Quantifiers on bounded integral residuated lattices

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monadic (Boolean) algebras (Halmos)

Algebraic counterpart of the one-variable fragment of the classical predicate logic; additional unary operation - algebraic counterpart of the existential quantifier.

monadic Heyting algebras (Monteiro, Varsavsky, Bezhanisvili, Harding, ...)

monadic *MV*-algebras (Rutledge, Grigolia, Di Nola, Belluce, Lettieri, Georgescu, Iorgulescu, Leustean)

monadic *GMV*-algebras (Rachůnek, Šalounová)

monadic *BL*-algebras (Grigolia)

Here: monadic bounded integral residuated lattices

$\mathcal{PBL}$ ... propositional calculus of a logic

connectives: $\odot, \rightarrow, \rightsquigarrow, \wedge, \vee,$

truth constant: $\bar{0}$

deduction rules (two modus ponens and implications):

(MP$\rightarrow$) $$\frac{\varphi,\ \varphi \rightarrow \psi}{\psi}$$

(MP$\rightsquigarrow$) $$\frac{\varphi,\ \varphi \rightsquigarrow \psi}{\psi}$$

(Imp$\rightarrow$) $$\frac{\varphi \rightarrow \psi}{\varphi \rightsquigarrow \psi}$$

(Imp$\rightsquigarrow$) $$\frac{\varphi \rightsquigarrow \psi}{\varphi \rightarrow \psi}$$

$\mathcal{MPBL}$ ... a monadic propositional logic

contains $\mathcal{PBL}$ with axioms:

(M1) $\varphi \to \exists\varphi, \ \ \varphi \rightsquigarrow \exists\varphi$;

(M2) $\forall\varphi \to \varphi, \ \ \forall\varphi \rightsquigarrow \varphi$;

(M3) $\forall(\varphi \to \exists\psi) \equiv \exists\varphi \to \exists\psi, \ \ \forall(\varphi \rightsquigarrow \exists\psi) \equiv \exists\varphi \rightsquigarrow \exists\psi$;

(M4) $\forall(\exists\varphi \to \psi) \equiv \exists\varphi \to \forall\psi, \ \ \forall(\exists\varphi \rightsquigarrow \psi) \equiv \exists\varphi \rightsquigarrow \forall\psi$;

(M5) $\forall(\varphi \vee \exists\psi) \equiv \forall\varphi \vee \exists\psi$;

(M6) $\exists\bar{0} \equiv \bar{0}$;

(M7) $\exists\forall\varphi \equiv \forall\varphi$;

(M8) $\forall\forall\varphi \equiv \varphi$;

(M9) $\exists(\exists\varphi \odot \exists\psi) \equiv \exists\varphi \odot \exists\psi$.

deduction rules: (MP$\to$), (MP$\rightsquigarrow$), (Imp$\to$), (Imp$\rightsquigarrow$) and necessitation

(Nec) $\quad \dfrac{\varphi}{\forall\varphi}$.

Bounded integral residuated lattice $A = (A; \odot, \vee, \wedge, \rightarrow, \rightsquigarrow, 0, 1)$, type $\langle 2, 2, 2, 2, 2, 0, 0, \rangle$

Axioms:

(i) $(A; \odot, 1)$ is a monoid (need not be commutative).

(ii) $(A; \vee, \wedge, 0, 1)$ is a bounded lattice.

(iii) $x \odot y \leq z$ iff $x \leq y \rightarrow z$ iff $y \leq x \rightsquigarrow z$ for any x, y, $z \in A$.

$x^{-} := x \rightarrow 0$, $x^{\sim} := x \rightsquigarrow 0$

a residuated lattice := a bounded integral residuated lattice

A residuated lattice is:

a) pseudo-MTL-algebra iff $(x \to y) \vee (y \to x) = 1 = (x \rightsquigarrow y) \vee (y \rightsquigarrow x)$;

b) $R\ell$-monoid iff $(x \to y) \odot x = x \wedge y = y \odot (y \rightsquigarrow x)$;

c) pseudo-BL-algebra iff b) + c);

d) involutive iff $x^{-\sim} = x = x^{\sim -}$;

e) GMV-algebra iff c) + d)

f) Heyting algebra iff $\odot$ and $\wedge$ coincide.

Monadic residuated lattice $A = (A;\ \odot, \vee, \wedge, \to, \rightsquigarrow, 0, 1, \exists, \forall) = (M;\ \exists, \forall)$, type $\langle 2, 2, 2, 2, 2, 0, 0, 1, 1 \rangle$

$A = (A;\ \odot, \vee, \wedge, \to, \rightsquigarrow, 0, 1)$ is a residuated lattice and for each $x, y \in A$:

(iv) $x \to \exists x = 1, \quad x \rightsquigarrow \exists x = 1;$

(v) $\forall x \to x = 1, \quad \forall x \rightsquigarrow x = 1;$

(vi) $\forall(x \to \exists y) = \exists x \to \exists y, \quad \forall(x \rightsquigarrow \exists y) = \exists x \rightsquigarrow \exists y;$

(vii) $\forall(\exists x \to y) = \exists x \to \forall y, \quad \forall(\exists x \rightsquigarrow y) = \exists x \rightsquigarrow \forall y;$

(viii) $\forall(x \vee \exists y) = \forall x \vee \exists y;$

(ix) $\exists \forall x = \forall x;$

(x) $\forall \forall x = \forall x;$

(xi) $\exists(\exists x \odot \exists y) = \exists x \odot \exists y;$

(xii) $\exists(x \odot x) = \exists x \odot \exists x.$

Theorem

If $A = (A; \exists, \forall)$ is a monadic residuated lattice, then $(x \in A)$:

(1) $(\exists x)^- = \forall(x^-)$, $(\exists x)^\sim = \forall(x^\sim)$;

(2) $(\exists x)^{-\sim} = (\forall(x^-))^\sim$, $(\exists x)^{\sim-} = (\forall(x^\sim))^-$;

(3) $(\exists(x^-))^\sim = \forall(x^{-\sim})$, $(\exists(x^\sim))^- = \forall(x^{\sim-})$;

(4) $(\forall(x^{-\sim}))^{-\sim} = \forall(x^{-\sim})$, $\forall(x^{\sim-}))^{\sim-} = \forall(x^{\sim-})$;

(5) $(\exists(x^{-\sim}))^{-\sim} = (\exists x)^{-\sim} \geq \exists(x^{-\sim})$, $(\exists(x^{\sim-}) = (\exists x)^{\sim-} \geq \exists(x^{\sim-})$.

Residuated lattice A ... good if $x^{-\sim} = x^{\sim-}$, for every $x \in A$.

Theorem

If $(A\,; \exists, \forall)$ is a monadic residuated lattice such that the residuated lattice A is good, then for every $x \in A$:

(6) $(\forall(x^-))^\sim = (\forall x^\sim)^-$;

(7) $(\exists(x^-))^\sim = (\exists(x^\sim))^-$.

$(A\,;\exists,\forall)$ ... monadic residuated lattice

$A_{\exists\forall} := \{x \in A \ : x = \exists x\} = \{x \in A \ : x = \forall x\}.$

Theorem

If $(A\,;\exists,\forall)$ is a monadic residuated lattice, then $A_{\exists\forall}$ is a subalgebra of A.

A=(A; $\oplus, \odot, ^{-}, ^{\sim}, 0, 1$) ... *GMV*-algebra

$x \to y := x^{-} \oplus y = (x \odot y^{\sim})^{-}$

$x \rightsquigarrow y := y \oplus x^{\sim} = (y^{-} \odot x)^{\sim}$

$x \vee y := x \oplus (y \odot x^{-}),\ \ x \wedge y := (x^{-} \oplus y) \odot x$

$(A;\ \odot, \vee, \wedge, \to, \rightsquigarrow, 0, 1)$ is a good residuated lattice.

$A = (A;\ \odot, \vee, \wedge, \to, \rightsquigarrow, 0, 1)$ ... a good residuated lattice

$x \oplus y := (x^{-} \odot y^{-})^{\sim} = (x^{\sim} \odot y^{\sim})^{-}$

$x^{-} := x \to 0,\ x^{\sim} := x \rightsquigarrow 0$

A ... GMV-algebra, $\exists : A \longrightarrow A$

$(A; \exists)$ ... monadic GMV-algebra if

(E1) $x \leq \exists x$;

(E2) $\exists(x \vee y) = \exists x \vee \exists y$;

(E3) $\exists((\exists x)^-) = (\exists x)^-$, $\exists((\exists x)^\sim) = (\exists x)^\sim$;

(E4) $\exists(\exists x \oplus \exists y) = \exists x \oplus \exists y$;

(E5) $\exists(x \odot x) = \exists x \odot \exists x$;

(E6) $\exists(x \oplus x) = \exists x \oplus \exists x$.

$\forall x := (\exists x^-)^\sim = (\exists x^\sim)^-$

Theorem

Let $(A; \odot, \vee, \wedge, \rightarrow, \rightsquigarrow, 0, 1, \exists, \forall)$ be a good monadic residuated lattice such that $(A; \oplus, \odot, ^-, ^\sim, 0, 1)$ is a GMV-algebra. Then $(A; \oplus, \odot, ^-, ^\sim, 0, 1, \exists) = (A; \exists)$ is a monadic GMV-algebra.

A ... Heyting algebra, $\exists : A \longrightarrow A$, $\forall : A \longrightarrow A$

$(A; \exists, \forall)$ ... monadic Heyting algebra if

(H1) $\forall x \leq x$;

(H2) $x \leq \exists x$

(H3) $\forall(x \wedge y) = \forall x \wedge \forall y$;

(H4) $\exists(x \vee y) = \exists x \vee \exists y$;

(H5) $\forall 1 = 1$;

(H6) $\exists 0 = 0$;

(H7) $\forall \exists x = \forall x$;

(H8) $\exists \forall x = \forall x$;

(H9) $\exists(\exists x \wedge y) = \exists x \wedge \exists y$.

Theorem

Let $(A; \exists, \forall)$ be a monadic residuated lattice such that A is a Heyting algebra. Then $(A; \exists, \forall)$ is a monadic Heyting algebra.

M ... residuated lattice, $X \neq \emptyset$ ... a set
M^X ... direct power of M, residuated lattice
M^X contains a subalgebra isomorphic to M.
$p \in M^X$, $R(p) := \{p(x) : x \in X\}$

A ... a subalgebra of M^X

A ... a functional monadic residuated lattice if

(i) for every $p \in A$ there exist

$$\sup_M R(p) = \bigvee R(p), \ \inf_M R(p) = \bigwedge R(p);$$

(ii) for every $p \in A$, the constant functions $\exists p$ and $\forall p$ defined by

$$\exists p(x) := \bigvee R(p), \ \forall p(x) := \bigwedge R(p),$$

for any $x \in X$, belong to A.

Theorem

If A is a functional residuated lattice and $p, q \in A$, then

$p \leq \exists p$;

$\forall p \leq p$;

$\exists \forall p = \forall p$;

$\forall \forall p = \forall p$;

$\exists(\exists p \odot \exists q) = \exists p \odot \exists q$;

$\forall(p \vee \exists q) = \forall p \vee \exists q$;

$\forall(p \to \exists q) = \exists p \to \exists q, \quad \forall(p \rightsquigarrow \exists q) = \exists p \rightsquigarrow \exists q$;

$\forall(\exists p \to q) = \exists p \to \forall q, \quad \forall(\exists p \rightsquigarrow q) = \exists p \rightsquigarrow \forall q$;

$\exists(p \odot p) = \exists p \odot \exists p$.

Theorem

If M is a residuated lattice, $X \neq \emptyset$ and $A \subseteq M^X$ is a functional monadic residuated lattice, then $(A; \exists, \forall)$ is a monadic residuated lattice.

A ... residuated lattice, B ... subalgebra of A

B ... relatively complete if for each $a \in A$, the set $\{b \in B : a \leq b\}$ has a least element $\bigwedge_{a \leq b \in B} b$, and the set $\{b \in B : b \leq a\}$ has a greatest element $\bigvee_{a \geq b \in B} b$.

B ... relatively complete subalgebra of A

B ... m-relatively complete if for every $a \in A$ and $x \in B$ such that $x \geq a \odot a$ there is $v \in B$ such that $v \geq a$ and $v \odot v \leq x$.

Theorem

If $(A; \exists, \forall)$ is a monadic residuated lattice, then $A_{\exists\forall}$ is an m-relatively complete subalgebra of A.

Theorem

There is a 1-1 correspondence between monadic residuated lattices and pairs (A, B), where B is an m-relatively complete subalgebra of A.

A ... residuated lattice

$\emptyset \neq F \subseteq A$ ... filter of A if

(i) $x, y \in F \Longrightarrow x \odot y \in F$;

(ii) $x \in F, y \in M, x \leq y \Longrightarrow y \in F$.

$D \subseteq A$ ... deductive system of A if

(iii) $1 \in D$;

(iv) $x \in D, x \to y \in D \Longrightarrow y \in D$.

filters = deductive systems

$\mathcal{F}(A)$ ... complete lattice of all filters of A

$(A; \exists, \forall)$ ... monadic residuted lattice, $F \in \mathcal{F}(A)$

F ... monadic filter (m-filter) of $(A; \exists, \forall)$ if $x \in F \Longrightarrow \forall x \in F$.

$\mathcal{F}(A; \exists, \forall)$ ... complete lattice of all m-filters of $(A; \exists, \forall)$

Theorem

If $(A; \exists, \forall)$ is a monadic residuated lattice, then the lattice $\mathcal{F}(A; \exists, \forall)$ is isomorphic to the lattice $\mathcal{F}(A_{\exists\forall})$ of all filters of the residuated lattice $A_{\exists\forall}$.