A General Approach to State-Morphism MV-Algebras

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- J. Łukasiewicz, 1922 many-valued logic

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- $(= p(a \vee b))$ test for a classical system

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- MV-algebras - compatibility

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- state or FAS on an algebraic structure $(M; +, ', 0, 1)$, $s :\ M \to [0,1]$ (i) $s(1) = 1$, (ii) $s(a+b) = s(a) + s(b)$ if $a + b \in M$

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$$s(M) = \sum_i \lambda_i s_{\phi_i}(M) = \mathrm{tr}(TP_M),\ M \in \mathcal{L}(H).$$

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- If s is a FAS $\mathcal{L}(H)$, Aarnes

$$s = \lambda s_1 + (1-\lambda)s_2$$

s_1 is a σ-additive, s_2 a FAS vanishing on each finite-dimensional subspace of H.

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- von Neumann algebra V - extension from FAS from $\mathcal{L}(V)$ to V.

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- extremal state $s = \lambda s_1 + (1-\lambda)s_2$ for $\lambda \in (0,1) \Rightarrow s = s_1 = s_2$.

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- $s \leftrightarrow \mathrm{Ker}(s)$, 1-1 correspondence

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- μ_s - unique Borel σ-additive probability measure on $\mathcal{B}(\mathcal{S}(M))$ such that $\mu_s(\partial_e \mathcal{S}(M)) = 1$

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- Flaminio-Montagna - introduce an algebraizable logic whose equivalent algebraic semantics is the variety of state MV-algebras
- A state MV-algebra is a pair (M, τ), M - MV-algebra, τ unary operation on A s.t.

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- τ -internal operator, state operator

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- s state on M, $[0,1] \otimes M$, $\tau_s(\alpha \otimes a) := \alpha \cdot s(a) \otimes 1$

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- Iff $\tau((n+1)x) = \tau(nx)$

State BL-algebras

- M - BL-algebra. A map $\tau : M \to M$ s.t.

$(1)_{BL}$ $\tau(0) = 0$;

$(2)_{BL}$ $\tau(x \to y) = \tau(x) \to \tau(x \wedge y)$;

$(3)_{BL}$ $\tau(x \odot y) = \tau(x) \odot \tau(x \to (x \odot y))$;

$(4)_{BL}$ $\tau(\tau(x) \odot \tau(y)) = \tau(x) \odot \tau(y)$;

$(5)_{BL}$ $\tau(\tau(x) \to \tau(y)) = \tau(x) \to \tau(y)$

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state-operator on M, pair (M, τ) - *state BL-algebra*

- If $\tau : M \to M$ is a BL-endomorphism s.t. $\tau \circ \tau = \tau$, - *state-morphism operator* and the couple (M, τ) - *state-morphism BL-algebra*.

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- **Example 0.2** Let M be a BL-algebra. On $M \times M$ we define two operators, τ_1 and τ_2, as follows

$$\tau_1(a,b) = (a,a), \quad \tau_2(a,b) = (b,b), \quad (a,b) \in M \times M. \tag{2.0}$$

Then τ_1 and τ_2 are two state-morphism operators on $M \times M$.

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- **Example 0.3** Let M be a BL-algebra. On $M \times M$ we define two operators, τ_1 and τ_2, as follows

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  Then τ_1 and τ_2 are two state-morphism operators on $M \times M$.
- Ker$(\tau) = \{a \in M : \tau(a) = 1\}$.

- We say that two subhoops, A and B, of a BL-algebra M have the *disjunction property* if for all $x \in A$ and $y \in B$, if $x \vee y = 1$, then either $x = 1$ or $y = 1$.

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- **Lemma 0.5** *Suppose that (M, τ) is a state BL-algebra. Then:*

  (1) *If τ is faithful, then (M, τ) is a subdirectly irreducible state BL-algebra if and only if $\tau(M)$ is a subdirectly irreducible BL-algebra.*

  Now let (M, τ) be subdirectly irreducible. Then:

- (2) Ker(τ) is (either trivial or) a subdirectly irreducible hoop.

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- **Theorem 0.7** *Let (M, τ) be a state BL-algebra satisfying conditions* (1), (2) *and* (3) *in the last Lemma. Then (M, τ) is subdirectly irreducible.*

- **Theorem 0.8** *A state-morphism BL-algebra (M, τ) is subdirectly irreducible irreducible if and only if one of the following three possibilities holds.*

- **Theorem 0.9** *A state-morphism BL-algebra (M, τ) is subdirectly irreducible irreducible if and only if one of the following three possibilities holds.*
- (i) M is linear, $\tau = \mathrm{id}_M$, and the BL-reduct M is a subdirectly irreducible BL-algebra.

- **Theorem 0.10** *A state-morphism BL-algebra (M, τ) is subdirectly irreducible irreducible if and only if one of the following three possibilities holds.*
- (i) M is linear, $\tau = \mathrm{id}_M$, and the BL-reduct M is a subdirectly irreducible BL-algebra.
- (ii) The state-morphism operator τ is not faithful, M has no nontrivial Boolean elements, and the BL-reduct M of (M, τ) is a local BL-algebra, $\mathrm{Ker}(\tau)$ is a subdirectly irreducible irreducible hoop, and $\mathrm{Ker}(\tau)$ and $\tau(M)$ have the disjunction property.

- **Theorem 0.11** *A state-morphism BL-algebra (M, τ) is subdirectly irreducible irreducible if and only if one of the following three possibilities holds.*
- (i) M is linear, $\tau = \mathrm{id}_M$, and the BL-reduct M is a subdirectly irreducible BL-algebra.
- (ii) The state-morphism operator τ is not faithful, M has no nontrivial Boolean elements, and the BL-reduct M of (M, τ) is a local BL-algebra, $\mathrm{Ker}(\tau)$ is a subdirectly irreducible irreducible hoop, and $\mathrm{Ker}(\tau)$ and $\tau(M)$ have the disjunction property.

- **Theorem 0.12** *A state-morphism BL-algebra* (M,τ) *is subdirectly irreducible irreducible if and only if one of the following three possibilities holds.*
- (i) M is linear, $\tau = \mathrm{id}_M$, and the BL-reduct M is a subdirectly irreducible BL-algebra.
- (ii) The state-morphism operator τ is not faithful, M has no nontrivial Boolean elements, and the BL-reduct M of (M,τ) is a local BL-algebra, $\mathrm{Ker}(\tau)$ is a subdirectly irreducible irreducible hoop, and $\mathrm{Ker}(\tau)$ and $\tau(M)$ have the disjunction property.

- **Theorem 0.13** *A state-morphism BL-algebra (M, τ) is subdirectly irreducible irreducible if and only if one of the following three possibilities holds.*
- (i) M is linear, $\tau = \mathrm{id}_M$, and the BL-reduct M is a subdirectly irreducible BL-algebra.
- (ii) The state-morphism operator τ is not faithful, M has no nontrivial Boolean elements, and the BL-reduct M of (M, τ) is a local BL-algebra, $\mathrm{Ker}(\tau)$ is a subdirectly irreducible irreducible hoop, and $\mathrm{Ker}(\tau)$ and $\tau(M)$ have the disjunction property.

- Moreover, M is linearly ordered if and only if $\mathrm{Rad}_1(M)$ is linearly ordered, and in such a case, M is a subdirectly irreducible BL-algebra such that if F is the smallest nontrivial state-filter for (M, τ), then F is the smallest nontrivial BL-filter for M.

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- If $\mathrm{Rad}(M) = \mathrm{Ker}(\tau)$, then M is linearly ordered.

- (iii) The state-morphism operator τ is not faithful, M has a nontrivial Boolean element. There are a linearly ordered BL-algebra A, a subdirectly irreducible BL-algebra B, and an injective BL-homomorphism $h: A \to B$ such that (M, τ) is isomorphic as a state-morphism BL-algebra with the state-morphism BL-algebra $(A \times B, \tau_h)$, where $\tau_h(x, y) = (x, h(x))$ for any $(x, y) \in A \times B$.

Varieties of SMMV-algebras

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- $\mathcal{P}_\tau = \mathsf{V}(D(C))$, $\mathcal{P}$ perfect MV-algebras, C-Chang

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$$(\tau(x) \leftrightarrow x)^* \leq (\tau(x) \leftrightarrow x).$$

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- $(A(X), \tau)$ is linearly ordered SMMV-algebra

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- Theorem: Between $\mathcal{MVI}$ and $\mathcal{MVR}$ there is uncountably many varieties

Generators of SMBL-algebras

- *t-norm*- function $t : [0,1] \times [0,1] \to [0,1]$ such that (i) t is commutative, associative, (ii) $t(x,1) = x$, $x \in [0,1]$, and (iii) t is nondecreasing in both components. Moreover, the variety of all BL-algebras is generated by all $\mathbb{I}_t$ with a continuous t-norm t.

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- If t is continuous, we define $x \odot_t y = t(x, y)$ and $x \to_t y = \sup\{z \in [0, 1] : t(z, x) \leq y\}$ for $x, y \in [0, 1]$, then $\mathbb{I}_t := ([0, 1], \min, \max, \odot_t, \to_t, 0, 1)$ is a BL-algebra.

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- Moreover, the variety of all BL-algebras is

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- **Theorem 0.15** *The variety of all state-morphism BL-algebras is generated by the class* $\mathcal{T}$.

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- $\phi \subseteq A^2$, $\Phi(\phi)$, $\Phi_\tau(\phi)$ congruence generated by ϕ on A and (A, τ)
- Lemma: For any $\phi \in \mathrm{Con}\,\tau(\mathbf{A})$, we have $\theta_\phi \in \mathrm{Con}\,(\mathbf{A}, \tau)$, and $\theta_\phi \cap \tau(A)^2 = \phi$. In addition, $\theta_\tau \in \mathrm{Con}\,(\mathbf{A}, \tau)$, $\phi \subseteq \theta_\phi$, and $\Theta_\tau(\phi) \subseteq \theta_\phi$.

- Lemma: Let $\theta \in \mathrm{Con}\,\mathbf{A}$ be such that $\theta \subseteq \theta_\tau$. Then $\theta \in \mathrm{Con}\,(\mathbf{A}, \tau)$ holds.

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- Lemma: Let $\theta \in \mathrm{Con}\,\mathbf{A}$ be such that $\theta \subseteq \theta_\tau$. Then $\theta \in \mathrm{Con}\,(\mathbf{A}, \tau)$ holds.
- Lemma: If $x, y \in \tau(\mathbf{A})$, then $\Theta(x, y) = \Theta_\tau(x, y)$. Consequently, $\Theta(\phi) = \Theta_\tau(\phi)$ whenever $\phi \subseteq \tau(A)^2$.
- if $(\mathbf{C}, \tau \hookrightarrow)(\mathbf{B} \times \mathbf{B}, \tau_B)$, $(\mathbf{C}, \tau)$ is said to be a *subdiagonal* state-morphism algebra

- **Theorem 0.16** *Let* (A, τ) *be a subdirectly irreducible state-morphism algebra such that* A *is subdirectly reducible. Then there is a subdirectly irreducible algebra* B *such that* (A, τ) *is* B*-subdiagonal.*

- **Theorem 0.18** *Let* (A, τ) *be a subdirectly irreducible state-morphism algebra such that* A *is subdirectly reducible. Then there is a subdirectly irreducible algebra* B *such that* (A, τ) *is* B*-subdiagonal.*
- **Theorem 0.19** *For every subdirectly irreducible state-morphism algebra* (A, τ)*, there is a subdirectly irreducible algebra* B *such that* (A, τ) *is* B*-subdiagonal.*

- **Theorem 0.20** *Let* (A, τ) *be a subdirectly irreducible state-morphism algebra such that* A *is subdirectly reducible. Then there is a subdirectly irreducible algebra* B *such that* (A, τ) *is* B*-subdiagonal.*
- **Theorem 0.21** *For every subdirectly irreducible state-morphism algebra* (A, τ)*, there is a subdirectly irreducible algebra* B *such that* (A, τ) *is* B*-subdiagonal.*
- $\mathcal{K}$ of algebras of the same type, I$(\mathcal{K})$, H$(\mathcal{K})$, S$(\mathcal{K})$ and P$(\mathcal{K})$ D$(\mathcal{K})$

- **Theorem 0.22** (1) *For every class* $\mathcal{K}$ *of algebras of the same type* F, $\mathsf{V}(\mathsf{D}(\mathcal{K})) = \mathsf{V}(\mathcal{K})_\tau$.
(2) *Let* $\mathcal{K}_1$ *and* $\mathcal{K}_2$ *be two classes of same type algebras. Then* $\mathsf{V}(D(\mathcal{K}_1)) = \mathsf{V}(D(\mathcal{K}_2))$ *if and only if* $\mathsf{V}(\mathcal{K}_1) = \mathsf{V}(\mathcal{K}_2)$.

- **Theorem 0.24** (1) *For every class* $\mathcal{K}$ *of algebras of the same type* F, $\mathsf{V}(\mathsf{D}(\mathcal{K})) = \mathsf{V}(\mathcal{K})_\tau$.
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- **Theorem 0.25** *If a system* $\mathcal{K}$ *of algebras of the same type* F *generates the whole variety* $\mathcal{V}(F)$ *of all algebras of type* F, *then the variety* $\mathcal{V}(F)_\tau$ *of all state-morphism algebras* $(\mathbf{A}, \tau)$, *where* $\mathbf{A} \in \mathcal{V}(F)$, *is generated by the class* $\{D(\mathbf{A}) : \mathbf{A} \in \mathcal{K}\}$.

- **Theorem 0.26** *If* A *is a subdirectly irreducible algebra, then any state-morphism algebra* (A, τ) *is subdirectly irreducible.*

- **Theorem 0.28** *If* A *is a subdirectly irreducible algebra, then any state-morphism algebra* (A, τ) *is subdirectly irreducible.*
- **Theorem 0.29** *A variety* $\mathcal{V}_\tau$ *satisfy the CEP if and only if* $\mathcal{V}$ *satisfies the CEP.*

Applications

- The variety of all state-morphism MV-algebras is generated by the diagonal state-morphism MV-algebra $D([0,1]_{MV})$.

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- The variety of all state-morphism naBL-algebras is generated by the class $\{D(\mathbb{I}_t^{na}) : \mathbb{I}_t \in na\mathcal{T}\}$.

- If a unital ℓ-group (G, u) is double transitive, then $D(\Gamma(G, u))$ generates the variety of state-morphism pseudo MV-algebras.

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Thank you for your attention