

Quantifier elimination, amalgamation,  
deductive interpolation and Craig  
interpolation in many-valued logic

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**Definition.** A logic  $L$  has the **deductive interpolation property (DIP)** if for any set  $\Sigma$  of formulas and for every formula  $\phi$ , if  $\Sigma \vdash_L \phi$ , then there is a formula  $\gamma$  (called **a deductive interpolant** of  $\Sigma$  and  $\phi$ ), such that  $\Sigma \vdash_L \gamma$ ,  $\gamma \vdash_L \phi$  and all variables in  $\gamma$  are common to  $\Sigma$  and to  $\phi$ .

A logic  $L$  has the **Craig interpolation property (CIP)** if for all formulas  $\psi, \phi$ , if  $L \vdash \phi \rightarrow \psi$ , then there is a formula  $\gamma$ , (called a **Craig interpolant of  $\phi$  and  $\psi$** ) such that  $L \vdash \phi \rightarrow \gamma$ ,  $L \vdash_L \gamma \rightarrow \psi$  and all variables in  $\gamma$  are common to  $\psi$  and to  $\phi$ .

**Example.** In classical logic, we have  $\vdash (p \wedge q) \rightarrow (q \vee r)$ . A Craig interpolant of  $p \wedge q$  and  $q \vee r$  is given by  $q$ .  $q$  is also a deductive interpolant.

For logics with the deduction theorem, CIP and DIP are equivalent. In fuzzy logic, CIP implies DIP but **the converse does not hold**:

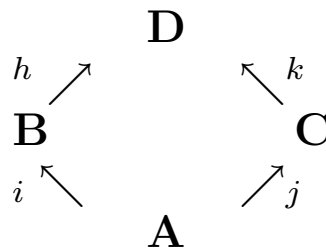
Łukasiewicz logic and product logic have DIP, but not CIP. If  $\phi = p \wedge (p \rightarrow q)$  and  $\psi = r \vee (r \rightarrow q)$ , then  $\phi \rightarrow \psi$  is provable in any fuzzy logic, but it does not have a Craig interpolant in Łukasiewicz or in product logic.

Among the most important fuzzy logics, only Gödel logic is known to have CIP.

**Definition.** A **V-formation** in a class  $\mathcal{K}$  of algebras of the same type is a system  $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$  where  $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$  and  $i, j$  are embeddings of  $\mathbf{A}$  into  $\mathbf{B}$  and into  $\mathbf{C}$ , respectively.



An **amalgam** in  $\mathcal{K}$  of a V-formation  $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$  is a triplet  $(\mathbf{D}, h, k)$  where  $\mathbf{D} \in \mathcal{K}$  and  $h, k$  are embeddings of  $\mathbf{B}$  and  $\mathbf{C}$ , respectively, into  $\mathbf{D}$  such that the diagram



commutes.

A class  $\mathcal{K}$  has the **amalgamation property (AP)** if any V-formation in  $\mathcal{K}$  has an amalgam in  $\mathcal{K}$ .

AP implies DIP (in fact, it implies a stronger property, namely, Robinson's property). In turn, AP is implied by quantifier elimination.

A first-order theory  $T$  has **quantifier elimination (QE)** if every formula  $\phi(x_1, \dots, x_n)$  is provably equivalent in  $T$  to a quantifier free formula  $\psi(x_1, \dots, x_n)$ .

We now present two theorems, the first one is well-known, the second one is a result by Metcalfe, Tsinakis and myself, to (dis)appear.

**Theorem 1.** If  $T$  has QE, then the class of all models of  $T_{\forall}$ , the universal fragment of  $T$ , has AP.

**Theorem 2.** Let  $\mathcal{V}$  be a variety of representable commutative residuated lattices. If  $\mathcal{V}_{lin}$ , the class of all chains in  $\mathcal{V}$ , has AP, then  $\mathcal{V}$  has AP.

From the theorems above we derive the following:

**Theorem 3.** Let  $\mathcal{V}$  a variety of commutative and representable residuated lattices. Suppose that some subclass  $\mathcal{K}$  of  $\mathcal{V}_{lin}$ , enjoys the following properties:

- (1)  $\mathcal{K}$  is elementary (first-order axiomatizable).
- (2)  $Th(\mathcal{K})$  has QE.
- (3) Every algebra in  $\mathcal{V}_{lin}$  can be extended to an algebra in  $\mathcal{K}$ .

Then  $\mathcal{V}$  has AP.

**Didactical examples.** (1) The class of all **divisible abelian o-groups** has QE. Every abelian o-group embeds into a divisible abelian o-group. Hence: The class of **commutative  $\ell$ -groups** has AP.

(2) Every MV-chain embeds into a divisible MV-chain and divisible MV-chains have QE. Hence, the class of **MV-algebras** has AP, see [MuAP] and Łukasiewicz logic has DIP. A similar result holds for the class of **product algebras**.

(3) The class of densely ordered Gödel chains has QE. Every Gödel chain embeds into a densely ordered Gödel chain. Hence: The class of **Gödel algebras** has AP and Gödel logic has DIP. Since Gödel logic has the deduction theorem, it also has CIP.

What about BL-algebras?

By Theorem 2, we can restrict ourselves to **BL-chains**.

Wanted: a class of BL-chains  $\mathcal{K}$  such that:

(1)  $Th(\mathcal{K})$  has QE.

(2) Every BL-chain embeds into an algebra from  $\mathcal{K}$ .



In [CMM], we found two examples of such classes, namely:

(1) The class of **strongly dense** BL-chains, that is, the class of BL-chains which are ordinal sums of divisible MV-algebras and the order of components is dense with minimum and without maximum.

(2) The class of BL-chains which are ordinal sums of divisible MV-algebras and the order of components is **discrete with minimum and without maximum**. In this second case, in order to have QE we need to add two new primitives: the function  $s$  associating to every element  $a < 1$  the minimum idempotent strictly greater than  $a$ , and the function  $p$  associating to every  $a$  not in the first component the minimum of the component immediately below the one  $a$  belongs to.

The class  $\mathcal{K}$  of strongly dense BL-algebras has QE and every BL-chain  $\mathbf{A}$  embeds into a chain in  $\mathcal{K}$ . Indeed, embed the order  $I$  of components of  $\mathbf{A}$  into a dense order  $J$ . Then replace every component  $\mathbf{A}_i$  of  $\mathbf{A}$  by a divisible MV-algebra  $\mathbf{B}_i$  in which  $\mathbf{A}_i$  embeds, and for every  $j \in J \setminus I$  add a divisible MV-chain as a new component. Therefore:

**Theorem 4.** The class of BL-algebras has AP, and BL has DIP.

**Craig interpolation.** We have seen that none of Łukasiewicz logic, product logic or BL has CIP. Apart from Gödel logic (and classical logic) the most interesting fuzzy logics do not have CIP. For instance, Nilpotent minimum NM, the logic induced by the t-norm  $x * y = 0$  if  $x + y \leq 1$  and  $x * y = \min\{x, y\}$  otherwise, has AP, but not CIP.

In [BV], it is shown that both **divisible Łukasiewicz logic  $\mathbf{L}_{div}$**  and **product logic with  $n^{th}$  roots  $\mathbf{\Pi}_{root}$**  have CIP. To prove this, they use an extension of  $\mathbf{L}_{div}$  (resp., of  $\mathbf{\Pi}_{root}$ ) with propositional quantifiers, and they show that such extensions have QE. Thus a Craig interpolant of  $\phi(P, Q)$  and  $\psi(Q, R)$ , where  $P, Q, R$  disjoint sequences of variables, is obtained by eliminating quantifiers in either  $\exists P(\phi(P, Q))$ , or in  $\forall R(\psi(Q, R))$ . Such interpolants are called **uniform** (the first one only depends on  $\phi$  and the second one only depends on  $\psi$ ).

$\mathcal{L}_{div}$  and  $\Pi_{root}$  are conservative extensions of  $\mathcal{L}$  and of  $\Pi$ , respectively. Hence, the following problem arises:

**Problem.** Given a fuzzy logic  $L$  which does not satisfy CIP, find a **conservative extension of it which satisfies CIP**.

The argument used by Baaz and Veith shows that for our problem it suffices to find a conservative extension  $L'$  of  $L$  with such that:

- (a) The extension  $QL'$  of  $L'$  by propositional quantifiers has **QE**.
- (b)  $QL'$  is a **conservative extension** of  $L$ . (Warning: it is possible that  $L'$  is conservative over  $L$ , but  $QL'$  is not conservative over  $L'$ ).

Our method only works for  $\Delta$ -core fuzzy logics, roughly, for logics having the Baaz-Monteiro operator  $\Delta$ .

Then under suitable additional assumptions, it is possible to interpret QL' into the first order theory,  $Th(L')$ , of all L'-chains and viceversa in such a way that  $Th(L')$  has QE iff QL' has QE.

In this way, we arrive to the following general theorem:

**Theorem 5.** Let  $L'$  be a conservative extension with  $\Delta$  of a fuzzy logic  $L$ , and let  $\mathcal{L}'$  be the class of  $L'$ -chains. Suppose that:

(a)  $Th(\mathcal{L}')$  is axiomatizable by universal formulas and has quantifier elimination.

(b)  $\mathcal{L}'$  has a model  $\mathbf{A}$  which is complete with respect to the order and a prime model  $\mathbf{B}$ . Then:

(1)  $QL'$  has **QE**.

(2)  $QL'$  is **conservative over  $L$** .

(3)  $L'$  has **CIP**.

**Applications.** Any **finitely valued fuzzy logic** falls under the scope of Theorem 5 . If we add a constant for each truth value and the Baaz-Monteiro operator  $\Delta$ , we obtain a conservative extension with CIP. But this example is straightforward, and  $\Delta$  is not necessary in this case.

A more interesting example is the following: let  $BL'$  be the logic of **BL-chains which are ordinal sums of divisible MV-algebras with a discrete order of components** added with operators  $s$  and  $p$  ( $s(x)$  is the minimum idempotent strictly above  $x$  and  $p(x)$  is the minimum of the component immediately below the component  $x$  belongs to). Then  $BL'$  satisfies the conditions of Theorem 5. Hence,  $BL'$  has CIP and is conservative over  $BL$ .

A third example is the following:  $NM$ , the **Nilpotent Minimum** logic, is strongly complete wrt the class of all discretely ordered  $NM$ -chains. Now let us add to  $NM$  a symbol  $\frac{1}{2}$  for the fixpoint of the negation, and two operators  $s$  and  $p$  such that, if  $x < 1$ , then  $s(x)$  is the minimum element  $> x$  and if  $0 < x < 1$ , then  $p(x)$  is the greatest element strictly below  $x$ ). In this way we obtain a logic  $L'$  which satisfies all assumptions of Theorem 5. Hence,  $L'$  is a conservative extension of  $NM$  which satisfies CIP.

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Nel caso dei reticoli residuati commutativi l'amalgama è equivalente alla **proprietà di Robinson**:

Una varietà  $\mathcal{V}$  di algebre universali ha la **proprietà di Robinson** se dati due insiemi  $\Pi$  e  $\Sigma$  di equazioni e un'equazione  $\varepsilon$ , se valgono le :

(1)  $Var(\varepsilon) \cap Var(\Pi) \subseteq Var(\Sigma)$ .

(2) Per ogni identità  $\delta$  nelle variabili comuni a  $\Pi$  e a  $\Sigma$  si ha  $\Sigma \models_{\mathcal{V}} \delta$  sse  $\Pi \models_{\mathcal{V}} \delta$ .

(3)  $\Pi, \Sigma \models \varepsilon$ ,

allora  $\Sigma \models \varepsilon$ .