# Admissible rules and Łukasiewicz logic

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# Admissible rules

# Basic concepts

Logical system $L$: specifies a consequence relation $\Gamma \vdash_L \varphi$ "formula $\varphi$ follows from a set $\Gamma$ of formulas"

Theorems of $L$: $\varphi$ such that $\varnothing \vdash_L \varphi$

(Inference) rule: a relation between sets of formulas $\Gamma$ and formulas $\varphi$

A rule $\varrho$ is derivable in $L \Leftrightarrow \Gamma \vdash_L \varphi$ for every $\langle \Gamma, \varphi \rangle \in \varrho$

A rule $\varrho$ is admissible in $L \Leftrightarrow$ the set of theorems of $L$ is closed under $\varrho$

# Propositional logics

Propositional logic $L$:

Language: formulas $\mathrm{Form}_L$ built freely from variables $\{p_n : n \in \omega\}$ using a fixed set of connectives of finite arity

Consequence relation $\vdash_L$: finitary structural Tarski-style consequence operator
I.e.: a relation $\Gamma \vdash_L \varphi$ between finite sets of formulas and formulas such that

- $\varphi \vdash_L \varphi$
- $\Gamma \vdash_L \varphi$ implies $\Gamma, \Gamma' \vdash_L \varphi$
- $\Gamma \vdash_L \varphi$ and $\Gamma, \varphi \vdash_L \psi$ imply $\Gamma \vdash_L \psi$
- $\Gamma \vdash_L \varphi$ implies $\sigma(\Gamma) \vdash_L \sigma(\varphi)$ for every substitution $\sigma$

# Propositional admissible rules

We consider rules of the form

$$\frac{\varphi_1,\dots,\varphi_n}{\psi} := \{\langle\{\sigma(\varphi_1),\dots,\sigma(\varphi_n)\},\sigma(\psi)\rangle : \sigma \text{ substitution}\}$$

This rule is

- derivable (valid) in $L$ iff $\varphi_1,\dots,\varphi_n \vdash_L \psi$
- admissible in $L$ (written as $\varphi_1,\dots,\varphi_n \mathrel{|\!\sim}_L \psi$) iff for all substitutions $\sigma$: if $\vdash_L \sigma(\varphi_i)$ for every $i$, then $\vdash_L \sigma(\psi)$

$\mathrel{|\!\sim}_L$ is the largest consequence relation with the same theorems as $\vdash_L$

$L$ is structurally complete if $\vdash_L = \mathrel{|\!\sim}_L$

# Examples

- Classical logic ($\mathbf{CPC}$) is structurally complete:
  a $0$–$1$ assignment witnessing $\Gamma \nvdash_{\mathbf{CPC}} \varphi$
  $\Rightarrow$ a ground substitution $\sigma$ such that $\vdash \bigwedge \sigma(\Gamma)$, $\nvdash \sigma(\varphi)$
- All normal modal logics $L$ admit

$$\Diamond q \wedge \Diamond \neg q \;/\; p$$

  $L$ is valid in a $1$-element frame $F$ (Makinson's theorem)
  $\Diamond q \wedge \Diamond \neg q$ is not satisfiable in $F$
- More generally: $\Gamma$ is unifiable $\Leftrightarrow \Gamma \not\mid\!\sim_L p$, where $p \notin \mathrm{Var}(\Gamma)$
- All superintuitionistic logics admit the Kreisel–Putnam rule [Prucnal]:

$$\neg p \to q \vee r \;/\; (\neg p \to q) \vee (\neg p \to r)$$

# Multiple-conclusion consequence relations

A (finitary structural) multiple-conclusion consequence:
a relation $\Gamma \vdash \Delta$ between finite sets of formulas such that

- $\varphi \vdash \varphi$
- $\Gamma \vdash \Delta$ implies $\Gamma, \Gamma' \vdash \Delta, \Delta'$
- $\Gamma \vdash \varphi, \Delta$ and $\Gamma, \varphi \vdash \Delta$ imply $\Gamma \vdash \Delta$
- $\Gamma \vdash \Delta$ implies $\sigma(\Gamma) \vdash \sigma(\Delta)$ for every substitution $\sigma$

# Multiple-conclusion rules

Multiple-conclusion rule: $\Gamma \,/\, \Delta$, where $\Gamma$ and $\Delta$ finite sets of formulas

- derivable in $L$ ($\Gamma \vdash_L \Delta$) iff $\Gamma \vdash_L \psi$ for some $\psi \in \Delta$
- admissible in $L$ ($\Gamma \mathrel{|\!\!\sim}_L \Delta$) iff for all substitutions $\sigma$:
  if $\vdash \sigma(\varphi)$ for every $\varphi \in \Gamma$, then $\vdash \sigma(\psi)$ for some $\psi \in \Delta$

$\vdash_L$ and $\mathrel{|\!\!\sim}_L$ are multiple-conclusion consequence relations

Example: disjunction property = $\dfrac{p \vee q}{p, q}$

# Algebraization

$L$ is finitely algebraizable wrt a class $K$ of algebras if there is a finite set $\Delta(x, y)$ of formulas and a finite set $E(p)$ of equations such that

- $\Gamma \vdash_L \varphi \Leftrightarrow E(\Gamma) \vDash_K^{\wedge} E(\varphi)$
- $\Theta \vDash_K t \approx s \Leftrightarrow \Delta(\Theta) \vdash_L^{\wedge} \Delta(t, s)$
- $p \dashv\vdash_L^{\wedge} \Delta(E(p))$
- $x \approx y \mathrel{=\!\!\!|\!\!\vDash}_K^{\wedge} E(\Delta(x, y))$

where $\Gamma \vdash_L^{\wedge} \Delta$ means $\Gamma \vdash_L \psi$ for all $\psi \in \Delta$

We may assume $K$ is a quasivariety

I will write $x \leftrightarrow y$ for $\Delta(x, y)$

# Admissibility and algebra

$L$ finitely algebraizable, $K$ its equivalent quasivariety

| logic | algebra |
| --- | --- |
| propositional formulas | terms |
| single-conclusion rules | quasi-identities |
| multiple-conclusion rules | clauses |
| $L$-derivable | valid in all $K$-algebras |
| $L$-admissible | valid in free $K$-algebras |

studying multiple-conclusion admissible rules
= studying the universal theory of free algebras

# Unification

Unifier of $\{t_i \approx s_i : i \in I\}$: a substitution $\sigma$ such that $\vDash_K \sigma(t_i) \approx \sigma(s_i)$ for all $i$

Dealgebraization: a unifier of a set of formulas $\Gamma$ is $\sigma$ such that $\vdash_L \sigma(\varphi)$ for every $\varphi \in \Gamma$

- $\Gamma \mathrel{|\!\!\sim}_L \Delta$ iff every unifier of $\Gamma$ also unifies some $\psi \in \Delta$
- $\Gamma$ is unifiable iff $\Gamma \mathrel{|\!\!\not\sim}_L p$ ($p \notin \mathrm{Var}(\Gamma)$) iff $\Gamma \mathrel{|\!\!\not\sim}_L$

$\sigma$ is more general than $\tau$ ($\tau \preceq \sigma$) if there is $\upsilon$ such that $\vdash_L \tau(\alpha) \leftrightarrow \upsilon(\sigma(\alpha))$ for every $\alpha$

# Properties of admissible rules

Typical questions about admissibility:

- structural completeness
- decidability
  - computational complexity
- semantic characterization
- description of a basis (= axiomatization of $\mathrel{|\!\sim}_L$ over $\vdash_L$)
  - finite basis? independent basis?
- inheritance of rules

# Admissibly saturated approximation

$\Gamma$ is admissibly saturated if $\Gamma \mathrel{|\!\sim}_L \Delta$ implies $\Gamma \vdash_L \Delta$ for any $\Delta$

Assume for simplicity that $L$ has a well-behaved conjunction.

Admissibly saturated approximation of $\Gamma$:
a finite set $\Pi_\Gamma$ such that

- each $\pi \in \Pi_\Gamma$ is admissibly saturated
- $\Gamma \mathrel{|\!\sim}_L \Pi_\Gamma$
- $\pi \vdash_L \varphi$ for each $\pi \in \Pi_\Gamma$ and $\varphi \in \Gamma$

# Application of admissible saturation

Reduction of $\mathrel|\!\!\sim_L$ to $\vdash_L$:

$$\Gamma \mathrel|\!\!\sim_L \Delta \quad \text{iff} \quad \forall \pi \in \Pi_\Gamma \, \exists \psi \in \Delta \, \pi \vdash_L \psi$$

Assuming every $\Gamma$ has an a.s. approximation $\Pi_\Gamma$:

- if $\Gamma \mapsto \Pi_\Gamma$ is computable and $\vdash_L$ is decidable, then $\mathrel|\!\!\sim_L$ is decidable
- if $\Gamma \,/\, \Pi_\Gamma$ is derivable in $\vdash_L +$ a set of rules $R \subseteq \mathrel|\!\!\sim_L$, then $R$ is a basis of admissible rules
- if each $\pi \in \Pi_\Gamma$ has an mgu $\sigma_\pi$, then $\{\sigma_\pi : \pi \in \Pi_\Gamma\}$ is a complete set of unifiers for $\Gamma$

# Projective formulas

$\pi$ is projective if it has a unifier $\sigma$ such that $\pi \vdash_L \varphi \leftrightarrow \sigma(\varphi)$ for every $\varphi$ (it's enough to check variables)

- $\sigma$ is an mgu of $\pi$: if $\tau$ is a unifier of $\pi$, then $\tau \equiv \tau \circ \sigma$
- projective formula = presentation of a projective algebra
- projective formulas are admissibly saturated
  projective approximation := admissibly saturated approximation consisting of projective formulas

If projective approximations exist:

- characterization of $\mathrel{|\!\!\sim}_L$ in terms of projective formulas
- finitary unification type

# Exact formulas

$\varphi$ is exact if there exists $\sigma$ such that

$$\vdash_L \sigma(\psi) \quad \text{iff} \quad \varphi \vdash_L \psi$$

for all formulas $\psi$

- projective $\Rightarrow$ exact $\Rightarrow$ admissibly saturated
- in general: can't be reversed
- if projective approximations exist:
  projective = exact = admissibly saturated
- exact formulas do not need to have mgu

# Known results

Admissibility well-understood for some superintuitionistic and transitive modal logics:

- logics with frame extension properties, e.g.:
  - $\mathbf{K4}$, $\mathbf{GL}$, $\mathbf{D4}$, $\mathbf{S4}$, $\mathbf{Grz}$ ($\pm.1$, $\pm.2$, $\pm$bounded branching)
  - $\mathbf{IPC}$, $\mathbf{KC}$
- logics of bounded depth
- linearly (pre)ordered logics: $\mathbf{K4.3}$, $\mathbf{S4.3}$, $\mathbf{S5}$; $\mathbf{LC}$
- some temporal logics: $\mathbf{LTL}$

Not much known for other nonclassical logics:

- structural (in)completeness of some substructural and fuzzy logics

# Methods in modal logic

Analysis of admissibility in modal and si logics:

- building models from reduced rules [Rybakov]
- combinatorial manipulation of universal frames [Rybakov]
- projective formulas and model extension properties [Ghilardi]
- Zakharyaschev-style canonical rules [J.]

# Projectivity in modal logics

Extension property: if $F$ is an $L$-model with a single root $r$ and $x \vDash \varphi$ for every $x \in F \smallsetminus \{r\}$, then we can change satisfaction of variables in $r$ to make $r \vDash \varphi$

Theorem [Ghilardi]: If $L \supseteq \mathbf{K4}$ has the finite model property, the following are equivalent:

- $\varphi$ is projective
- $\varphi$ has the extension property
- $\theta_\varphi$ is a unifier of $\varphi$

where $\theta_\varphi$ is an explicitly defined substitution

# Extensible modal logics

$L \supseteq \mathbf{K4}$ with FMP is extensible if a finite transitive frame $F$ is an $L$-frame whenever

- $F$ has a unique root $r$
- $F \smallsetminus \{r\}$ is an $L$-frame
- $r$ is (ir)reflexive and $L$ admits a finite frame with an (ir)reflexive point

Theorem [Ghilardi]: If $L$ is extensible, then any $\varphi$ has a projective approximation $\Pi_\varphi$ whose modal degree is bounded by $\mathrm{md}(\varphi)$.

# Admissibility in extensible logics

Let $L$ be an extensible modal logic:

- if $L$ is finitely axiomatizable, $\mid\!\sim_L$ is decidable
- $\mid\!\sim_L$ is complete wrt $L$-frames where all finite subsets have appropriate tight predecessors
- it is possible to construct an explicit basis of admissible rules of $L$
  ($L$ has an independent basis, but no finite basis)
- any logic inheriting admissible multiple-conclusion rules of $L$ is itself extensible
- $L$ has finitary unification type

# Łukasiewicz logic

# Admissibility in basic fuzzy logics

Fuzzy logics: multivalued logics using a linearly ordered algebra of truth values

The three fundamental continuous t-norm logics are:

- Gödel–Dummett logic ($\mathbf{LC}$): superintuitionistic; structurally complete
- Product logic ($\mathbf{\Pi}$): also structurally complete [Cintula & Metcalfe]
- Łukasiewicz logic (**Ł**): structurally incomplete $\Rightarrow$ nontrivial admissibility problem

# Łukasiewicz logic

Connectives: $\to$, $\neg$, $\cdot$, $\oplus$, $\wedge$, $\vee$, $\bot$, $\top$ (not all needed as basic)

Semantics: $[0,1]_{\text{Ł}} = \langle [0,1], \{1\}, \to, \neg, \cdot, \oplus, \min, \max, 0, 1\rangle$, where

- $x \to y = \min\{1, 1 - x + y\}$
- $\neg x = 1 - x$
- $x \cdot y = \max\{0, x + y - 1\}$
- $x \oplus y = \min\{1, x + y\}$

$[0,1]_{\mathbb{Q}}$ suffices instead of $[0,1]$

Calculus: Modus Ponens + finitely many axiom schemata

# Algebraization

Ł is finitely algebraizable:

$K$ = the variety of $MV$-algebras

$\Rightarrow$ we are interested in the universal theory of free $MV$-algebras

# McNaughton functions

Free $MV$-algebra $F_n$ over $n$ generators, $n$ finite:

- The algebra of formulas in $n$ variables modulo **Ł**-provable equivalence (Lindenbaum–Tarski algebra)
- Explicit description by McNaughton: the algebra of all continuous piecewise linear functions

$$f \colon [0,1]^n \to [0,1]$$

  with integer coefficients, with operations defined pointwise (i.e., as a subalgebra of $[0,1]_{\text{Ł}}^{[0,1]^n}$)

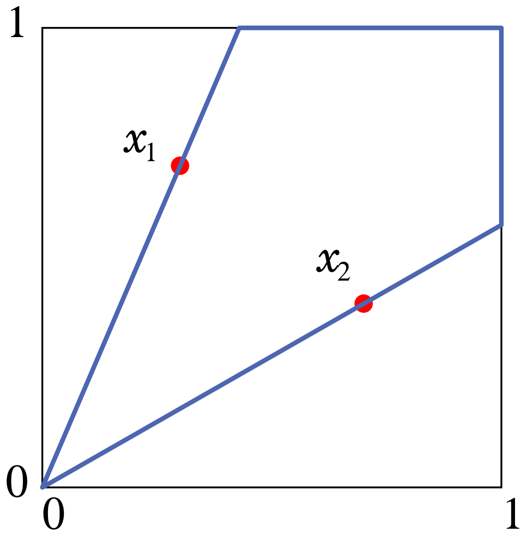
$k$-tuples of elements of $F_n$: piecewise linear functions $f \colon [0,1]^n \to [0,1]^k$

# 1-reducibility

Theorem [J.]: $\Gamma \mathrel{|\!\!\sim_{\text{Ł}}} \Delta$ iff $F_1 \vDash \Gamma \,/\, \Delta$

IOW: all free $MV$-algebras except $F_0$ have the same universal theory

Proof idea:
Finitely many points in $[0,1]^n_{\mathbb{Q}}$ can be connected by a suitable piecewise linear curve



# Reparametrization

Recall: valuation to $m$ variables in $F_1$ = continuous piecewise linear $f\colon [0,1] \to [0,1]^m$ with integer coefficients

Validity of a formula under $f$ only depends on $\mathrm{rng}(f)$
$\Rightarrow$ Question: which piecewise linear curves can be reparametrized to have integer coefficients?

Observation: Let

$$f(t) = a + tb, \quad t \in [t_i, t_{i+1}],$$

where $a, b \in \mathbb{Z}^m$. Then the lattice point $a$ lies on the line connecting the points $f(t_i)$, $f(t_{i+1})$. This is independent of parametrization.

# Anchoredness

If $X \subseteq \mathbb{R}^m$, let $A(X)$ be its affine hull and $C(X)$ its convex hull

$X$ is anchored if $A(X) \cap \mathbb{Z}^m \neq \varnothing$

Using Hermite normal form, we obtain:

- $X \subseteq \mathbb{Q}^m$ is anchored iff

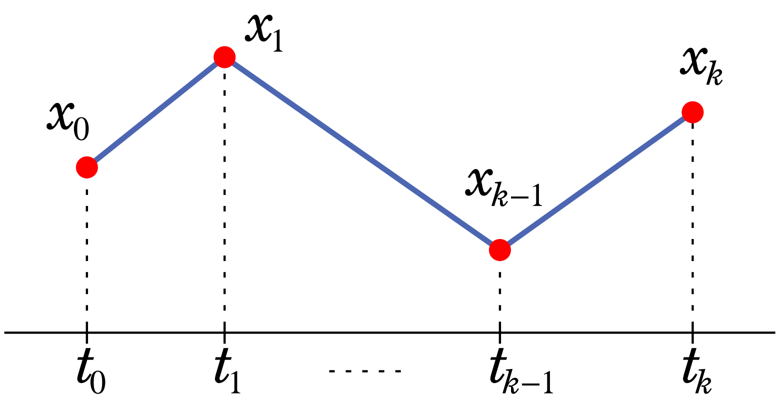
$$\forall u \in \mathbb{Z}^m \, \forall a \in \mathbb{Q} \, [\forall x \in X \, (u^{\mathsf{T}} x = a) \Rightarrow a \in \mathbb{Z}]$$

  (Whenever $X$ is contained in a hyperplane defined by an affine function with integral linear coefficients, its constant coefficients must be integral, too.)

- Given $x_0, \dots, x_k \in \mathbb{Q}^m$, it is decidable in polynomial time whether $\{x_0, \dots, x_k\}$ is anchored

# Reparametrization (cont'd)

Notation: $L(t_0, x_0; t_1, x_1; \ldots; t_k, x_k) =$




Lemma [J.]: If $x_0, \ldots, x_k \in \mathbb{Q}^m$, TFAE:

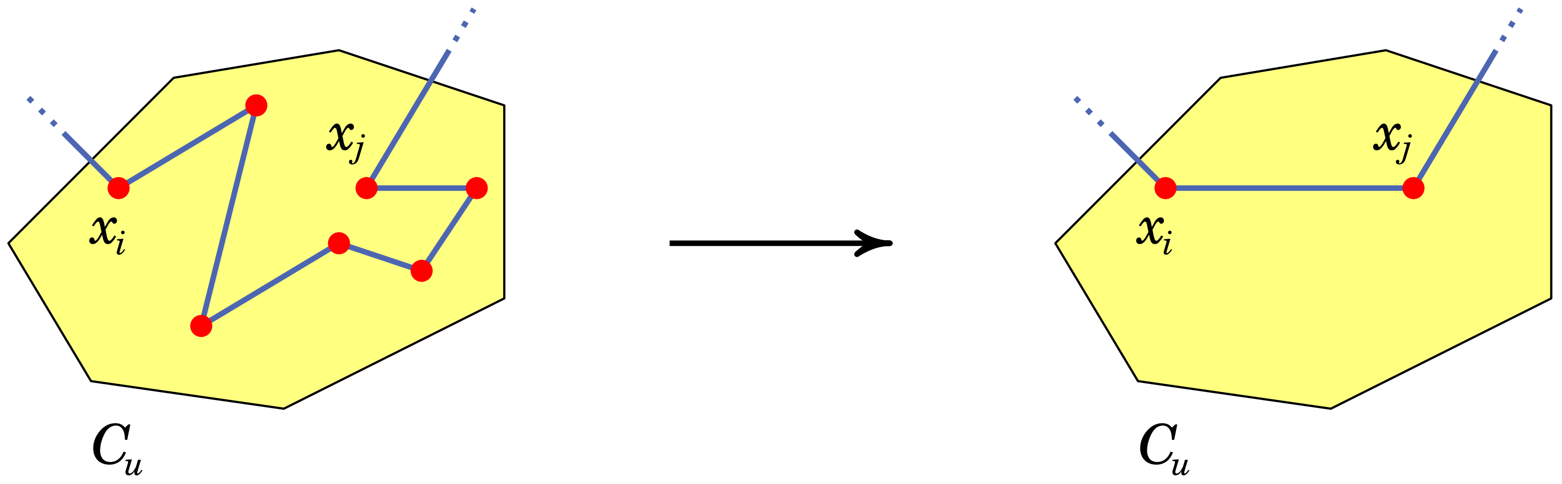
- there exist rationals $t_0 < \cdots < t_k$ such that $L(t_0, x_0; \ldots; t_k, x_k)$ has integer coefficients
- $\{x_i, x_{i+1}\}$ is anchored for each $i < k$

# Simplification of counterexamples

Goal: Given a counterexample $L(t_0, x_0; \dots ; t_k, x_k)$ for $\Gamma \ / \ \Delta$ in $F_1$, simplify it so that its parameters (e.g., $k$) are bounded

$\{x \in [0,1]^m : \Gamma(x) = 1\}$ is a finite union $\bigcup_{u<r} C_u$ of polytopes

Idea: If $\mathrm{rng}(L(t_i, x_i; \dots ; t_j, x_j)) \subseteq C_u$, replace $L(t_i, x_i; t_{i+1}, x_{i+1}; \dots ; t_j, x_j)$ with $L(t_i, x_i; t_j, x_j)$

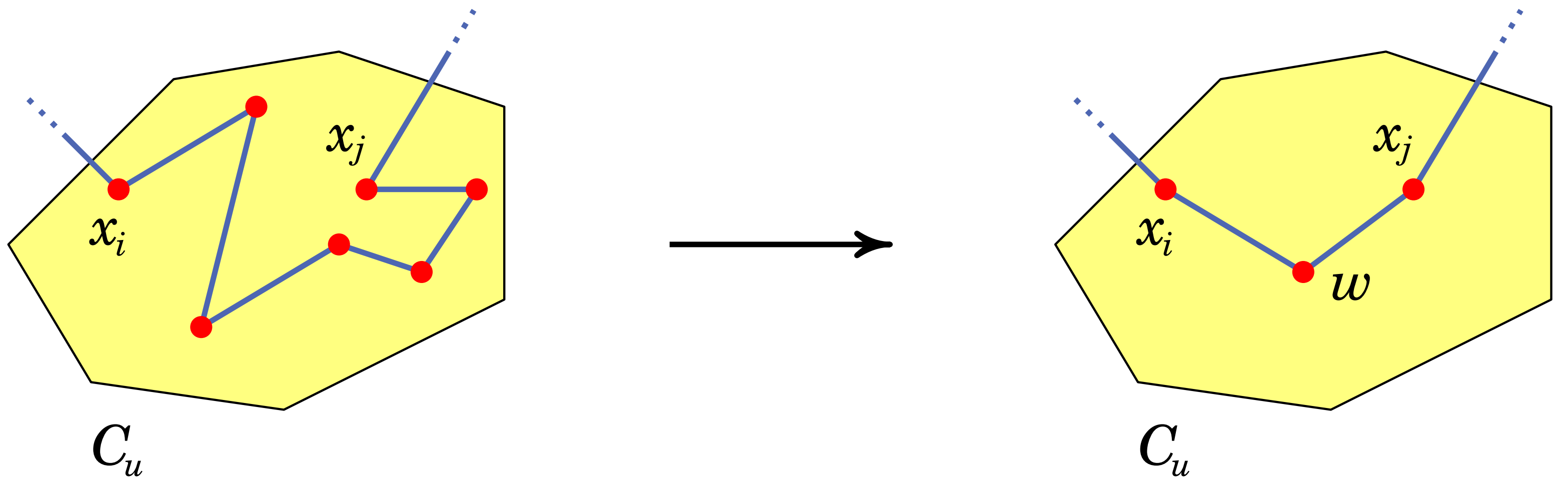



Trouble: $\{x_i, x_j\}$ needn't be anchored: $L(t_i, \frac{1}{2}; t_{i+1}, 0; t_{i+2}, \frac{1}{2})$

# Simplification of counterexamples (cont'd)

What cannot be done in one step can be done in two steps:

Lemma [J.]: If $X \subseteq \mathbb{Q}^m$ is anchored and $x, y \in \mathbb{Q}^m$, there exists $w \in C(X)$ such that $\{x, w\}$ and $\{w, y\}$ are anchored.



# Characterization of admissibility in Ł

Theorem [J.]: Write $t(\Gamma) = \{x \in [0,1]^m : \forall \varphi \in \Gamma\ \varphi(x) = 1\}$ as a union of rational polytopes $\bigcup_{j<r} C_j$.

Then $\Gamma \not\mid\!\sim_{\text{Ł}} \Delta$ iff $\exists a \in \{0,1\}^m\ \forall \psi \in \Delta\ \exists j_0, \dots, j_k < r$ such that

- $a \in C_{j_0}$
- each $C_{j_i}$ is anchored
- $C_{j_i} \cap C_{j_{i+1}} \neq \varnothing$
- $\psi(x) < 1$ for some $x \in C_{j_k}$

Corollary: Admissibility in **Ł** is decidable

# Complexity

Theorem [J.]: If $\Gamma \mathrel{/} \Delta$ in $m$ variables and length $n$ is not **Ł**-admissible, it has a counterexample

$$L(0,x_0;t_1,x_1;\dots;t_{k-1},x_{k-1};1,x_k)\in F_1^m$$

such that

- $k = O(n2^n)$
- $h(x_i) = O(nm)$
- $h(t_i) = O(nmk)$

where $h(x)$, $x \in \mathbb{Q}^m$, denotes the logarithmic height

# Computational complexity

- $\Gamma \not\mathrel{|\!\!\sim}_{\text{Ł}} \Delta$ is reducible to reachability in an exponentially large graph with poly-time edge relation:
  - vertices: anchored polytopes in $t(\Gamma)$
  - edges: $C, C'$ connected iff $C \cap C' \neq \varnothing$

  $\Rightarrow \mathrel{|\!\!\sim}_{\text{Ł}} \in \mathit{PSPACE}$
- $\mathrel{|\!\!\sim}_{\text{Ł}}$ trivially $co\mathit{NP}$-hard:

$$\vdash_{\mathbf{CPC}} \varphi(p_1, \ldots, p_m) \Leftrightarrow p_1 \vee \neg p_1, \ldots, p_m \vee \neg p_m \mathrel{|\!\!\sim}_{\text{Ł}} \varphi$$

  (Aside: both $\mathrm{Th}(\text{Ł})$ and $\vdash_{\text{Ł}}$ are $co\mathit{NP}$-complete [Mundici])
- In fact: $\mathrel{|\!\!\sim}_{\text{Ł}}$ is $\mathit{PSPACE}$-complete (?)

All of this also applies to the universal theory of free $MV$-algebras

# Complexity in context

Examples of known completeness results:

| logic | $\vdash$ | $\mid\!\sim$ |
|---|---|---|
| $\mathbf{CPC, LC, S5}$ | $coNP$ | $coNP$ |
| $\mathbf{GL} + \Box^2\bot$ | $coNP$ | $\Pi^P_3$ |
| **Ł** | $coNP$ | $PSPACE$ |
| $\mathbf{BD_3}, \mathbf{GL} + \Box^3\bot$ | $coNP$ | $coNEXP$ |
| $\mathbf{IPC}_{\to,\bot}$ | $PSPACE$ | $PSPACE$ |
| $\mathbf{IPC, K4, S4, GL}$ | $PSPACE$ | $coNEXP$ |
| $\mathbf{K4_u}$ | $PSPACE$ | $\Pi^0_1$ |
| $\mathbf{K_u}$ | $EXP$ | $\Pi^0_1$ |

# Admissibly saturated formulas

The characterization of $\mathrel{|\!\sim}_{\text{Ł}}$ easily implies:

- $\varphi \in F_m$ is admissibly saturated in **Ł** iff $t(\varphi)$
  - is connected,
  - hits $\{0,1\}^m$, and
  - is piecewise anchored
    (i.e., a finite union of anchored polytopes)
- In **Ł**, every formula $\varphi$ has an admissibly saturated approximation $\Pi_\varphi$:
  - throw out nonanchored polytopes
  - throw out connected components with no lattice point
  - each remaining component gives $\pi \in \Pi_\varphi$

# Strong regularity

A rational polyhedron $P$ is piecewise anchored $\Leftrightarrow$ it has a strongly regular triangulation $\Delta$ (simplicial complex):

- $x \in \mathbb{Q}^m$: $\tilde{x} = \mathrm{den}(x)\langle x, 1\rangle \in \mathbb{Z}^{m+1}$
- simplex $C(x_0, \dots, x_k)$ regular:
  $\tilde{x}_0, \dots, \tilde{x}_k$ included in a basis of $\mathbb{Z}^{m+1}$
- $\Delta$ strongly regular: every maximal $C(x_0, \dots, x_k) \in \Delta$ is regular and $\gcd(\mathrm{den}(x_0), \dots, \mathrm{den}(x_k)) = 1$

Theorem [Cabrer & Mundici]:

$t(\varphi)$ collapsible, hits $\{0,1\}^m$, strongly regular
$\Rightarrow \varphi$ projective
$\Rightarrow t(\varphi)$ contractible, hits $\{0,1\}^m$, strongly regular

# Exact formulas

Theorem [Cabrer]: $\varphi$ exact iff $t(\varphi)$ connected, hits $\{0,1\}^m$, strongly regular

Corollary: The following are equivalent:

- $\varphi$ is admissibly saturated
- $\varphi$ is exact
- $t(\varphi)$ is connected and $\vdash_{\text{Ł}} \varphi \leftrightarrow \bigvee_i \pi_i$ for some projective $\pi_i$

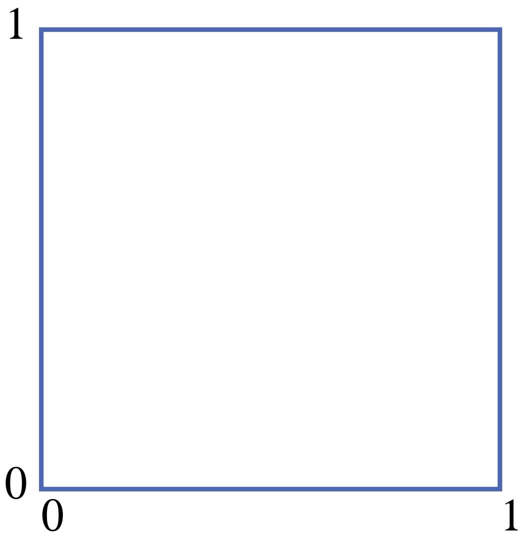
OTOH: some admissibly saturated formulas are not projective

# Projective approximations

Ł has nullary unification type [Marra & Spada]
$\Rightarrow$ it can't have projective approximations
i.e., some admissibly saturated formulas are not projective

Example: $\varphi = p \vee \neg p \vee q \vee \neg q$

- $t(\varphi) = \partial[0,1]^2$
- $\varphi$ is admissibly saturated
- $\pi$ projective
  $\Rightarrow t(\pi)$ retract of $[0,1]^n$
  $\Rightarrow$ contractible
  $\Rightarrow$ simply connected



# Multiple-conclusion basis

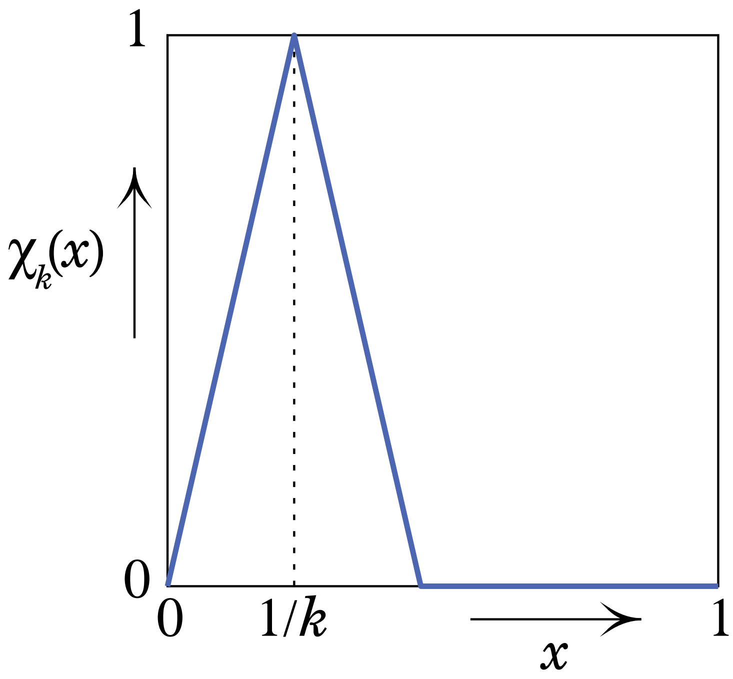
The three steps in the construction of $\Pi_\varphi$ can be simulated by simple rules:

Theorem [J.]: $\{NA_p : p$ is a prime$\} + CC_3 + WDP$ is an independent basis of multiple-conclusion Ł-admissible rules

$$NA_k = \frac{p \vee \chi_k(q)}{p}$$

$$CC_n = \frac{\neg(q \vee \neg q)^n}{}$$

$$WDP = \frac{p \vee \neg p}{p, \neg p}$$



# Conservativity

$\vdash_1$ single-conclusion consequence relation:

Define

$$\Pi \vdash_m \Lambda \quad \text{iff} \quad \forall \Gamma, \varphi, \sigma \, (\forall \psi \in \Lambda \; \Gamma, \sigma(\psi) \vdash_1 \varphi \; \Rightarrow \; \Gamma, \sigma(\Pi) \vdash_1 \varphi)$$

Observation: $\vdash_m$ is the largest multiple-conclusion consequence relation whose s.-c. fragment is $\vdash_1$

Then one can show:
Lemma: If $X$ is a set of s.-c. rules, TFAE:

- **Ł** $+ X + WDP$ is conservative over **Ł** $+ X$
- $\Gamma \,/\, \varphi \in X \; \Rightarrow \; \Gamma \vee \alpha, \neg\alpha \vee \alpha \vdash_{\text{Ł}+X} \varphi \vee \alpha$ for any $\alpha$

# Single-conclusion basis

Theorem [J.]: $\{NA_p : p$ is a prime$\} + RCC_3$ is an independent basis of single-conclusion **Ł**-admissible rules

$$RCC_n = \frac{(q \vee \neg q)^n \to p \quad p \vee \neg p}{p}$$

# Thank you for attention!

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