Measuring Uncertainty and Vagueness on MV-algebras

Tomáš Kroupa

Institute of Information Theory and Automation
Academy of Sciences of the Czech Republic

Motivation: Probability

- **probability on BAs** provides tools for dealing with events like

  "Manchester United will score in the 1st half of a match"

- **MV-probability** is dealing with infinite-valued events like

  "Manchester United will score early in a match"

Motivation: Probability

- **probability on BAs** provides tools for dealing with events like

  "Manchester United will score in the 1st half of a match"

- **MV-probability** is dealing with infinite-valued events like

  "Manchester United will score early in a match"

How to combine degrees of belief with truth degrees?

1. Axiomatization
2. Representation
3. Semantics (de Finetti-style theorems)

Motivation: Belief Functions

...when only a special lower estimate of probability is available!

How to combine degrees of belief with truth degrees?

1. Axiomatization
2. Representation
3. Semantics (de Finetti-style theorems)

Motivation: Möbius Transform

- cumulative representation of a real function on a locally finite poset (G.-C. Rota)

Motivation: Möbius Transform

- cumulative representation of a real function on a locally finite poset (G.-C. Rota)
- totally monotone function
  - function on a finite BA with all n-th order differences nonnegative
  - function possessing nonnegative Möbius transform

Motivation: Möbius Transform

- cumulative representation of a real function on a locally finite poset (G.-C. Rota)
- totally monotone function
  - function on a finite BA with all n-th order differences nonnegative
  - function possessing nonnegative Möbius transform

Generalization of Möbius transform to MV-algebras?

The Framework

Combining degrees of truth/belief

Φ := set of (equivalence classes of) formulas in Łukasiewicz logic
Φ := k-generated free MV-algebra L_k
$p : \Phi \to [0, 1]$

The Framework

Combining degrees of truth/belief

Φ := set of (equivalence classes of) formulas in Łukasiewicz logic
$\Phi := k$-generated free MV-algebra L_k
$p : \Phi \to [0, 1]$

p can be a

- **probability** (Mundici,Riečan,...)
- **belief/plausibility** (TK, Flaminio, Godo, Marchioni)
- **CLP/CUP** (Montagna, Keimel,...)

States

Definition (Mundici, Riečan)

A state s on L_k is a function $L_k \to [0,1]$ with

- $s(f \oplus g) = s(f) + s(g)$, for every $f, g \in L_k$ s.t. $f \odot g = 0$
- $s(0) = 0, s(1) = 1$

States

Definition (Mundici, Riečan)

A state s on L_k is a function $L_k \to [0,1]$ with

- $s(f \oplus g) = s(f) + s(g)$, for every $f, g \in L_k$ s.t. $f \odot g = 0$
- $s(0) = 0, s(1) = 1$

Every state is

- monotone $f \leqslant g$ implies $s(f) \leqslant s(g)$
- modular $s(f \oplus g) + s(f \odot g) = s(f) + s(g)$

States are Integrals

Theorem

For every state s there exists a unique Borel probability measure μ on $[0,1]^k$ such that $s(f) = \int f \, \mathrm{d}\mu$, for each $f \in L_k$.

States are Integrals

Theorem

For every state s there exists a unique Borel probability measure μ *on* $[0,1]^k$ *such that* $s(f) = \int f \,\mathrm{d}\mu$*, for each* $f \in L_k$*.*

Equivalently:

$$\int_{[0,1]^k} f \,\mathrm{d}\mu = \int_0^1 \mu\left(f^{-1}([t,1])\right)\,\mathrm{d}t$$

Measuring upper level sets determines the integral.

Averaging the Truth Value

- ϕ formula ($f \in L_k$)
- V_1, V_2 truth valuations

Averaging the Truth Value

- ϕ formula ($f \in L_k$)
- V_1, V_2 truth valuations ($x_1, x_2 \in [0,1]^k$)

Averaging the Truth Value

- ϕ formula ($f \in L_k$)
- V_1, V_2 truth valuations ($x_1, x_2 \in [0,1]^k$)
- s "averaged" truth valuation with $c \in [0,1]$:

$$s(\phi) := cV_1(\phi) + (1-c)V_2(\phi) = cf(x_1) + (1-c)f(x_2)$$

Averaging the Truth Value

- ϕ formula ($f \in L_k$)
- V_1, V_2 truth valuations ($x_1, x_2 \in [0,1]^k$)
- s "averaged" truth valuation with $c \in [0,1]$:

$$s(\phi) := cV_1(\phi) + (1-c)V_2(\phi) = cf(x_1) + (1-c)f(x_2)$$

- (s_n) convergent sequence (in $[0,1]^{L_k}$) of "averaged" TVs:

$$s(\phi) := \lim_{n\to\infty} s_n(\phi)$$

Averaging the Relative Truth Value

- ϕ formula ($f \in L_k$)
- A closed set of truth valuations (closed set in $[0,1]^k$)
- Pavelka-style truth degree of ϕ over A:

$$\|\phi\|_A := \inf \{ V(\phi) \mid V \in A \} = \inf \{ f(x) \mid x \in A \}$$

Averaging the Relative Truth Value

- ϕ formula ($f \in L_k$)
- A closed set of truth valuations (closed set in $[0,1]^k$)
- Pavelka-style truth degree of ϕ over A:

$$\|\phi\|_A := \inf \{ V(\phi) \mid V \in A \} = \inf \{ f(x) \mid x \in A \}$$

Question

Which function on L_k is obtained by

- averaging $\|\phi\|_{A_1}, \|\phi\|_{A_2}$
- taking limits of such averages

BFs on BAs

Definition (Dempster, Shafer)

Let X be a finite nonempty set. A function

$$\beta : \mathcal{P}(X) \to [0,1]$$

is a belief measure if there is a mapping (basic assignment)

$$m : \mathcal{P}(X) \to [0,1]$$

with $m(\emptyset) = 0$ and $\sum_{A \in \mathcal{P}(X)} m(A) = 1$ such that

$$\beta(A) = \sum_{B \subseteq A} m(B), \quad A \in \mathcal{P}(X).$$

BFs on BAs: Examples

Example (MU wins, loses, or a TV-set was switched off?)

$X = \{W, L\}$

$$m(A) = \begin{cases} w, & A = \{W\} \\ \ell, & A = \{L\} \\ 1 - w - \ell, & A = X \end{cases} \qquad w + \ell < 1,\ w, \ell \geqslant 0$$

BFs on BAs: Examples

Example (MU wins, loses, or a TV-set was switched off?)

$X = \{W, L\}$

$$m(A) = \begin{cases} w, & A = \{W\} \\ \ell, & A = \{L\} \\ 1 - w - \ell, & A = X \end{cases} \qquad w + \ell < 1,\ w, \ell \geqslant 0$$

Example (Laplace principle of insufficient reason)

$$m(A) = \begin{cases} 1, & A = X \\ 0, & \text{otherwise} \end{cases}$$

Total Monotonicity

Theorem

The FAE:

1. β *is a belief measure*
2. $\beta : \mathcal{P}(X) \to [0,1]$ *satisfies* $\beta(\emptyset) = 0$, $\beta(X) = 1$ *and*
   - *it is monotone*
   - *for each* $n \geqslant 2$ *and every* $A_1, \dots, A_n \in \mathcal{P}(X)$*:*

$$\beta\left(\bigcup_{i=1}^{n} A_i\right) \geqslant \sum_{\substack{I \subseteq \{1,\dots,n\} \\ I \neq \emptyset}} (-1)^{|I|+1} \beta\left(\bigcap_{i \in I} A_i\right).$$

The function m_β constructed in (2) $\Rightarrow$ (1) is called the Möbius transform of β and $\beta(A) = \sum_{B \subseteq A} m_\beta(B)$

Belief Measures: From BAs to MVs

Belief measures	**Belief functions**
belief measure on $\mathcal{P}(X)$	belief function on L_k
basic assignment	?
TM set function	?

Belief Measures: From BAs to MVs

Belief measures	**Belief functions**
belief measure on $\mathcal{P}(X)$	belief function on L_k
basic assignment	?
TM set function	?

- the mapping $A \mapsto \{B \in \mathcal{P}(X) \mid B \subseteq A\}$ sends the event A to a set of all sets of possible worlds rendering A true:

$$\begin{aligned} \|A\|_B &:= \min\{\, A(x) \mid x \in B \,\} \\ \|A\| &= \{\, B \in \mathcal{P}(X) \mid B \subseteq A \,\} \end{aligned}$$

- **belief** of A = **probability** of $\|A\|$:

$$\beta(A) = m(\|A\|)$$

Belief Measures: From BAs to MVs (ctnd.)

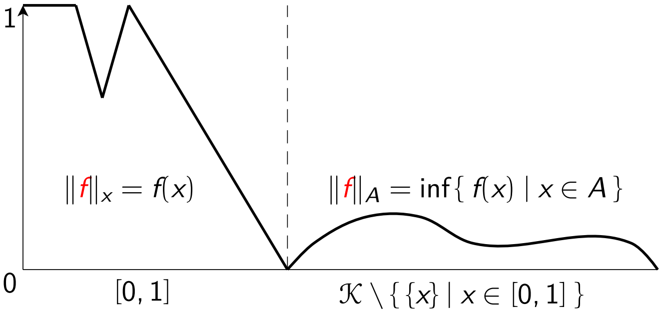
- $f \in L_k$, k-variable McNaughton function
- $A \in \mathcal{K}$, nonempty closed subset of $[0,1]^k$
- define $\|f\|_A := \inf\{\, f(x) \mid x \in A \,\}$
- **belief** of f = **state** of $\|f\|$:

$$\mathrm{Bel}(f) = \mathbf{s}(\|f\|)$$

Belief Measures: From BAs to MVs (ctnd.)

- $f \in L_k$, k-variable McNaughton function
- $A \in \mathcal{K}$, nonempty closed subset of $[0,1]^k$
- define $\|f\|_A := \inf\{ f(x) \mid x \in A \}$
- **belief** of f = **state** of $\|f\|$:

$$\mathrm{Bel}(f) = \mathbf{s}(\|f\|)$$



Space of Closed Subsets

Definition

Let $\mathcal{K}$ be the set of all nonempty closed subsets of $[0,1]^k$ equipped with the Hausdorff metric d_H given by

$$d_H(A,B) = \max\left\{\sup_{a\in A}\inf_{b\in B}\|a-b\|, \sup_{b\in B}\inf_{a\in A}\|a-b\|\right\}, \quad A,B\in\mathcal{K}.$$

Theorem

The metric space $(\mathcal{K}, d_H)$ is ***compact****.*

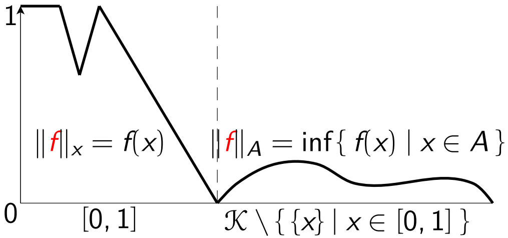
Continuation of McNaughton Functions

$C(\mathcal{K})$ the MV-algebra of all continuous functions $\mathcal{K} \to [0, 1]$

Proposition

The mapping $\|\cdot\| : L_k \to [0, 1]^{\mathcal{K}}$ is

- into $C(\mathcal{K})$
- injective
- preserving any existing infima from L_k to $C(\mathcal{K})$



Belief Functions

A state assignment is any state on $C(\mathcal{K})$.

Definition

Let $\mathbf{s}$ be a state assignment on $C(\mathcal{K})$. A belief function is a mapping $\mathrm{Bel} : L_k \to [0,1]$ given by

$$\mathrm{Bel}(f) = \mathbf{s}(\|f\|), \quad f \in L_k.$$

Example

For each $A \in \mathcal{K}$, the function $\mathrm{Bel}_A(f) = \|f\|_A$ is a belief function whose state assignment is $\mathbf{s}_A$, where

$$\mathbf{s}_A(h) = h(A), \quad h \in C(\mathcal{K}).$$

Properties

Proposition

Let Bel be a belief function on L_k. Then:

- $\mathrm{Bel}(0) = 0$, $\mathrm{Bel}(1) = 1$
- if $f \odot g = 0$, then $\mathrm{Bel}(f \oplus g) \geqslant \mathrm{Bel}(f) + \mathrm{Bel}(g)$
- $\mathrm{Bel}(f) + \mathrm{Bel}(\neg f) \leqslant 1$
- Bel is totally monotone on the lattice reduct of L_k:
  - it is monotone
  - for each $n \geqslant 2$ and every $f_1, \ldots, f_n \in L_k$:

$$\mathrm{Bel}\left(\bigvee_{i=1}^{n} f_i\right) \geqslant \sum_{\substack{I \subseteq \{1,\ldots n\} \\ I \neq \emptyset}} (-1)^{|I|+1} \, \mathrm{Bel}\left(\bigwedge_{i \in I} f_i\right).$$

Representation of BFs

Theorem

For every belief function Bel *on* L_k *there exists a unique*

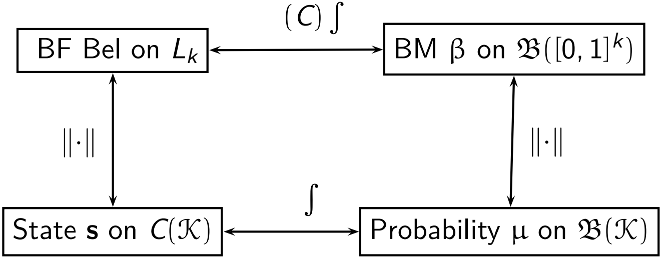
1. ***Borel probability measure*** μ *on* $\mathfrak{B}(\mathcal{K})$ *such that*

$$\mathrm{Bel}(f) = \int_{\mathcal{K}} \|f\| \, \mathrm{d}\mu, \quad f \in L_k$$

2. ***belief measure*** β *on* $\mathfrak{B}([0,1]^k)$ *such that*

$$\mathrm{Bel}(f) = (C) \int_{[0,1]^k} f \mathrm{d}\beta, \quad f \in L_k$$

Scheme


BF Bel on L_k
$(C)\int$
BM β on $\mathfrak{B}([0,1]^k)$
$\|\cdot\|$
$\|\cdot\|$
$\int$
State $\mathbf{s}$ on $C(\mathcal{K})$
Probability μ on $\mathfrak{B}(\mathcal{K})$

Space of Belief Functions

Theorem

Let Bel *be a belief function on* L_k. *Then the FAE:*

- Bel *is an extreme point of* $BEL(L_k)$
- *there exists* $A \in \mathcal{K}$ *with* $\mathrm{Bel} = \mathrm{Bel}_A$
- $\{f \in L_k \mid \mathrm{Bel}(f) = 1\}$ *is a filter that is* $\bigcap$ *of maximal filters*
- $\{f \in L_k \mid \mathrm{Bel}(f) = 0\}$ *is an ideal that is* $\bigcap$ *of maximal ideals*

Compare:

D. Mundici.
Averaging the truth-value in Łukasiewicz logic.
Studia Logica, 55(1):113–127, 1995.

BF as a Lower Probability

Theorem

For every $f \in L_k$:

$$\mathrm{Bel}(f) = \min\left\{ s(f) \mid s \textit{ state on } L_k \textit{ with } s \geqslant \mathrm{Bel} \right\}$$

BF as a Lower Probability

Theorem

For every $f \in L_k$:

$$\mathrm{Bel}(f) = \min \left\{ s(f) \mid s \text{ state on } L_k \text{ with } s \geqslant \mathrm{Bel} \right\}$$

In the spirit of:

M. Fedel, K. Keimel, F. Montagna, and W. Roth.
Imprecise probabilities, bets and functional analytic methods in Łukasiewicz logic.
To appear in Forum Mathematicum.

Second Encounter with Upper Level Sets

- every **probability** (**state**) of $f \in L_k$ is

$$\int_{[0,1]^k} f \, \mathrm{d}\mu = \int_0^1 \mu\left(f^{-1}([t,1])\right) \, \mathrm{d}t$$

for a Borel probability measure μ on $[0,1]^k$

Second Encounter with Upper Level Sets

- every **probability** (**state**) of $f \in L_k$ is

$$\int_{[0,1]^k} f \, \mathrm{d}\mu = \int_0^1 \mu \left(f^{-1}([t, 1]) \right) \, \mathrm{d}t$$

  for a Borel probability measure μ on $[0, 1]^k$

- every **belief** of $f \in L_k$ is

$$(C) \int_{[0,1]^k} f \, \mathrm{d}\nu = \int_0^1 \nu \left(f^{-1}([t, 1]) \right) \, \mathrm{d}t$$

  for a TM capacity ν on $[0, 1]^k$

Analyzing Upper Level Sets

- upper level sets of McNaughton functions:

$$\mathcal{U} := \left\{ f^{-1}([t,1]) \;\middle|\; f \in L_k, t \in [0,1] \right\}$$

- measuring this family determines the state/belief function

Analyzing Upper Level Sets

- upper level sets of McNaughton functions:

$$\mathcal{U} := \left\{ f^{-1}([t,1]) \;\middle|\; f \in L_k, t \in [0,1] \right\}$$

- measuring this family determines the state/belief function
- but $\mathcal{U}$ is NOT a natural domain of a measure/valuation

Analyzing Upper Level Sets

- upper level sets of McNaughton functions:

$$\mathcal{U} := \left\{ f^{-1}([t,1]) \;\middle|\; f \in L_k, t \in [0,1] \right\}$$

- measuring this family determines the state/belief function
- but $\mathcal{U}$ is NOT a natural domain of a measure/valuation

Is it enough to take $\mathcal{R} := \left\{ f^{-1}(1) \;\middle|\; f \in L_k \right\}$ **?**

One-sets of McNaughton Functions

Definition

A rational polyhedron is a finite union of simplices in $[0,1]^k$ with rational coordinates.

One-sets of McNaughton Functions

Definition

A rational polyhedron is a finite union of simplices in $[0,1]^k$ with rational coordinates.

Theorem

There is a 1-1 correspondence between one-sets of McNaughton functions and rational polyhedra.

Rational Polyhedra

$\mathcal{R}$ = the set of all rational polyhedra

- $\mathcal{R}$ is a lattice of subsets of $[0, 1]^k$ closed under pointwise $\cup$, $\cap$

Rational Polyhedra

$\mathcal{R}$ = the set of all rational polyhedra

- $\mathcal{R}$ is a lattice of subsets of $[0,1]^k$ closed under pointwise $\cup$, $\cap$
- there are many disjoint pairs $A_1, A_2 \in \mathcal{R}$

Rational Polyhedra

$\mathcal{R}$ = the set of all rational polyhedra

- $\mathcal{R}$ is a lattice of subsets of $[0,1]^k$ closed under pointwise $\cup$, $\cap$
- there are many disjoint pairs $A_1, A_2 \in \mathcal{R}$
- there are enough $A \in \mathcal{R}$ to approximate $K \in \mathcal{K}$:

$$K = \bigcap \{ A \in \mathcal{R} \mid A \supseteq K \}$$

Lattices of Subsets

- the usual algebras for measures are Boolean (σ)-algebras
- the usual extension procedures exist for Boolean rings...
- ...but extension of set functions from lattices is possible!

Lattices of Subsets

- the usual algebras for measures are Boolean (σ)-algebras
- the usual extension procedures exist for Boolean rings...
- ...but extension of set functions from lattices is possible!

Example

$\mathcal{K}$ compact subsets of $[0,1]^k$
$\mathcal{R}$ rational polyhedra in $[0,1]^k$

$\emptyset, [0,1]^k \in \mathcal{K}, \mathcal{R} \;\Rightarrow\;$ the generated Boolean ring is a BA

Is a function on $\mathcal{R}$ extendable to a Borel probability measure?

Not Hopeless at All!

Theorem (Mundici)

For each $k = 1, 2, \ldots,$ the k-dimensional rational measure (of k-dimensional rational polyhedra) on $\mathcal{R}$ extends to Lebesgue measure on $[0, 1]^k$.

D. Mundici.
Measure theory in the geometry of $GL(n, \mathbb{Z}) \ltimes \mathbb{Z}^n$
arXiv:1102.0897v1 [math.GN]

Not Hopeless at All!

Theorem (Mundici)

For each $k = 1, 2, \ldots,$ the k-dimensional rational measure (of k-dimensional rational polyhedra) on $\mathcal{R}$ extends to Lebesgue measure on $[0, 1]^k$.

D. Mundici.
Measure theory in the geometry of $GL(n, \mathbb{Z}) \ltimes \mathbb{Z}^n$
arXiv:1102.0897v1 [math.GN]

Which functions on $\mathcal{R}$ extend to Borel probability measures?

Functions on Lattices

Definition

Let $\mu : \mathcal{R} \to [0,1]$ be s.t. $\mu(\emptyset) = 0$, $\mu\left([0,1]^k\right) = 1$. Then μ is

- modular if $\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$
- monotone if $A \subseteq B \Rightarrow \mu(A) \leqslant \mu(B)$
- valuation if it is modular and monotone

Functions on Lattices

Definition

Let $\mu : \mathcal{R} \to [0,1]$ be s.t. $\mu(\emptyset) = 0$, $\mu\left([0,1]^k\right) = 1$. Then μ is

- modular if $\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$
- monotone if $A \subseteq B \Rightarrow \mu(A) \leqslant \mu(B)$
- valuation if it is modular and monotone
- continuous if, for each $A_1 \supseteq A_2 \supseteq \cdots$ with $\bigcap_{n=1}^{\infty} A_n \in \mathcal{R}$

$$\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n \to \infty} \mu(A_n)$$

Functions on Lattices

Definition

Let $\mu : \mathcal{R} \to [0,1]$ be s.t. $\mu(\emptyset) = 0$, $\mu\left([0,1]^k\right) = 1$. Then μ is

- modular if $\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$
- monotone if $A \subseteq B \;\Rightarrow\; \mu(A) \leqslant \mu(B)$
- valuation if it is modular and monotone
- continuous if, for each $A_1 \supseteq A_2 \supseteq \cdots$ with $\bigcap_{n=1}^{\infty} A_n \in \mathcal{R}$

$$\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n)$$

- tight if, for each pair s.t. $A \subseteq B$,

$$\mu(A) + \sup\{\, \mu(C) \mid C \subseteq B \setminus A,\; C \in \mathcal{R} \,\} = \mu(B)$$

Intermezzo 1

Theorem (Smiley-Horn-Tarski Theorem)

Every valuation on $\mathcal{R}$ has a unique extension to a finitely additive probability measure on the least algebra of subsets containing $\mathcal{R}$.

Intermezzo 1

Theorem (Smiley-Horn-Tarski Theorem)

Every valuation on $\mathcal{R}$ has a unique extension to a finitely additive probability measure on the least algebra of subsets containing $\mathcal{R}$.

- extension of real modular functions
- the extension is essentially finitely additive
- it does NOT preserve continuity of valuations!

The continuity must be incorporated into the extension.

Intermezzo 2: Extension from $\mathcal{K}$

Theorem

If $\mu : \mathcal{K} \to [0, \infty)$ *is tight, then it has a unique extension to a Borel probability measure.*

Intermezzo 2: Extension from $\mathcal{K}$

Theorem

If $\mu : \mathcal{K} \to [0, \infty)$ is tight, then it has a unique extension to a Borel probability measure.

Extension

1. for each $A \subseteq [0,1]^k$ let

$$\hat{\mu}(A) := \sup\{\, \mu(B) \mid B \subseteq A,\ B \in \mathcal{K} \,\}$$

2. verify that the restriction of $\hat{\mu}$ to Borel sets is a measure

Extension from $\mathcal{R}$: Finally!

Theorem

If $\mu : \mathcal{R} \to [0, 1]$ *is tight, then:*

Extension from $\mathcal{R}$: Finally!

Theorem

If $\mu : \mathcal{R} \to [0, 1]$ *is tight, then:*

- μ *is monotone, modular, and upper continuous*
- μ *has a unique extension to a Borel probability measure*

Extension from $\mathcal{R}$: Finally!

Theorem

If $\mu : \mathcal{R} \to [0,1]$ is tight, then:

- *μ is monotone, modular, and upper continuous*
- *μ has a unique extension to a Borel probability measure*

Extension

1. for each $A \in \mathcal{K}$ let

$$\bar{\mu}(A) := \inf\{\, \mu(B) \mid B \supseteq A,\ B \in \mathcal{R}\,\}$$

2. verify that $\bar{\mu}$ is tight on the lattice $\mathcal{K}$
3. use the theorem of Kisyński

Summary

Theorem (Representing states)

There is a 1-1 correspondence between

- *states on L_k*
- *tight measures on $\mathcal{R}$*

Summary

Theorem (Representing states)

There is a 1-1 correspondence between

- *states on L_k*
- *tight measures on $\mathcal{R}$*

Theorem (Representing BFs)

There is a 1-1 correspondence between

- *BFs on L_k*
- *TM capacities on $\mathcal{K}$*

Open Problems

AXIOMATIZATION

- Does every totally monotone function on an MV-algebra possess generalized Möbius transform?

Open Problems

AXIOMATIZATION

- Does every totally monotone function on an MV-algebra possess generalized Möbius transform?

REPRESENTATION

- Can we extend the duality states/Borel probabilities to MV-CLPs/BA-CLPs?

Open Problems

AXIOMATIZATION

- Does every totally monotone function on an MV-algebra possess generalized Möbius transform?

REPRESENTATION

- Can we extend the duality states/Borel probabilities to MV-CLPs/BA-CLPs?
- By using capacities on upper level sets?

Open Problems

AXIOMATIZATION

- Does every totally monotone function on an MV-algebra possess generalized Möbius transform?

REPRESENTATION

- Can we extend the duality states/Borel probabilities to MV-CLPs/BA-CLPs?
- By using capacities on upper level sets?
- Is state on any MV-algebra M determined by measuring $\{ f^{-1}(1) \mid f \in M \}$?

References

G. Gierz, K.H. Hofmann, K. Keimel et al.
Continuous Lattices and Domains.
Cambridge University Press, 2003.

J. Kisyński.
On the generation of tight measures.
Studia Mathematica T. XXX. (1968)

T. Kroupa.
Generalized Möbius transform of games on MV-algebras and its application to Cimmino-type algorithm for the core.
To appear in Contemporary Mathematics/AMS (volume on Optimization Theory and Related Topics), 2011.

D. Mundici.
Measure theory in the geometry of $GL(n, \mathbb{Z}) \ltimes \mathbb{Z}^n$
arXiv:1102.0897v1 [math.GN]