On Balanced Implications

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Introduction

- We contribute to the theory of implications related by adjointness, in multiple-valued logics.
- Our aim is to investigate implications in (L, P)-valued propositional logic.
- $(L, \leq)$ and $(P, \leq)$ stand for two (complete) lattices, interpreting two, possibly different, types of *truth values*. In most applications, they are the real unit interval under its usual order.
- We expound some motivations behind the use of two lattices L and P in the definition of implications, as a generalization of the usual one-lattice approach.

Implications and their adjoints

An implication, on two lattices P and L, is a function $\Rightarrow: P \times L \to L$ with the following basic intuitive demands:

- $\Rightarrow$ is antitone in the left and isotone in the right argument,
- $\Rightarrow$ satisfies the boundary condition $1 \Rightarrow z = z$,
- $\Rightarrow$ has an adjoint $\supset$ in the left argument, i.e., $\supset: L \times L \to P$ satisfies $\forall a \in P,\ \forall y, z \in L :$

$$\textbf{Adjointness:} \quad y \leq_L a \Rightarrow z \quad \text{iff} \quad a \leq_P y \supset z. \tag{1}$$

The ordered pair $(\Rightarrow, \supset)$ is called an *adjoint pair* on (L, P).

Implications and their adjoints

- The adjoint $\supset$ of $\Rightarrow$ exists iff the implication $\Rightarrow$ satisfies for all indexed families $\{a_j\}$ in P

$$\sup_j a_j \Rightarrow z = \inf_j (a_j \Rightarrow z) , \tag{2}$$

- It is then uniquely given by

$$y \supset z = \sup \{a \in P \mid y \leq_L a \Rightarrow z\} . \tag{3}$$

- The function $\supset: L \times L \to P$ is called a *comparator*.
- The comparator satisfies a condition analogous to (2) and

$$\textbf{Comparator axiom} : \quad y \supset z = 1_P \quad \text{iff} \quad y \leq_L z. \tag{4}$$

Implications and their adjoints

- Numerous articles and books are concerned with residuation (= adjointness) in the multiple-valued-logic component of fuzzy logic.
- On the one hand, the condition Adjointness is a main tool in building a useful calculus for implications; generating universally valid inequalities.
- On the other hand, it is now known that there are different notions of truth; e.g., the "local" concept of truth was adopted by Boldrin and Sossai (1995-1997) where the authors considered the (least informative) possibility distribution where the property is true as the semantical meaning of the property, i.e. its truth value.
- This local notion of truth appears when dealing with incomplete information, as is the case with plausibility-measure based systems. While the usual classical semantics of truth appears when dealing with complete information, as is the case with fuzzy systems.

Implications and their adjoints

- It is also known that the membership values have different semantics (see, e.g., D. Dubois, H. Prade (1997)), which frequently coexist in the same application
- It is natural to request an implication between truth values with same semantics to satisfy the comparator axiom (4).
- It is equally natural to restrict the need for the comparator axiom (4) of implications between truth values of differing semantics.
- Those belong to independently chosen lattices, which may or may not coincide.
- Our approach admits more concrete examples, that would otherwise have been excluded.
- By allowing P to differ from L, we gain some flexibility in its handling.
- The two-posets approach comes at no price at all. An algebraic proof in this framework is an exact replica of the corresponding proof in the one-poset situation.

Faithful implications

Definition

An implication $\Rightarrow$ is said to be *faithful* if it satisfies:

$$(\forall a, b \in P) \ \ (\text{if } a \neq b, \text{ then } (\exists z \in L): \ a \Rightarrow z \neq b \Rightarrow z)\,. \tag{5}$$

Faithfulness is closely related to the following *closure operator* on P:

$$\hat{a} = \inf_{z \in L} ((a \Rightarrow z) \supset z)\,, \quad a \in P. \tag{6}$$

Theorem

Let $(\Rightarrow, \supset)$ be an adjoint pair on complete lattices (L, P). The operator $\hat{}$ becomes id_P if and only if $\Rightarrow$ is faithful.

Lemma

Let $(\Rightarrow, \supset)$ be an adjoint pair on (L, P). If the comparator $\supset$ is surjective, then the implication $\Rightarrow$ is faithful.

Balanced implications

Definition

We say that an adjoint pair $(\Rightarrow, \supset)$ on (L, P) is *balanced* if the following inequality holds for all $x, y, z, w \in L$:

$$x \supset y \leq_P ((x \supset z) \Rightarrow w) \supset ((y \supset z) \Rightarrow w) \tag{7}$$

We also say that the implication $\Rightarrow$ is balanced.

Definition

An adjoint pair $(\Rightarrow, \supset)$ on (L, P) is said to satisfy the *mixed exchange principle* (MEP for short) if there exists an implication-like operation $\rightarrow$ on P (i.e., $\rightarrow$ is antitone in the left argument and isotone in the right argument) such that the following identity holds for all $y, z \in L$ and $a \in P$:

$$y \supset (a \Rightarrow z) = a \rightarrow (y \supset z) \tag{8}$$

We also say that the adjoint pair $(\Rightarrow, \supset)$ with $\rightarrow$ satisfy MEP.

Theorem

Let $(\Rightarrow, \supset)$ be an adjoint pair on (L, P). Then

(i) *There is an implication-like operation $\twoheadrightarrow$ satisfies the following:*

(ii) *For all a, b in P: $b \leq_P a \twoheadrightarrow b$.*

(iii) $1_P \twoheadrightarrow b = b$ *if and only if* $\Rightarrow$ *is faithful.*

(iv) *If* $\Rightarrow$ *is faithful, then* $a \twoheadrightarrow b = 1_P$ *if and only if* $a \leq_P b$.

(v) $\Rightarrow$ *is balanced iff the following MEP holds:* $\forall a \in P$ *and* $y, z \in L$

$$y \supset (a \Rightarrow z) = a \twoheadrightarrow (y \supset z) \tag{9}$$

(vi) *If* $\Rightarrow$ *is balanced then* $\Rightarrow$ *satisfies the exchange principle iff* $\twoheadrightarrow$ *satisfies the following adjointness condition,* $\forall a, b, c \in P$:

$$a \leq_P b \twoheadrightarrow c \text{ iff } b \leq_P a \twoheadrightarrow c \tag{10}$$

(vii) *If* $\Rightarrow$ *is balanced and satisfies the exchange principle, and* $\supset$ *is surjective, then the tuple* $(P, \twoheadrightarrow, 1_P)$ *is a BCK-algebra.*

Balanced implications and the law of importation

Definition

A binary operation $\star$ on P is said to *tie* an implication $\Rightarrow: P \times L \to L$ if the following identity holds:

$$(\forall a, b \in P)\,(\forall z \in L) \quad (((a \star b) \Rightarrow z) = (a \Rightarrow (b \Rightarrow z))), \tag{11}$$

We say that $\Rightarrow$ is *tied*.

- Tiedness extends to multiple-valued logic the following equivalence in classical logic:

$$((X \& Y) \Longrightarrow Z) \equiv (X \Longrightarrow (Y \Longrightarrow Z)), \tag{12}$$

  known as the *law of importation*.
- It holds for several types of implications used in fuzzy logic, among which the *residuated implications* and *S-implications* are two types.

Balanced implications and the law of importation

- There have been many papers, both theoretical and showing usefulness of tied implications in approximate reasoning in the recent past.
- We point out that the study tied implications was started algebrically by Abdel-Hamid and Morsi (2003).
- It was then adopted and formulated syntactically, within the first order logic of tied implications, by El-Zekey (joint work with Morsi, Lotfallah) in FSS (2006).
- However, in this work, we don't request the implication $\Rightarrow$ to have an adjoint $\& : P \times L \to L$ in the right argument; that is,

$$\forall a \in P, \ \forall y, z \in L : y \leq_L a \Rightarrow z \quad \textit{iff} \quad a \& y \leq_L z. \tag{13}$$

- This work is a continuation of the mentioned work.

Balanced implications and the law of importation

Theorem

Let $(\Rightarrow, \supset)$ be an adjoint pair on (L, P) (completeness is not assumed). If $\Rightarrow$ is tied, then it is balanced.

Theorem

Suppose an implication $\Rightarrow$, of an adjoint pair $(\Rightarrow, \supset)$ on (L, P), is tied, faithful (and satisfies the exchange principle), then there exists a partially ordered (commutative) residuated integral monoid $(P, \leq, \otimes, \rightarrow, 1_P)$ on P such that its conjunction $\otimes$ ties $\Rightarrow$ and its residuum $\rightarrow$ satisfies with $(\Rightarrow, \supset)$ the MEP.

- Balance is the strongest necessary condition we could formulate for the tiedness of an implication $\Rightarrow$.
- The following question is well justified: **Is balance equivalent to tiedness?**

Balanced implications and the law of importation

Definition

An implication-like operation with condition (P) (i.e. *with product*), is an implication-like operation $\rightarrow$ on P satisfying the condition (P):

(P) For all $a, b \in P$ there exists

$$a \odot b \overset{notation}{=} \min \{c \in P | a \leq_P b \rightarrow c\} .$$

Proposition

Let $\rightarrow$ be a binary operation on a poset $(P, \leq_P)$. Then the following statments are equivalent:

(i) *The operation $\rightarrow$ is an implication-like operation with condition (P).*

(ii) *There exists an isotone binary operation $\odot$ on P such that the condition (RP) holds, where:*

(RP) *for all a, b, c in $P : a \odot b \leq_P c$ iff $a \leq_P b \rightarrow c$.*

Balanced implications and the law of importation

Theorem

Let $(\Rightarrow, \supset)$ be an adjoint pair on (L, P) (completeness is not assumed) in which the comparator $\supset$ is surjective, and let $\rightarrow$ be an implication-like operation on P. If the adjoint pair $(\Rightarrow, \supset)$ with $\rightarrow$ satisfy the MEP, then the following two statements are equivalent:

- **(i)** *The implication-like operation $\rightarrow$ satisfies the condition (P).*
- **(ii)** *The implication $\Rightarrow$ is tied.*

Balanced implications and the law of importation

Example

Let $P = [0,1]$ and $L = P^W$, where W is a finit set of elements.
Consider the following operations: $\forall a, b \in P$ and $\forall y, z \in L$,

$$a \rightarrow b = \begin{cases} 1 & a \leq b \\ 1 - a + b & 0 < b < a \leq 1 \\ 0 & b = 0, 0 < a \leq 1 \end{cases}$$

$$\begin{aligned} (a \Rightarrow z)(w) &= a \rightarrow z(w) \\ y \supset z &= \inf_{w \in W} (y(w) \rightarrow z(w)) \end{aligned}$$

$(\Rightarrow, \supset)$ is a balanced adjoint pair on (L, P) and satisfies the MEP with $\rightarrow$ but the implication $\Rightarrow$ is not tied.

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THANKS