(\mathcal{G})$ of the combinatorial laplacian on the Cayley graph of $\mathcal{G} = \mathcal{G}(G,S)$ of G:

$$\frac{1}{|S|}\lambda_1(\mathcal{G}) \leqslant \kappa(G,S)^2 \leqslant \lambda_1(\mathcal{G})$$

Spectral gaps for finite groups and expanders

Definition (Expander)

A Cayley graph $\mathcal{G} = \mathcal{G}(G, S)$ is said to be an ε-expander if $\lambda_1(\mathcal{G}) > \varepsilon$.
A family of Cayley graphs $\mathcal{G}_{n \geqslant 0} = \mathcal{G}(G_n, S_n)$ is said to be a family of expanders if there is $\varepsilon > 0$ such that $\lambda_1(\mathcal{G}_n) > \varepsilon$ for all $n \geqslant 0$.

Definition (Expander)

A Cayley graph $\mathcal{G} = \mathcal{G}(G, S)$ is said to be an ε-expander if $\lambda_1(\mathcal{G}) > \varepsilon$.
A family of Cayley graphs $\mathcal{G}_{n \geqslant 0} = \mathcal{G}(G_n, S_n)$ is said to be a family of expanders if there is $\varepsilon > 0$ such that $\lambda_1(\mathcal{G}_n) > \varepsilon$ for all $n \geqslant 0$.

- first appears in a somewhat different form in a paper of Kolmogorov and Barzdin about random graphs.

- first explicit construction by Margulis (1970s). E.g. take $G_n = \mathrm{SL}(3, \mathbb{Z}/n\mathbb{Z})$ and $S_n = S \bmod n$, for a fixed generating set S of $\mathrm{SL}(3, \mathbb{Z})$. A consequence of Kazhdan's property (T).

- Lubotzky (1990's) asks: which Cayley graphs are expanders ? How does this relate to the group structure ?

Spectral gaps for finite groups and expanders

A big breakthrough came with the work of Bourgain-Gamburd (2005) who worked with Cayley graphs of $\mathrm{SL}(2, \mathbb{Z}/p\mathbb{Z})$, p a prime, and partly reduced Lubotzky's question to classifying approximate subgroups of $\mathrm{SL}(2, \mathbb{Z}/p\mathbb{Z})$.

Definition (approximate subgroup)

A finite subset A of a group G is said to be a K-approximate subgroup of G if $A = A^{-1}$, $1 \in A$ and there is a subset $X \subset G$ of cardinality $\leqslant K$ such that

$$AA \subset XA.$$

Spectral gaps for finite groups and expanders

A big breakthrough came with the work of Bourgain-Gamburd (2005) who worked with Cayley graphs of $\mathrm{SL}(2, \mathbb{Z}/p\mathbb{Z})$, p a prime, and partly reduced Lubotzky's question to classifying approximate subgroups of $\mathrm{SL}(2, \mathbb{Z}/p\mathbb{Z})$.

Definition (approximate subgroup)

A finite subset A of a group G is said to be a K-approximate subgroup of G if $A = A^{-1}$, $1 \in A$ and there is a subset $X \subset G$ of cardinality $\leqslant K$ such that

$$AA \subset XA.$$

Helfgott (2005) classified approximate subgroups of $\mathrm{SL}(2, \mathbb{Z}/p\mathbb{Z})$. This was later generalized by Hrushovski, Pyber-Szabo and Breuillard-Green-Tao to $\mathbf{G}(\mathbb{F}_q)$, $\mathbf{G}$ fixed simple algebraic group over a finite field $\mathbb{F}_q$.

Definition (approximate subgroup)

A finite subset A of a group G is said to be a K-approximate subgroup of G if $A = A^{-1}$, $1 \in A$ and there is a subset $X \subset G$ of cardinality $\leqslant K$ such that

$$AA \subset XA.$$

Helfgott (2005) classified approximate subgroups of $\mathrm{SL}(2, \mathbb{Z}/p\mathbb{Z})$. This was later generalized by Hrushovski, Pyber-Szabo and Breuillard-Green-Tao to $\mathbf{G}(\mathbb{F}_q)$, $\mathbf{G}$ fixed simple algebraic group over a finite field $\mathbb{F}_q$.

The upshot is: there are no non-trivial generating approximate subgroups of $\mathbf{G}(\mathbb{F}_q)$. They are either very small ($|A| = O(K^{O(1)})$ or very large $|\mathbf{G}(\mathbb{F}_q)|/|A| = O(K^{O(1)})$).

Spectral gaps for finite groups and expanders

The Bourgain-Gamburd method then yields:

Theorem (Super-strong approximation)

Suppose $\mathbf{G}$ *is a semisimple* $\mathbb{Q}$*-group, and* $S \subset \mathbf{G}(\mathbb{Q})$ *is a finite subset s.t.* $\langle S \rangle$ *is Zariski dense. Then* $\mathcal{G}_p := \mathcal{G}(\mathbf{G}(\mathbb{Z}/p\mathbb{Z}), S_p)$*, where* $S_p = S \bmod p$*, is a family of expanders.*

That S_p generates $\mathbf{G}(\mathbb{Z}/p\mathbb{Z})$ for all but finitely many p's is the strong-approximation theorem of Matthews-Vaserstein-Weisfeiler and Nori (also Hrushovski-Pillay).

Spectral gaps for finite groups and expanders

The Bourgain-Gamburd method then yields:

Theorem (Super-strong approximation)

Suppose $\mathbf{G}$ *is a semisimple* $\mathbb{Q}$*-group, and* $S \subset \mathbf{G}(\mathbb{Q})$ *is a finite subset s.t.* $\langle S \rangle$ *is Zariski dense. Then* $\mathcal{G}_p := \mathcal{G}(\mathbf{G}(\mathbb{Z}/p\mathbb{Z}), S_p)$*, where* $S_p = S \bmod p$*, is a family of expanders.*

That S_p generates $\mathbf{G}(\mathbb{Z}/p\mathbb{Z})$ for all but finitely many p's is the strong-approximation theorem of Matthews-Vaserstein-Weisfeiler and Nori (also Hrushovski-Pillay).

Open problem: is the family of all Cayley graphs of $\mathbf{G}(\mathbb{Z}/p\mathbb{Z})$ a family of expanders ?

The Bourgain-Gamburd method then yields:

Theorem (Super-strong approximation)

Suppose $\mathbf{G}$ *is a semisimple* $\mathbb{Q}$*-group, and* $S \subset \mathbf{G}(\mathbb{Q})$ *is a finite subset s.t.* $\langle S \rangle$ *is Zariski dense. Then* $\mathcal{G}_p := \mathcal{G}(\mathbf{G}(\mathbb{Z}/p\mathbb{Z}), S_p)$*, where* $S_p = S \bmod p$*, is a family of expanders.*

That S_p generates $\mathbf{G}(\mathbb{Z}/p\mathbb{Z})$ for all but finitely many p's is the strong-approximation theorem of Matthews-Vaserstein-Weisfeiler and Nori (also Hrushovski-Pillay).

Open problem: is the family of all Cayley graphs of $\mathbf{G}(\mathbb{Z}/p\mathbb{Z})$ a family of expanders ?
[B-Gamburd 2010: yes for SL_2 and a family of primes with density 1].

A characterization of weak expansion

Building on work of Hrushovski, B-Green-Tao proved in 2011 a structure theorem for approximate subgroups of arbitrary groups: they are contained in at most $O_K(1)$ translates of a finite-by-nilpotent subgroup.

A characterization of weak expansion

Building on work of Hrushovski, B-Green-Tao proved in 2011 a structure theorem for approximate subgroups of arbitrary groups: they are contained in at most $O_K(1)$ translates of a finite-by-nilpotent subgroup.
Here is a consequence for non (weak) expanders (answers a question of Ellenberg-Hall-Kowalski):

Theorem (BGT)

Let $\varepsilon > 0$. Assume that G is a finite group and $\mathcal{G} = \mathcal{G}(G, S)$ is a Cayley graph of G such that

$$\lambda_1(\mathcal{G}) \leqslant \frac{1}{|G|^{\varepsilon}},$$

then G has a quotient of size $\geqslant |G|^{\varepsilon/2}$ which has a nilpotent subgroup of index $\leqslant C(\varepsilon)$.

Expanders and Bogomolov

A differential geometric inequality of Li-Yau relates the gonality $\gamma(U)$ of an algebraic curve U to the its λ_1 (when the curve is viewed as a hyperbolic surface):

$$\frac{1}{8\pi}\lambda_1(U) \cdot vol(U) \leqslant \gamma(U)$$

$\gamma(U)$ is the smallest degree of a non-constant meromorphic function on U.

A differential geometric inequality of Li-Yau relates the gonality $\gamma(U)$ of an algebraic curve U to the its λ_1 (when the curve is viewed as a hyperbolic surface):

$$\frac{1}{8\pi}\lambda_1(U) \cdot vol(U) \leqslant \gamma(U)$$

$\gamma(U)$ is the smallest degree of a non-constant meromorphic function on U.

Another basic geometric comparison principle, due to Brooks and Burger states that if U' is a finite cover of U, then

$$\lambda_1(U') \geqslant c(U) \cdot \lambda_1(\mathcal{G}(Gal(U'|U)))$$

where $\mathcal{G}(Gal(U'|U))$ is the Cayley-Schreier graph of the covering group.

Expanders and Bogomolov

On the field of meromorphic functions, the natural height is the degree. So the Li-Yau inequality can be interpreted as giving a height lower bound in terms of a spectral gap.

Indeed, using the super-strong-approximation theorem above (more precisely its extension to square free modulus by Peter Varju), and the Brooks-Burger principle, one obtains:

Theorem (Ellenberg-Hall-Kowalski)

Let $A \to U$ be an abelian scheme whose monodromy in Sp_{2g} is Zariski-dense. Then the compositum of all fields of meromorphic functions on the covers U_ℓ associated to the ℓ-torsion of A (ℓ prime) has the Bogomolov property.

Ellenberg's question

A field has the Bogomolov property if there is a uniform positive lower bound on the (absolute) height of every element, which is not a root of unity.

Examples: totally real numbers (Schinzel), $\mathbb{Q}^{ab}$ (Amoroso-Zannier), $\mathbb{Q}(E_{tors})$ E elliptic curve over $\mathbb{Q}$ (Habegger).

Ellenberg's question

A field has the Bogomolov property if there is a uniform positive lower bound on the (absolute) height of every element, which is not a root of unity.

Examples: totally real numbers (Schinzel), $\mathbb{Q}^{ab}$ (Amoroso-Zannier), $\mathbb{Q}(E_{tors})$ E elliptic curve over $\mathbb{Q}$ (Habegger).

Ellenberg question: Let $F|\mathbb{Q}$ be a Galois extension such that the family of all Cayley graphs of all finite quotients of $Gal(F\mathbb{Q}^{ab}|\mathbb{Q}^{ab})$ is an expander family. Does F have the Bogomolov property ?

A spectral criterion for Salem numbers

A Salem number is an algebraic integer $\alpha \in \mathbb{R}$, $\alpha > 1$, all of whose conjugates lie inside the unit disc with at least one on the unit circle.

A spectral criterion for Salem numbers

A Salem number is an algebraic integer $\alpha \in \mathbb{R}$, $\alpha > 1$, all of whose conjugates lie inside the unit disc with at least one on the unit circle.

Salem's conjecture: The set of Salem numbers is bounded away from 1.

Rk: This is a special case of the Lehmer conjecture.

A spectral criterion for Salem numbers

A Salem number is an algebraic integer $\alpha \in \mathbb{R}$, $\alpha > 1$, all of whose conjugates lie inside the unit disc with at least one on the unit circle.

Salem's conjecture: The set of Salem numbers is bounded away from 1.

Rk: This is a special case of the Lehmer conjecture.

In the early 1980's Sury gave a beautiful geometric reformulation of Salem's conjecture:

Sury: Salem's conjecture holds iff there is a uniform lower bound on the systole of arithmetic surfaces.

A spectral criterion for Salem numbers

Sury: Salem's conjecture holds iff there is a uniform lower bound on the systole of arithmetic surfaces.

An *arithmetic surface* is a quotient $\Sigma := \mathbb{H}^2/\Gamma$, where $\mathbb{H}^2$ is the hyperbolic plane, and Γ is a lattice in $PSL_2(\mathbb{R})$ commensurable to some an arithmetic group $\mathbf{G}(\mathcal{O}_K)$, ($K$ a number field, $\mathbf{G}$ a K-form of PGL_2).

The *systole* is the length of the shortest geodesic in $\mathbb{H}^2/\Gamma$.

Rk. Sury's surface is congruence.

A spectral criterion for Salem numbers

The systole of a hyperbolic surface Σ is related to its the Cheeger constant $h(\Sigma)$ and to the first eigenvalue of the Laplacian $\lambda_1(\Sigma)$.

$$h(\Sigma) := \inf_{A \subset \Sigma} \frac{L(\partial A)}{\min\{area(A), area(\Sigma \setminus A)\}}$$

Cheeger-Buser inequality:

$$\frac{1}{4} h(\Sigma)^2 \leqslant \lambda_1(\Sigma) \leqslant c(h(\Sigma) + h(\Sigma)^2)$$

For congruence arithmetic surfaces (i.e. those containing a congruence subgroup) we have:

Theorem (Selberg, Vigneras)

If Σ is a congruence arithmetic surface, then

$$\lambda_1(\Sigma) \geqslant \frac{3}{16}$$

A spectral criterion for Salem numbers

Observation (Breuillard-Deroin)

Salem's conjecture holds iff given some (or any) $d \geqslant 2$ *there is* $c(d) > 0$ *such that*

$$\lambda_1(\widetilde{\Sigma}) \geqslant \frac{c(d)}{vol(\widetilde{\Sigma})}$$

for every d*-cover* $\widetilde{\Sigma}$ *of a congruence surface.*

A spectral criterion for Salem numbers

Observation (Breuillard-Deroin)

Salem's conjecture holds iff given some (or any) $d \geqslant 2$ there is $c(d) > 0$ such that

$$\lambda_1(\widetilde{\Sigma}) \geqslant \frac{c(d)}{vol(\widetilde{\Sigma})}$$

for every d-cover $\widetilde{\Sigma}$ of a congruence surface.

• In one direction the point is that a congruence surface with small systole yields a 2-cover $\widetilde{\Sigma}$ with $h_1(\widetilde{\Sigma}) \cdot vol(\widetilde{\Sigma})$ small.

• In the other direction, it turns out (B-D 2013) that Cheeger's inequality can be improved for d-covers $\widetilde{\Sigma}$ of a surface (even any manifold) Σ:

$$\lambda_1(\widetilde{\Sigma}) \geqslant \frac{1}{Cd^2} h(\widetilde{\Sigma}) \sqrt{\lambda_1(\Sigma)}$$

So this together with Selberg-Vigneras concludes.

A spectral criterion for Salem numbers

• In one direction the point is that a congruence surface with small systole yields a 2-cover $\widetilde{\Sigma}$ with $h_1(\widetilde{\Sigma}) \cdot vol(\widetilde{\Sigma})$ small.

• In the other direction, it turns out (B-D 2013) that Cheeger's inequality can be improved for d-covers $\widetilde{\Sigma}$ of a surface (even any manifold) Σ:

$$\lambda_1(\widetilde{\Sigma}) \geqslant \frac{1}{Cd^2} h(\widetilde{\Sigma}) \sqrt{\lambda_1(\Sigma)}$$

So this together with Selberg-Vigneras concludes.

• Further question: More generally can one interpret the full Lehmer conjecture in spectral terms ?

Thank you!