Quantum B-Algebras and their Spectrum

Wolfgang Rump

In propositional logic, conjunction $A \wedge B$ is related to implication $A \to B$ by an adjunction

$$A \wedge B \leqslant C \iff A \leqslant B \to C,$$

where $\leqslant$ stands for the implication of propositions. If the commutativity of $\wedge$ is dropped, implication splits into a left and right implication, according to the maps $A \mapsto A \wedge B$ and $A \mapsto B \wedge A$.

Algebraic semantics of such a non-commutative logic have been studied by

- Ward and Dilworth 1939 (residuated lattices)
- Bosbach 1965 (pseudo-hoops)
- Bosbach 1982 (cone algebras, bricks)
- Georgescu, Iorgulescu 2001 (pseudo BCK-alg.)
- Dvurečenskij, Vetterlein 2001 (GPE-algebras)
- Galatos, Tsinakis 2005 (GBL-algebras)

Quantum B-algebras form a common framework for such structures. Their unifying principle comes from their spectrum which is a quantale.

The lecture consists of three parts:

A. Genesis of quantum B-algebras from a quantalic approach of algebraic semantics;
B. Main examples and prototypes of logical algebras with two implications (residuals);
C. Structural results.

1. Quantales and non-commutative logic

Quantales were introduced on a 1984 conference in Taormina (Sicily) by C. J. Mulvey. His paper carries the shortest title ever seen in mathematics, namely:

&

which refers to the non-commutative conjunction.

Definition 1. A *quantale* Q is a partially ordered semigroup with arbitrary joins $\bigvee A$ (for $A \subset Q$) so that multiplication (& or $\cdot$) distributes over joins:

$$a \cdot \Big(\bigvee_{i \in I} a_i\Big) = \bigvee_{i \in I}(a \cdot a_i), \qquad \Big(\bigvee_{i \in I} a_i\Big) \cdot a = \bigvee_{i \in I}(a_i \cdot a).$$

Q is *unital* if $(Q, \cdot)$ admits a unit element u.

Quantales Q were conceived as *non-commutative spaces*: Elements $a \in Q$ are open sets, $\bigvee A$ is the union, $a \cdot b$ generalizes the intersection. Examples:

- The spectrum of a C^*-algebra,
- The space of a Penrose tiling.

There is always a smallest element $0 := \bigvee \varnothing$ and a greatest element $1 := \bigvee Q$.

The multiplication gives rise to binary operations (residuals $\twoheadrightarrow$ and $\rightarrowtail$) which satisfy

$$\boxed{a \leqslant b \twoheadrightarrow c \iff a \cdot b \leqslant c \iff b \leqslant a \rightarrowtail c} \quad (1)$$

The corresponding "logic" suggests itself: The non-commutative conjunction $\cdot$ gives rise to a pair of implications, a left one $\rightarrowtail$, and a right one $\twoheadrightarrow$.

Definition 2. A *residuated poset* is a po-semigroup with two operations $\rightarrowtail$ and $\twoheadrightarrow$ satisfying (1).

Every residuated poset X naturally embeds into a quantale Q such that X can be recovered as the set Q^{sc} of supercompact elements (H. Ono 1993, Ono and Komori 1985). An element $c \in Q$ is said to be *supercompact* if for subsets $A \subset Q$,

$$c \leqslant \bigvee A \implies \exists a \in A \colon c \leqslant a.$$

For algebras $(X; \rightarrow, \rightsquigarrow)$ without a multiplication, an embedding into a quantale is sometimes possible. For example, if X is a pseudo BCK-algebra, this has been shown by J. Kühr (2005) in two steps:

1. Embed the algebra X into a $\wedge$-ordered monoid.
2. Embed this monoid into a residuated lattice.

To associate a quantale as a "spectrum" to X, such an indirect way seems to be not appropriate. We propose a different method.

Since every quantale Q is a complete lattice, the following operations are well-defined:

$$a \to b := \bigwedge\{x \in Q \mid x \cdot a \geqslant b\}$$

$$a \rightsquigarrow b := \bigwedge\{x \in Q \mid a \cdot x \geqslant b\}$$

Of course, the "inverse residuals" are not adjoint to the product. They merely satisfy the implications

$$\boxed{a \geqslant b \to c} \;\Leftarrow\; \boxed{a \cdot b \geqslant c} \Rightarrow \boxed{b \geqslant a \rightsquigarrow c} \qquad (2)$$

However, it will be sufficient that equivalence holds among the supercompact elements!

Definition 3. Let Q be a quantale. An element $c \in Q$ is *balanced* if is satisfies

$$c \cdot \Big(\bigwedge_{i\in I} a_i\Big) = \bigwedge_{i\in I}(c \cdot a_i), \qquad \Big(\bigwedge_{i\in I} a_i\Big) \cdot c = \bigwedge_{i\in I}(a_i \cdot c).$$

Equivalently, c is balanced if and only if c satisfies

$$a \cdot c \geqslant b \iff a \geqslant c \to b$$

$$c \cdot a \geqslant b \iff a \geqslant c \rightsquigarrow b$$

for all $a, b \in Q$. The product of balanced elements is balanced, and there is a kind of duality between balanced and supercompact elements:

If c is balanced and d supercompact, then $c \to d$ and $c \rightsquigarrow d$ are supercompact. Furthermore:

$$c \to \bigvee_{i \in I} a_i = \bigvee_{i \in I} (c \to a_i), \qquad (\bigvee_{i \in I} a_i) \to d = \bigwedge_{i \in I} (a_i \to d).$$

Definition 4. A quantale Q is *logical* if $Q = \bigvee Q^{sc}$ and every supercompact element is balanced.

For a logical quantale Q, the set $X := Q^{sc}$ of supercompact elements is an algebra $(X; \to, \rightsquigarrow)$. It is the most general two-implication algebra coming from a quantale. The associated quantale $Q = U(X)$ can thus be viewed as the *spectrum* of X.

Questions arise:

- How general are these "quantalic" algebras X?
- Are the residuated posets of this type?

We will show that

1. virtually all important non-commutative logical algebras $(X; \to, \rightsquigarrow)$ are covered in this way and thus have a spectrum;

2. the spectrum $U(X)$ provides an efficient tool for the structural analysis of logical algebras X;

The algebras $X = Q^{sc}$ coming from a logical quantale Q will be called *quantum B-algebras*.

2. Quantum B-algebras

Our terminology (concerning "B") refers to the basic inequalities

$$\begin{aligned} y \to z &\leqslant (x \to y) \to (x \to z) \\ y \rightsquigarrow z &\leqslant (x \rightsquigarrow y) \rightsquigarrow (x \rightsquigarrow z) \end{aligned} \tag{3}$$

similar to the implication

$$y \leqslant z \implies x \to y \leqslant x \to z. \tag{4}$$

Definition 5. A *quantum B-algebras* is a poset X with two binary operations $\to$ and $\rightsquigarrow$ satisfying (3), (4), and the equivalence

$$x \leqslant y \to z \iff y \leqslant x \rightsquigarrow z. \tag{5}$$

The counterpart of (4) holds for every quantum B-algebra, i. e. quantum B-algebras are self-dual with respect to $\to$ and $\rightsquigarrow$. Furthermore, the implications

$$\begin{aligned} x \leqslant y &\implies y \to z \leqslant x \to z \\ x \leqslant y &\implies y \rightsquigarrow z \leqslant x \rightsquigarrow z \end{aligned}$$

hold for any quantum B-algebra.

Theorem 1. *Up to isomorphism, there is a one-to-one correspondence between logical quantales and quantum B-algebras.*

The two operations of a quantum B-algebra are related by the pair of equations

$$x \rightsquigarrow y = \big((x \rightsquigarrow y) \to y\big) \rightsquigarrow y$$
$$x \to y = \big((x \to y) \rightsquigarrow y\big) \to y$$

and the equation

$$x \to (y \rightsquigarrow z) = y \rightsquigarrow (x \to z).$$

Definition 6. A quantum B-algebra X is *unital* if X admits an element u, the *unit element*, which satisfies $u \to x = u \rightsquigarrow x = x$ for all $x \in X$.

A unit element is unique. If such an element u exists, the axioms can be written as inequalities:

$$x \rightsquigarrow (y \to z) = y \to (x \rightsquigarrow z)$$
$$y \to z \leqslant (x \to y) \to (x \to z)$$
$$y \rightsquigarrow z \leqslant (x \rightsquigarrow y) \rightsquigarrow (x \rightsquigarrow z)$$

The unit element partially reduces the relation $\leqslant$ to the operations $\to$ and $\rightsquigarrow$:

$$x \leqslant y \iff u \leqslant x \to y \iff u \leqslant x \rightsquigarrow y.$$

Thus, if u the greatest element of X, the relation $x \leqslant y$ just means that $x \to y$ is true. In general, this need not be the case.

In terms of the quantale $U(X)$, an element $u \in X$ is a unit element of X if and only if u is a unit element of $U(X)$.

3. Examples

We consider three prototypes of logical algebras X with two implications $\to$ and $\rightsquigarrow$ and show that they can be regarded as quantum B-algebras. In what follows, we denote a greatest (smallest) element of X (if it exists) by 1 and 0, respectively.

a) Pseudo BCK-algebras. For a set X with a binary operation $\to$, an element u is called a *logical unit* if the equations

$$\boxed{u \to x = x\,, \quad x \to u = x \to x = u}$$

hold for all $x \in X$. Such an element u is unique.

A logical unit u stands for the “true” proposition.

Definition 7. An algebra $(X; \to, \rightsquigarrow, 1)$ is a *pseudo BCK-algebra* if 1 is a simultaneous logical unit for the operations $\to$ and $\rightsquigarrow$ such that the equations

$$(x \to y) \rightsquigarrow \big((y \to z) \rightsquigarrow (x \to z)\big) = 1$$
$$(x \rightsquigarrow y) \to \big((y \rightsquigarrow z) \to (x \rightsquigarrow z)\big) = 1$$

and the implication

$$x \to y = y \rightsquigarrow x = 1 \implies x = y$$

are satisfied.

Every pseudo BCK-algebra is a unital quantum B-algebra. Precisely:

Proposition 1. *A unital quantum B-algebra X is a pseudo BCK-algebra if and only if $u = 1$.*

In other words, a pseudo BCK-algebra is a unital quantum B-algebra where the truth value $u =$"true" is the top value!

b) Partially ordered groups give an important case where the "truth" is located in the middle: For a partially ordered group G with unit element u, we define

$$\boxed{x \to y := yx^{-1}, \qquad x \rightsquigarrow y := x^{-1}y} \tag{6}$$

Then G becomes a unital quantum B-algebra. The multiplication is determined by each of the residuals:

$$x \cdot y = \big(y \to (x \to x)\big) \to x.$$

Proposition 2. *A quantum B-algebra X is a partially ordered group if and only if*

$$(x \to y) \rightsquigarrow y = (x \rightsquigarrow y) \to y = x$$

for all $x, y \in X$.

By the above equations (6), a partially ordered group is commutative if and only if the operations $\to$ and $\rightsquigarrow$ coincide.

The tradition of BCK-algebras produced another concept of "commutativity":

c) Pre-cone algebras. Assume that a pseudo BCK-algebra X satisfies

$$(x \to y) \rightsquigarrow y = (y \rightsquigarrow x) \to x =: x \vee y. \qquad (7)$$

Then (7) makes X into a semilattice.

Definition 8. A *pre-cone algebra* is an algebra $(X; \to, \rightsquigarrow)$ with a simultaneous logical unit which satisfies Eq. (7) and

$$x \to (y \rightsquigarrow z) = y \rightsquigarrow (x \to z).$$

Pre-cone algebras are special pseudo BCK-algebras. They are implicit in Bosbach's 1982 paper and have been studied in 2009 by J. Kühr where they are called *commutative pseudo BCK-algebras*.

Bosbach's *cone algebras* (i. e. algebras which can be embedded into an l-group cone) form a special case:

Proposition 3. *For a pre-cone algebra X, the equations*

$$(x \to y) \to (x \to z) = (y \to x) \to (y \to z)$$
$$(x \rightsquigarrow y) \rightsquigarrow (x \rightsquigarrow z) = (y \rightsquigarrow x) \rightsquigarrow (y \rightsquigarrow z)$$

are equivalent. They hold if and only if X is a cone algebra.

d) Residuated posets are quantum B-algebras. (The multiplication can be regarded as a derived operation, as it is expressible by the residuals.) We call a residuated poset X *unital* if the semigroup of X has a unit element u.

Proposition 4. *A residuated poset X is unital if and only if X is a unital quantum B-algebra.*

Proof. Assume that $x \cdot u = x$ holds for all $x \in X$. Then

$$x \leqslant u \to y \iff x \cdot u \leqslant y \iff x \leqslant y$$

holds for all $x \in X$, and thus $u \to y = y$. Similarly, $\forall x \colon u \cdot x = x$ implies $u \rightsquigarrow y = y$.

Conversely, assume that $u \to y = y$ holds for all $y \in X$. Then

$$x \cdot u \leqslant y \iff x \leqslant u \to y \iff x \leqslant y,$$

which yields $x \cdot u = x$. □

For residuated posets X, Theorem 1 tells us that $U(X)$ can be made into a quantale in two essentially different ways:

- The obvious way: $\to, \rightsquigarrow$ are just the restrictions of the residuals $\twoheadrightarrow, \rightarrowtail$ of $U(X)$;
- The natural way: $\to, \rightsquigarrow$ do not extend to $U(X)$.

e) Quantales. In particular, residuated lattices are quantum B-algebras, and thus, every quantale Q is a quantum B-algebra. However, the spectrum $U(Q)$ is not Q itself, but a bigger quantale. By Proposition 4, a quantale Q is a unital iff Q is unital as a quantum B-algebra iff $U(Q)$ is a unital quantale.

f) Pseudo effect-algebras. In 1994, Foulis and Bennett introduced *effect algebras* for the study of quantum effects in physics. A non-commutative version (*pseudo effect-algebras*) was introduced in 2001 by Dvurečenskij and Vetterlein. By dropping the greatest element, they arrived at the concept of *generalized pseudo effect-algebra* (=*GPE-algebra*).

Definition 9. A *GPE-algebra* is a set E with a constant u and a partially defined multiplication $\cdot$ such that the following are satisfied.

(1) $(a \cdot b) \cdot c = d \iff a \cdot (b \cdot c) = d$
(2) $a \cdot b = c \implies \exists a', b' \in E \colon b \cdot a' = b' \cdot a = c$
(3) $a \cdot b = a \cdot c \implies b = c$
$b \cdot a = c \cdot a \implies b = c$
(4) $a \cdot b = u \implies a = b = u$
(5) $a \cdot u = u \cdot a = a.$

The equations are to be understood so that the products occurring in them exist.

Every GPE-algebra E has a natural partial order given by left or right divisibility:

$$a \leqslant b \;:\Longleftrightarrow\; \exists c \in E\colon c \cdot a = b$$

so that u is the smallest element of E.

The elements a and b in a product $a \cdot b = c$ are unique. We write $b \rightarrow c := a$ and $a \rightsquigarrow c := b$. Thus $a \rightarrow b$ and $a \rightsquigarrow b$ are defined if $a \leqslant b$, and then

$$(a \rightarrow b) \cdot a = a \cdot (a \rightsquigarrow b) = b.$$

In other words, the equation $a \cdot b = c$ can be expressed in three different ways:

$$\boxed{a \cdot b = c \iff a = b \rightarrow c \iff b = a \rightsquigarrow c}$$

In terms of residuals, the associativity (1) can be expressed by the equation

$$\boxed{a \rightsquigarrow (c \rightarrow d) = c \rightarrow (a \rightsquigarrow d)}$$

with the proviso that the left-hand side exists if and only if the right-hand side exists.

The partial operations on E can be totalized: We adjoin two elements $0, 1$ with $0 < a < 1$ for all $a \in E$:

$$\widetilde{E} := E \sqcup \{0, 1\}$$

and extend the operations as follows.

For $x, y \in \widetilde{E}$ with $x \nleqslant y$, we set

$$x \to y = x \rightsquigarrow y = 0.$$

Furthermore, we define

$$0 \to x = 0 \rightsquigarrow x = x \to 1 = x \rightsquigarrow 1 = 1.$$

Proposition 5. *Let E be a GPE-algebra. Then $\widetilde{E}$ is a unital residuated poset, hence a unital quantum B-algebra.*

The product of E can be extended to $\widetilde{E}$ as follows. If $a \cdot b$ with $a, b \in E$ is undefined, we set $a \cdot b := 1$. For any $x \in \widetilde{E}$, we set $0 \cdot x = x \cdot 0 = 0$, and for $y \in \widetilde{E} \smallsetminus \{0\}$, we set $y \cdot 1 = 1 \cdot y = 1$.

Definition 10. A *pseudo effect-algebra* is a GPE-algebra with a greatest element v.

By Proposition 5, pseudo effect-algebras E are equivalent to a special type of quantum B-algebra. We call these quantum B-algebras $\widetilde{E}$ *effective*.

Definition 11. We call a quantum B-algebra X *bounded* if X admits a smallest element.

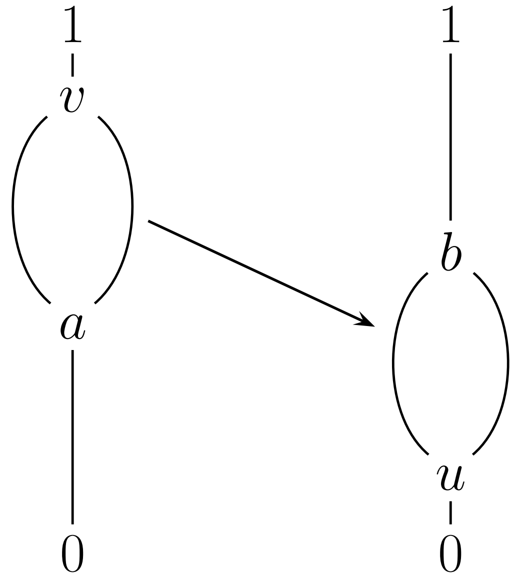
If a smallest element (denoted by 0) exists, then X also has a greatest element 1. In fact, $0 \leqslant y \rightsquigarrow x \Leftrightarrow y \leqslant 0 \to x$ yields $0 \to x = 1$ for any $x \in X$.

Proposition 6. *A unital quantum B-algebra X is effective (i. e. $X \cong \widetilde{E}$ for a pseudo effect-algebra E) if and only if*

(a) *X is bounded, has a greatest element $v < 1$, and $1 \rightarrow 1 = 1$.*

(b) *u is the smallest element > 0.*

(c) *For $a \in X \smallsetminus \{0, 1\}$, the maps $x \mapsto (a \rightarrow x)$ and $x \mapsto (a \rightsquigarrow x)$ are isotone from the interval $[a, v]$ onto some interval $[u, b]$ with $b < 1$.*



A similar characterization holds for arbitrary GPE-algebras. Further examples arise by combining the above prototypes.

4. The category of quantum B-algebras.

We have seen that up to isomorphism, there is a one-to-one correspondence between quantum B-algebras and logical quantales. What about the morphisms?

Of course, a morphism of quantum B-algebras is a monotonous map which respects the residuals.

Definition 12. We call a morphism $f\colon X \to Y$ of quantum B-algebras *spectral* if for all $y \in Y$ and $z \in f(X)$, the element $y \to z$ belongs to $f(X)$. In short: $Y \to f(X) \subset f(X)$.

The concept of spectral morphism is symmetric:

Proposition 7. *Let $f\colon X \to Y$ be a spectral morphism of quantum B-algebras. Then*

$$Y \rightsquigarrow f(X) \subset f(X).$$

Spectral morphisms are closed under composition.

Let **qB** denote the category of quantum B-algebras with spectral morphisms.

Now we turn our attention to logical quantales. Here is the counterpart to Definition 12.

Definition 13. We call a morphism $g\colon Q \to L$ of quantales *logical* if g respects arbitrary meets and

$$g(Q) \twoheadrightarrow L \subset g(Q), \qquad g(Q) \rightarrowtail L \subset g(Q). \quad (8)$$

In contrast to Proposition 7, the two inclusions (8) are not equivalent.

By **LQuant** we denote the category of logical quantales with logical morphisms. We get a functor

$$U\colon \mathbf{qB}^{\mathrm{op}} \to \mathbf{LQuant} \tag{9}$$

which maps a quantum B-algebra to its spectrum.

Theorem 2. *The functor U is an equivalence.*

Now let us indicate how the theory of quantum B-algebras takes profit from the theory of quantales.

5. Structural results.

We have mentioned three basic types of quantum B-algebras with a unit element u:

1. Pseudo BCK-algebras;
2. partially ordered groups;
3. GPE-algebras.

In the sequel: X is a unital quantum B-algebra.

We will show that every quantum B-algebra has a largest subalgebra of either type.

Definition 14. We call an element $x \in X$ *integral* if $x \to u = x \rightsquigarrow u = u$. The subset of integral elements in X will be denoted by $I(X)$.

Note that u is the greatest element of $I(X)$, and $I(X)$ is a subalgebra of X. Moreover,

Proposition 8. *$I(X)$ is the largest pseudo-BCK subalgebra of X. In particular, X is a pseudo-BCK algebra if and only if $I(X) = X$.*

Secondly, we consider the class of partially ordered groups. For a unital quantale Q, the invertible elements form a partially ordered group, the *unit group* $Q^{\times}$ of Q. The inverse of an element $a \in Q$ will be denoted by a^{-1}. If $a \in Q^{\times}$, the inverse of a can be expressed by the inverse residuals:

$$a^{-1} = a \rightarrow u = a \rightsquigarrow u.$$

Definition 15. We say that an element $a \in X$ is *invertible* if it satisfies the equations

$$(a \rightarrow u) \rightarrow (a \rightarrow x) = x$$
$$(a \rightsquigarrow u) \rightsquigarrow (a \rightsquigarrow x) = x.$$

The following result shows that the unit group of the quantale $U(X)$ is completely contained in X:

Theorem 3. *The invertible elements of X form a subalgebra $X^{\times}$ of X, the largest partially ordered subgroup of X. Furthermore, $X^{\times}$ coincides with the unit group of the quantale $U(X)$.*

Corollary. *X is a partially ordered group if and only if $X^{\times} = X$.*

Thirdly, let us consider GPE-algebras. Instead of introducing some formalism, we give an explicit definition of effective elements:

Definition 16. Let X be bounded. We call $a \in X$ *effective* if $a \to 1 = a \rightsquigarrow 1 = 1$ and the following implications hold for all $x, y \in X$.

$$\begin{aligned} u \leqslant a \to x \leqslant a \to y &\implies x \leqslant y \\ u \leqslant a \rightsquigarrow x \leqslant a \rightsquigarrow y &\implies x \leqslant y \\ u \leqslant x \leqslant a \to y < 1 &\implies \exists z \in X \colon a \to z = x \\ u \leqslant x \leqslant a \rightsquigarrow y < 1 &\implies \exists z \in X \colon a \rightsquigarrow z = x. \end{aligned}$$

Let $E^+(X)$ be the set of effective elements $a \geqslant u$.

Proposition 9. *Let X be bounded. Then $E^+(X)$ is a GPE-algebra such that for $a, b, c \in E^+(X)$,*

$$a \cdot b = c \iff a = b \to c.$$

Furthermore, $X \cong \widetilde{E}$ for some GPE-algebra E if and only if $E^+(X) = X \smallsetminus \{0, 1\}$ and $1 \to 1 = 1$.

A GPE-algebra with a total multiplication is the same as the positive cone of a partially ordered group.

We have indicated how quantum B-algebras X specialize into pseudo BCK-algebras, partially ordered groups, or GPE-algebras, and that X contains a largest subalgebra of each of these types.

Accidentally, the tree types can be distinguished by the position of their unit element u: For a pseudo BCK-algebra, u is the largest element, for a partially ordered group, u is in the middle, and for a GPE-algebra, u is the smallest element. Our next theorem deals with compounds of the first two types.

Galatos and Tsinakis (2005) consider *generalized BL-algebras* (= *GBL-algebras*), that is, residuated lattices X which satisfy the equations

$$\big(y \to (x \wedge y)\big)y = x \wedge y = y\big(y \rightsquigarrow (x \wedge y)\big).$$

They prove that such a GBL-algebra splits into a cartesian product $G \times Y$ of a lattice-ordered group G with a lattice-ordered pseudo BCK-algebra Y. A generalization to certain residuated posets was given by Jónsson and Tsinakis (2004). Let us extend these results to algebras without a product.

Definition 17. A *quantum BL-algebra* is a unital quantum B-algebra X such that $x \to u$ and $x \rightsquigarrow u$ are invertible for all $x \in X$.

Every GBL-algebra is a quantum BL-algebra. In addition, a GBL-algebra is a residuated lattice with

$$x \to x = x \rightsquigarrow x = u,$$

and every $x \geqslant u$ is invertible.

Example. For a lattice-ordered group G, let $\Delta(G)$ be the set of non-empty lower sets $A \subset G$ generated by finitely many maximal elements. For a pair of elements $A, B \in \Delta(G)$,

$$A \to B := \{c \in G \mid cA \subset B\}$$
$$A \rightsquigarrow B := \{c \in G \mid Ac \subset B\}$$

again belong to $\Delta(G)$. This makes $\Delta(G)$ into a residuated poset. The unit group $\Delta(G)^{\times}$ consists of the lower sets $\downarrow a := \{c \in G \mid c \leqslant a\}$ with $a \in G$. In particular, $E := \downarrow u$ is the unit element of $\Delta(G)$. For any $A \in \Delta(G)$,

$$A \to E = A \rightsquigarrow E = \downarrow(\sup A)^{-1}$$

is invertible. Hence $\Delta(G)$ is a quantum BL-algebra. In particular,

$$\Delta(G)^{\times} \cong G,$$

and $I(\Delta(G))$ consists of the A with $\sup A = u$.

In general, $\Delta(G)$ is not a GBL-algebra because positive elements need not be invertible.

Let X be a unital quantum B-algebra, G be a partially ordered group with a group homomorphism $\gamma \colon G \to \mathrm{Aut}(X)$ and a map $\delta \colon X \to U(G^{\mathrm{op}})$ with certain properties which will not be stated explicitly.

Then we can form a twisted semidirect product $G \ltimes_{\delta} X$ which is again a unital quantum B-algebra.

Moreover, there are natural embeddings

$$G \hookrightarrow G \ltimes_\delta X \hookleftarrow X$$

which turn G and X into subalgebras of $G \ltimes_\delta X$. The unit group and integral part of $G \ltimes_\delta X$ are

$$\boxed{(G \ltimes_\delta X)^\times = G \ltimes_\delta X^\times, \qquad I(G \ltimes_\delta X) = I(X).}$$

The structure of quantum BL-algebras can now be determined explicitly:

Theorem 4. *Every quantum BL-algebra X is of the form $X \cong X^\times \ltimes_\delta I(X)$. Conversely, every twisted semidirect product $G \ltimes_\delta Y$ with a partially ordered group G and a pseudo-BCK algebra Y is a quantum BL-algebra.*

Note that a quantum BL-algebra X need not have a multiplication. However, the elements of the unit group $X^\times$ operate on X from the left and right via multiplication in the quantale $U(X)$. Therefore, Theorem 4 implies, in particular, that any element $x \in X$ can be written uniquely in the form

$$\boxed{x = a \cdot y}$$

with $a \in X^\times$ and $y \in I(X)$.

Question. How does a general twisted product $X \times_\delta Y$ of quantum B-algebras look like? ...