Conversations among Inference Relations

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The method of studying inter-relations between logical systems by the analysis of translations between them was originally introduced by Kolmogorov, in 1925.

Kolmogorov, A.N. (1977) On the principle of excluded middle (1925). In: HEIJENOORT, J. (Ed.) **From Frege to Gödel:** a source book in mathematical logic 1879-1931. Cambridge: Harvard University Press, p. 414-437.

The first known ‘translations’ involving classical logic, intuitionistic logic and modal logic were presented by Kolmogorov (1925), Glivenko (1929), Lewis and Langford (1932), Gödel (1933) and Gentzen (1933).

Some of them were developed mainly in order to show the relative consistency of classical logic with respect to intuitionistic logic.

In spite of Kolmogorov, Glivenko, Gödel and Gentzen dealing with inter-relations between the systems studied by them, they are not interested in the meaning of the concept of translation between logics.

Since then, interpretations between logics have been used to different purposes.

PRAWITZ AND MALMNÄS

Prawitz and Malmnäs (1968) survey these historical papers and this is the first paper in which a general definition for the concept of translation between logical systems is introduced.

Prawitz, D., Malmnäs, P.E. (1968) A survey of some connections between classical, intuitionistic and minimal logic. In: SCHMIDT, H. *et alii*. (Ed.) **Contributions to mathematical logic**. Amsterdam: North-Holland, p. 215-229.

Wójcicki (1988) and Epstein (1990) are the first works with a general systematic study on translations between logics.

Both study inter-relations between propositional calculi in terms of translations.

The ”Campinas CLE - Group“ Definition of Translation

Carnielli, W.A., D′ Ottaviano, I.M.L., Alves, E.H. (1997) Translation between logics: a manifesto. *Logique et Analyse*, v. 40, p. 67-81.

LOGICS AND TRANSLATIONS

Da Silva, D′ Ottaviano and Sette (1999), explicitly interested in the study of inter-relations between logic systems in general, propose a general definition for the concept of translation between logics, in order to single out what seems to be in fact the essential feature of a logical translation.

Logics are characterized as pairs constituted by a set (ignoring the fact that in general a logic deals with formulas of a language) and a consequence operator, and translations between logics are defined as maps preserving consequence relations.

Definition: A *logic* **A** is a pair $\langle A, C \rangle$, where the set A is the *domain* of **A** and C is a *consequence operator* in A, that is, $C: \mathscr{P}(A) \to \mathscr{P}(A)$ is a function that satisfies, for $X, Y \subseteq A$:

(i) $X \subseteq C(X)$

(ii) $X \subseteq Y$, then $C(X) \subseteq C(Y)$

(iii) $C(C(X)) \subseteq C(X)$

Definition: A *translation* from a logic A into a logic B is a map

$$t : A \to B$$

such that, for any $X \subseteq A$

$$t(C_A(X)) \subseteq C_B(t(X)).$$

If A and B are formal languages, with associated syntactic consequence relations $\vdash_{C_A}$ and $\vdash_{C_B}$, respectively, then *t* is a *translation* if, and only if, for $\Gamma \cup \{\alpha\} \subseteq$ Form(A):

$$\Gamma \vdash_{C_A} \alpha \text{ implies } t(\Gamma) \vdash_{C_B} t(\alpha).$$

An initial treatment of a theory of translations between logics is presented by da Silva, D′ Ottaviano and Sette (1999).

da Silva, J.J., D′ Ottaviano, I.M.L., Sette, A.M. (1999) Translations between logics. In: CAICEDO, X., MONTENEGRO, C.H. (Eds.) **Models, algebras and proofs.** New York: Marcel Dekker, p. 435-448. (Lectures Notes in Pure and Applied Mathematics, v. 203)

An important subclass of translations, the conservative translations, was investigated by Feitosa and D′Ottaviano.

Definition: Let **A** and **B** be logics. A *conservative translation* from **A** into **B** is a function $t: A \to B$ such that, for every set $X \cup \{x\} \subseteq A$:

$$x \in C_A(X) \text{ if, and only if, } t(x) \in C_B(t(X))$$

Feitosa, H.A., D′ Ottaviano, I.M.L. (2001) Conservative translations. *Annals of Pure and Applied Logic*, Amsterdam, v. 108, p. 205-227.

D′ Ottaviano, I.M.L., Feitosa, H.A. (1999) Conservative translations and model-theoretic translations – Revista Internacional de Filosofia, v XXII, n.2, p. 117-132.

Feitosa, H.A. (1997) **Traduções conservativas** (Conservative translations). Doctorate Thesis. Campinas: Institute of Philosophy and the Human Sciences, State University of Campinas.

Note that, in terms of consequence relations, t :**Form**(L_1) $\rightarrow$ **Form**(L_2) is a conservative translation when, for every $\Gamma \cup \{\alpha\} \subseteq$ **Form** (L_1):

$$\Gamma \vdash_{C_1} \alpha \text{ if, and only if, } t(\Gamma) \vdash_{C_2} t(\alpha).$$

Our notion of translation accommodates certain maps that seem to be intuitive examples of translations, such as the identity map from intuitionistic into classical logic and the forgetful map from modal logics into classical logic.

Such cases would be ruled out if the stricter notion of conservative translation were imposed.

In this sense, the more abstract notion and general concept of translation that we have assumed is a genuine advance in the scope of relating logic systems, based upon which further unfoldings can be devised.

Translations in the sense of Prawitz and Malmnäs do not coincide with translations in our sense.

Translation in Wójcicki s sense are particular cases of our conservative translations.

Epstein's translations are instances of our conservative translations.

Example 1

The identity function i : IPC $\rightarrow$ CPC, both logics considered in the connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, is a translation from IPC into CPC: for every $\Gamma \subseteq \mathbf{Form}(L)$, $C_{\mathrm{IPC}}(\Gamma) \subseteq C_{\mathrm{CPC}}(\Gamma)$.

But i is not a conservative translation: it suffices to observe that

$$p \vee \neg p \notin C_{\mathrm{IPC}}(\varnothing)$$

while

$$I(p \vee \neg p) = (p \vee \neg p) \in C_{\mathrm{CPC}}(\varnothing).$$

However

$$i : \mathbf{CPC} \to \mathbf{IPC}$$

is not a translation

Kolmogorov s, Glivenko s and Gentzen s interpretations are conservative translations from classical into intuitionistic logic.

Both Gödel s (1933) interpretations are not translations in our sense, even in the propositional level.

D'Ottaviano, I.M.L., Feitosa, H.A. (2011) On Gödel's modal interpretation of intuitionistic logic. **Anthology of Universal Logic**: from Paul Hertz to Dov Gabbay. Springer Basel: *Studies in Universal Logic*.

Some General Results on Conservative Translations

The next results are relevant to the study of general properties of logic systems from the point of view of translations between them.

Proposition: If $t : L_1 \rightarrow L_2$ is a literal translation relatively to $\neg$ and L_2 is $\neg$ – consistent, then L_1 is $\neg$– consistent.

When A_1 and A_2 are strongly complete logic systems, the next result corresponds to the compactness of the systems.

Theorem: If A_1 and A_2 are logics with finitary consequence operators, $t: A_1 \to A_2$ is a conservative translation if, and only if, for every finite $A \cup \{x\} \subseteq A_1$,

$$x \in C_1(A) \text{ is equivalent to } t(x) \in C_2(t(A)).$$

The following theorem supplies a necessary and sufficient condition for a translation between deductive systems being conservative.

Theorem*: A translation $t : \mathcal{A}_1 \to \mathcal{A}_2$ is conservative if, and only if, for every $A \subseteq A_1$,

$$t^{-1}(C_2(t(A))) \subseteq C_1(A).$$

Proposition: There is no translation from a non-vacuum system into a vacuum system.

Theorem: If there is a recursive and conservative translation from a logic system L_1 into a decidable logic system L_2, then L_1 is decidable.

As an easy consequence, there is no recursive conservative translation from first-order logic into CPC.

Proposition: If L_1 is a logic system with an axiomatic Λ and there is a surjective and conservative translation $t: L_1 \rightarrow L_2$, then $t(\Lambda)$ is an axiomatic for L_2.

Conservative translations preserve non-triviality.

Preservation of Deduction Meta-Theorems

Theorem: **Conservative translations** preserve the Deduction Theorem.

An Important Algebraic Result

By dealing with the Lindenbaum algebraic structures associated to logics, Feitosa and D Ottaviano obtained a useful method to define conservative translations.

Given a logic A, consider the equivalence relation on A

$$x \sim y =_{\text{def}} C(x) = C(y)$$

and the quotient map

$$Q: A \to A/_{\sim}$$

Theorem: Let A_1 and A_2 be logics, with the domain of A_2 being denumerable; and let $A_{1/\sim_1}$ and $A_{2/\sim_2}$ be the logics co-induced by A_1, Q_1 and A_2, Q_2 respectively. Then there is a conservative translation $t: A_1 \to A_2$ if, and only if, there is a conservative translation $t^*: A_{1/\sim_1} \to A_{2/\sim_2}$.

Moreover, if such t^* exists, then it is injective.

Families of Conservative Translations

Based on the previous results, Feitosa and D Ottaviano, dealing with syntactic results, algebraic semantics and matrix semantics, have introduced conservative translations involving:

- Classical logic
- Intuitionistic logics
- A Modal logics
- Lukasiewicz and Post logics
- Paraconsistent logics
- Predicate logics

D′ Ottaviano, I.M.L., Feitosa, H.A. (1999) Many-valued logics and translations. *Journal of Applied Non-Classical Logics*, v. 9, n.1, p. 121-140.

D′ Ottaviano, I.M.L., Feitosa, H.A. (2000) Paraconsistent logics and translations. *Synthèse*, Dordrecht, v. 125, n. 1-2, p. 77-95.

D′ Ottaviano, I.M.L., Feitosa, H.A. (2007) Deductive systems and translations. In: Béziau, J-Y, Costa-Leite, A. (Org.) Perspectives on Universal Logic. Itália: Polimétrica Internationl Scientific Publisher, p. 125-157.

Conservative Translations from L_n into CPC

D Ottaviano and Feitosa (2006) present a (non-constructive) proof of the existence of a conservative translation from the finite Lukasiewicz s logics into CPC.

D′ Ottaviano, I.M.L., Feitosa, H.A. (2006) Is there a translation from Lukasiewicz logics into classical logic? Poznan Studies in Philosophy of Sciences and the Humanities. Amsterdam/New York, vol. 91, p. 157-168.

Conservative Translation from IPC into CPC

If the language of CPC has an infinite and denumerable set of propositional variables then, differentely of what has been supposed in the literature, there is a conservative translation from IPC into CPC – our proof is non-constructive.

D′Ottaviano, I.M.L., Feitosa, H.A. (2007) Is there a translation from intuitionistic logic into classical logic?

Non-Monotonic Logics and Translations

Scheer (2002) initiates the study of conservative translations involving cumulative non-monotonic logics.

Scheer, M.C. (2002) **Para uma teoria de traduções entre lógicas cumulativas** (Towards a theory of translations between cumulative logics). Master Dissertation. Campinas: Institute of Philosophy and the Human Sciences. State University of Campinas.

Scheer, M.C., D′Ottaviano, I.M.L.(2005) Operadores de conseqüência cumulativos e traduções entre lógicas cumulativas. Revista Informação e Cognição, v. 4, p. 47-60.

Conservative Translations Do Not Exist in all Cases

There is no conservative translation from a cumulative non-monotonic logic into a Tarskian logic.

There is no surjective conservative translation from a Tarskian logic into a non-monotonic cumulatve logic.

New Dimensions on Translations between Logics

Carnielli, Coniglio and D'Ottaviano (2009) introduce the concept of *contextual translations*.

Carnielli, W.A., Coniglio, M.E., D'Ottaviano, I.M.L. (2009) New dimensions on translations between logic. *Logica Universalis,* v.3, p.1-19.

Contextual translations are mappings between languages preserving certain meta-properties of the source logics, that are defined in a formal first-order meta-language.

Contextual translations are translations in our general sense, but contextual and conservative translations are independent concepts.

Categories of Logics and Translations

Da Silva, D Ottaviano and Sette proved that the class of logics and translations between them is a bi-complete category.

Da Silva, J.J., D′ Ottaviano, I.M.L., Sette, A.M. (1999) Translations between logics. In: CAICEDO, X., MONTENEGRO, C.H. (Eds.) **Models, algebras and proofs.** New York: Marcel Dekker, p. 435-448. (*Lectures Notes in Pure and Applied Mathematics*, v. 203)

Scheer (2002) proved that this bi-complete category of Tarskian logics is a full sub-category of the category of the cumulative non-monotonic logics and translations.

Scheer, M.C., D′ Ottaviano, I.M.L.(2005) Operadores de conseqüência cumulativos e traduções entre lógicas cumulativas. Revista Informação e Cognição, v. 4, p. 47-60.

Feitosa and D Ottaviano proved that the co-complete category of logics and conservative translations between them is a sub-category of the bi-complete category of logics and translations.

Feitosa, H.A., D′ Ottaviano, I.M.L. (2001) Conservative translations. *Annals of Pure and Applied Logic*, Amsterdam, v. 108, p. 205-227.

The category whose objects are topological spaces and whose morphisms are the continuous functions between them is a full sub-category of the bi-complete category of logics and translations.

THANK YOU!

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