Conversations among Inference Relations

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The method of studying inter-relations between logical systems by the analysis of translations between them was originally introduced by Kolmogorov, in 1925.

The first known ‘translations’ involving classical logic, intuitionistic logic and modal logic were presented by Kolmogorov (1925), Glivenko (1929), Lewis and Langford (1932), Gödel (1933) and Gentzen (1933).

Some of them were developed mainly in order to show the relative consistency of classical logic with respect to intuitionistic logic.
In spite of Kolmogorov, Glivenko, Gödel and Gentzen dealing with inter-relations between the systems studied by them, they are not interested in the meaning of the concept of translation between logics.

Since then, interpretations between logics have been used to different purposes.
PRAWITZ AND MALMNAS

Prawitz and Malmnas (1968) survey these historical papers and this is the first paper in which a general definition for the concept of translation between logical systems is introduced.

Wójcicki (1988) and Epstein (1990) are the first works with a general systematic study on translations between logics.

Both study inter-relations between propositional calculi in terms of translations.
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Definition of Translation

Translation between logics: a manifesto. *Logique et
LOGICS AND TRANSLATIONS

Da Silva, D’ Ottaviano and Sette (1999), explicitly interested in the study of inter-relations between logic systems in general, propose a general definition for the concept of translation between logics, in order to single out what seems to be in fact the essential feature of a logical translation.
Logics are characterized as pairs constituted by a set (ignoring the fact that in general a logic deals with formulas of a language) and a consequence operator, and translations between logics are defined as maps preserving consequence relations.
Definition: A logic \( A \) is a pair \( \langle A, C \rangle \), where the set \( A \) is the domain of \( A \) and \( C \) is a consequence operator in \( A \), that is, \( C: \mathcal{P}(A) \to \mathcal{P}(A) \) is a function that satisfies, for \( X, Y \subseteq A \):

(i) \( X \subseteq C(X) \)

(ii) \( X \subseteq Y \), then \( C(X) \subseteq C(Y) \)

(iii) \( C(C(X)) \subseteq C(X) \)
Definition: A *translation* from a logic $A$ into a logic $B$ is a map

$$t : A \rightarrow B$$

such that, for any $X \subseteq A$

$$t(C_A(X)) \subseteq C_B(t(X)).$$
If $A$ and $B$ are formal languages, with associated syntactic consequence relations $\vdash_{C_A}$ and $\vdash_{C_B}$, respectively, then $t$ is a translation if, and only if, for $\Gamma \cup \{\alpha\} \subseteq \text{Form}(A)$:

$$\Gamma \vdash_{C_A} \alpha \text{ implies } t(\Gamma) \vdash_{C_B} t(\alpha).$$
An initial treatment of a theory of translations between logics is presented by da Silva, D’ Ottaviano and Sette (1999).

An important subclass of translations, the conservative translations, was investigated by Feitosa and D’Ottaviano.
Definition: Let $A$ and $B$ be logics. A \textit{conservative translation} from $A$ into $B$ is a function $t : A \rightarrow B$ such that, for every set $X \cup \{x\} \subseteq A$:

$$x \in C_A(X) \text{ if, and only if, } t(x) \in C_B(t(X))$$


Note that, in terms of consequence relations, \( t : \text{Form}(\mathcal{L}_1) \rightarrow \text{Form}(\mathcal{L}_2) \) is a conservative translation when, for every \( \Gamma \cup \{ \alpha \} \subseteq \text{Form}(\mathcal{L}_1) \):

\[
\Gamma \vdash_{\mathcal{C}_1} \alpha \text{ if, and only if, } t(\Gamma) \vdash_{\mathcal{C}_2} t(\alpha).
\]
Our notion of translation accommodates certain maps that seem to be intuitive examples of translations, such as the identity map from intuitionistic into classical logic and the forgetful map from modal logics into classical logic.

Such cases would be ruled out if the stricter notion of conservative translation were imposed.
In this sense, the more abstract notion and general concept of translation that we have assumed is a genuine advance in the scope of relating logic systems, based upon which further unfoldings can be devised.
Translations in the sense of Prawitz and Malmsås do not coincide with translations in our sense.

Translation in Wójcicki’s sense are particular cases of our conservative translations.

Epstein’s translations are instances of our conservative translations.
Example 1

The identity function $i : \text{IPC} \rightarrow \text{CPC}$, both logics considered in the connectives $\neg$, $\land$, $\lor$, $\rightarrow$, is a translation from IPC into CPC: for every $\Gamma \subseteq \text{Form}(\mathcal{L})$, $C_{\text{IPC}}(\Gamma) \subseteq C_{\text{CPC}}(\Gamma)$.

But $i$ is not a conservative translation: it suffices to observe that

$$p \lor \neg p \not\in C_{\text{IPC}}(\emptyset)$$

while

$$I(p \lor \neg p) = (p \lor \neg p) \in C_{\text{CPC}}(\emptyset).$$
However

\[ i : \text{CPC} \rightarrow \text{IPC} \]

is not a translation
Kolmogorov's, Glivenko's and Gentzen's interpretations are conservative translations from classical into intuitionistic logic.
Both Gödel’s (1933) interpretations are not translations in our sense, even in the propositional level.

Some General Results on Conservative Translations

The next results are relevant to the study of general properties of logic systems from the point of view of translations between them.
Proposition: If $t : L_1 \rightarrow L_2$ is a literal translation relatively to $\neg$ and $L_2$ is $\neg \neg \neg$ consistent, then $L_1$ is $\neg \neg \neg$ consistent.
When $A_1$ and $A_2$ are strongly complete logic systems, the next result corresponds to the compactness of the systems.

**Theorem:** If $A_1$ and $A_2$ are logics with finitary consequence operators, $t : A_1 \rightarrow A_2$ is a conservative translation if, and only if, for every finite $A \cup \{x\} \subseteq A_1$,

$$x \in C_1(A) \text{ is equivalent to } t(x) \in C_2(t(A)).$$
The following theorem supplies a necessary and sufficient condition for a translation between deductive systems being conservative.

**Theorem**: A translation $t : A_1 \rightarrow A_2$ is conservative if, and only if, for every $A \subseteq A_1$,

$$t^{-1}(C_2(t(A))) \subseteq C_1(A).$$
Proposition: There is no translation from a non-vacuum system into a vacuum system.
Theorem: If there is a recursive and conservative translation from a logic system $L_1$ into a decidable logic system $L_2$, then $L_1$ is decidable.

As an easy consequence, there is no recursive conservative translation from first-order logic into CPC.
Proposition: If $L_1$ is a logic system with an axiomatic $\land$ and there is a surjective and conservative translation $t: L_1 \rightarrow L_2$, then $t(\land)$ is an axiomatic for $L_2$.

Conservative translations preserve non-triviality.
Preservation of Deduction Meta-Theorems

Theorem: Conservative translations preserve the Deduction Theorem.
An Important Algebraic Result

By dealing with the Lindenbaum algebraic structures associated to logics, Feitosa and D'Ottaviano obtained a useful method to define conservative translations.
Given a logic $A$, consider the equivalence relation on $A$

\[ x \sim y \overset{\text{def}}{=} C(x) = C(y) \]

and the quotient map

\[ Q : A \rightarrow \frac{A}{\sim} \]
Theorem: Let $A_1$ and $A_2$ be logics, with the domain of $A_2$ being denumerable; and $A_1, Q_1$ and $A_2, Q_2$ respectively. Then there is a conservative translation $t: A_1 \rightarrow A_2$ if, and only if, there is a conservative translation $t^*: A_1/\cong \rightarrow A_2/\sim$.

Moreover, if such $t^*$ exists, then it is injective.
Families of Conservative Translations

Based on the previous results, Feitosa and D Ottaviano, dealing with syntactic results, algebraic semantics and matrix semantics, have introduced conservative translations involving:
- Classical logic
- Intuitionistic logics
- Modal logics
- Lukasiewicz and Post logics
- Paraconsistent logics
- Predicate logics


Conservative Translations from $L_n$ into CPC

D’Ottaviano and Feitosa (2006) present a (non-constructive) proof of the existence of a conservative translation from the finite Lukasiewicz’s logics into CPC.

Conservative Translation from IPC into CPC

If the language of CPC has an infinite and denumerable set of propositional variables then, differently of what has been supposed in the literature, there is a conservative translation from IPC into CPC – our proof is non-constructive.

D’ Ottaviano, I.M.L., Feitosa, H.A. (2007) Is there a translation from intuitionistic logic into classical logic?
Non-Monotonic Logics and Translations


Conservative Translations
Do Not Exist in all Cases

There is no conservative translation from a cumulative non-monotonic logic into a Tarskian logic.

There is no surjective conservative translation from a Tarskian logic into a non-monotonic cumulative logic.
New Dimensions on Translations between Logics

Carnielli, Coniglio and D’Ottaviano (2009) introduce the concept of contextual translations.

Contextual translations are mappings between languages preserving certain meta-properties of the source logics, that are defined in a formal first-order meta-language.

Contextual translations are translations in our general sense, but contextual and conservative translations are independent concepts.
Categories of Logics and Translations
Da Silva, D’Ottaviano and Sette proved that the class of logics and translations between them is a bi-complete category.

Scheer (2002) proved that this bi-complete category of Tarskian logics is a full sub-category of the category of the cumulative non-monotonic logics and translations.

Feitosa and D’Ottaviano proved that the co-complete category of logics and conservative translations between them is a sub-category of the bi-complete category of logics and translations.

The category whose objects are topological spaces and whose morphisms are the continuous functions between them is a full subcategory of the bi-complete category of logics and translations.
THANK YOU!

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