L-relations and Galois triangles

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May 20, 2011

Preliminaries
Galois connections
Extended-order algebras
Basic notions
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Symmetry
Commutativity and associativity
L-relations
Relation algebras
MV-relation algebras
Dedekind categories
Extended-order algebras and L-relations
L-Galois triangles
Weak L-Galois triangle
Symmetrical L-Galois triangle
Strong L-Galois triangle

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- M.E.D.S., C. Guido: *Associativity, commutativity and symmetry in residuated structures*, (submitted).

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- (o_4) $a \rightarrow b = \top$ *and* $b \rightarrow c = \top \Rightarrow a \rightarrow c = \top$ *(weak transitivity condition).*

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- $(L, \to, \top)$ *is a cdeo algebra;*
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- $\top \otimes a = a$;
- $(\bigvee B) \otimes a = \bigvee (B \otimes a)$.

Remark

- *The adjoint product* $\tilde{\otimes}$ *of the cdeo algebra* $(L, \rightsquigarrow, \top)$ *is the opposite* $\otimes^{op}$ *of* $\otimes$, *i. e.* $a \tilde{\otimes} b = b \otimes a$.

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- *The cdeo algebras need not to be symmetrical. In fact in the cdeo algebras* $\top \otimes a \neq a$, *in general.*
- *Symmetrical cdeo algebras are complete integral residuated lattices, hence equivalent to pseudo BCK-algebras, without associativity.*

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A w-ceo algebra $(L, \rightarrow, \top)$ *is commutative iff*
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- *$(L, \to, \top)$ is commutative;*
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- *$(L, \to, \top)$ is symmetrical and $\rightsquigarrow$ coincides with $\to$ (and, of course, $\tilde{\otimes} = \otimes$).*

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B. Jónsson, A. Tarski: *Representation problems for relation algebras*, Bull. Amer. Math. Soc. **54** (1948), 79-80.

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Definition

A (classical) relation algebra is a structure $(A, \vee, \wedge, ^{-}, 0, 1, \circ, ^{\smile}, \Delta)$ *of type* $(2, 2, 1, 0, 0, 2, 1, 0)$ *such that:*

- $(A, \vee, \wedge, ^{-}, 0, 1)$ *is a Boolean algebra;*
- $(A, \circ, \Delta)$ *is a monoid;*
- $(x \circ y) \wedge z = 0 \Leftrightarrow (x^{\smile} \circ z) \wedge y = 0 \Leftrightarrow (z \circ y^{\smile}) \wedge x = 0$ *(cycle law).*

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Example

Rel(X)*, the algebra of classical binary relations on a set* X*, is the standard example of relation algebra.*

A. Popescu: *Many-valued relation algebras*, Algebra Universalis **53** (2005), 73-108.

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An MV-relation algebra is a structure $(A, \oplus, \odot, ^{-}, 0, 1, \circ, ^{\smile}, \Delta)$ *of type* $(2, 2, 1, 0, 0, 2, 1, 0)$ *such that:*

- $(A, \oplus, \odot, ^{-}, 0, 1)$ *is an MV-algebra;*
- $(A, \circ, \Delta)$ *is a monoid;*
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Example

$MVRel(X) = ([0,1]^{X \times X}, \oplus, \odot, ^{-}, 0, 1, \circ, ^{\smile}, \Delta)$ *is the classical algebra of binary* $[0,1]$*-relations on a set* X *where:*

- *0 and 1 are the constant relations;*
- $\oplus$, $\odot$ *and* $^{-}$ *are the pointwise operation from* $([0,1], \oplus, \odot, ^{-})$;
- $\mathcal{R} \circ \mathcal{S}(x,y) = \bigvee_{z \in X} \mathcal{R}(x,z) \odot \mathcal{S}(z,y)$;
- $\mathcal{R}^{\smile}(x,y) = \mathcal{R}(y,x)$;
- $\Delta(x,y) = 1$ *if* $x = y$ *and* $\Delta(x,y) = 0$ *if* $x \neq y$.

H. Furusawa, Y. Kawahara, M. Winter: *Dedekind Categories with Cutoff Operators*, Fuzzy Sets Syst. (article in press).

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Definition

A Dedekind category $\mathcal{D}$ is a category with composition $\cdot$ such that:

- $\mathcal{D}(X,Y) = (\mathcal{D}(X,Y), \subseteq, \cup, \cap, \Rightarrow, 0_{XY}, \nabla_{XY})$ *is an Heyting algebra,* $\forall X, Y \in Obj(\mathcal{D})$*,where:*
  - $\alpha \subseteq \beta$ *iff* $\alpha = \alpha \cap \beta$ *iff* $\beta = \alpha \cup \beta$*;*
  - $\alpha \Rightarrow \beta$ *is the relative pseudo-complement of* α *relative to* β *i.e.* $\gamma \subseteq \alpha \Rightarrow \beta$ *iff* $\alpha \cap \gamma \subseteq \beta$*;*
  - 0_{XY} *e* ∇_{XY} *are the least and the greatest element.*
- *There exists a converse operation* $\# : \mathcal{D}(X,Y) \to \mathcal{D}(Y,X)$ *such that* $\forall \alpha, \alpha' : X \rightharpoonup Y$, $\forall \beta : Y \rightharpoonup Z$*:*
  - $(\alpha \cdot \beta)^{\#} = \beta^{\#} \cdot \alpha^{\#}$*;*
  - $(\alpha^{\#})^{\#} = \alpha$*;*
  - *if* $\alpha \subseteq \alpha'$*, then* $\alpha^{\#} \subseteq \alpha'^{\#}$*.*

- $\forall \alpha : X \rightharpoonup Y, \beta : Y \rightharpoonup Z, \gamma : X \rightharpoonup Z$:
  $\alpha \cdot \beta \cap \gamma \subseteq \alpha \cdot (\beta \cap \alpha^{\#} \cdot \gamma)$. *(modular law)*
- $\forall \alpha : X \rightharpoonup Y, \beta : Y \rightharpoonup Z$ *the residual composition* $\alpha \ominus \beta : X \rightharpoonup Z$ *is a morphism such that* $\forall \delta : X \rightharpoonup Z$, $\delta \subseteq \alpha \ominus \beta$ *iff* $\alpha^{\#} \cdot \delta \subseteq \beta$.

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Example

$Rel(L)(X,Y) = (Rel(L)(X,Y), \subseteq, \cup, \cap, \Rightarrow, 0_{XY}, \nabla_{XY})$ is the Dedekind category of binary heterogeneous L-relations taking value in an Heyting algebra $(L, \wedge, \vee, 0, 1, \rightarrow)$, where:

- $\subseteq$, $\cup$, $\cap$ *and* $\Rightarrow$ *are pointwise induced by* $\vee$, $\wedge$, $\leq$ *and* $\rightarrow$ *of* L;
- $\mathcal{R}^{\#}(y,x) = \mathcal{R}(x,y)$;
- $\mathcal{R} \cdot \mathcal{S}(x,z) = \bigvee_{y \in Y} \mathcal{R}(x,y) \wedge \mathcal{S}(y,z)$;
- $0_{XY} = 0$ *and* $\nabla_{XY} = 1$;
- $\mathcal{R} \ominus \mathcal{S}(x,z) = \bigwedge_{y \in Y} \mathcal{R}(x,y) \rightarrow \mathcal{S}(y,z)$.

Let $L = (L, \to, \top)$ be a *w-ceo* algebra and consider $\forall X, Y \in |\mathbf{Set}|$ the set $\mathbf{R}(L)(X, Y)$ of L-relations $\mathcal{R} : X \times Y \to L \equiv X \rightharpoondown Y$.

What about $\mathbf{R}(L)(X, Y)$?

Definition

- $\perp\!\!\!\perp_{XY}: X \rightharpoonup Y \ : \ \perp\!\!\!\perp_{XY}(x,y) = \perp;$

Definition

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- $\mathcal{I}_X : X \rightharpoonup X \; : \; \mathcal{I}_X(x,x) = \top$ *and* $\mathcal{I}_X(x,x') = \bot, \; \forall x \neq x' \in X;$

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- $\mathcal{R} \leq \mathcal{R}' \Leftrightarrow \mathcal{R}(x,y) \leq \mathcal{R}'(x,y);$
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Consider

$$\mathbf{R}(L)(X,Y) = (\mathbf{R}(L)(X,Y), \to, \top\!\!\!\top_{XY}).$$

Proposition

- *If L is a w-ceo algebra, then* $\mathbf{R}(L)(X, Y)$ *is a w-ceo algebra;*
- *If L is a ceo algebra, then* $\mathbf{R}(L)(X, Y)$ *is a ceo algebra;*
- *If L is a right/left-distributive w-ceo algebra, then* $\mathbf{R}(L)(X, Y)$ *is a right/left-distributive w-ceo algebra;*
- *If L is a cdeo algebra, then* $\mathbf{R}(L)(X, Y)$ *is a cdeo algebra;*
- *If L is symmetrical, then* $\mathbf{R}(L)(X, Y)$ *is symmetrical;*
- *If L is commutative, then* $\mathbf{R}(L)(X, Y)$ *is commutative;*
- *If L is associative, then* $\mathbf{R}(L)(X, Y)$ *is associative.*

R.Bělohlávek: *Fuzzy Relational Systems: Foundations and Principles*, Kluwer Academic, Plenum Press, Dordrecht, NewYork, 2002.
L. Běhounek, M. Daňková: *Relational compositions in Fuzzy Class Theory*, Fuzzy Sets Syst. **160** (2008), 1005-1036.

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- $\mathcal{R} \wedge_{\leftarrow} \mathcal{S} : X \rightharpoonup Z$: $\mathcal{R} \wedge_{\leftarrow} \mathcal{S}(x, z) = \bigwedge_{y \in Y} \mathcal{S}(y, z) \to \mathcal{R}(x, y)$;

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- $\mathcal{R}_{-} : Y \rightharpoonup X$: $\mathcal{R}_{-}(y, x) = \mathcal{R}(x, y)$;
- $\mathcal{R} \otimes \mathcal{R}' : X \rightharpoonup Y$: $\mathcal{R} \otimes \mathcal{R}'(x, y) = \mathcal{R}(x, y) \otimes \mathcal{R}'(x, y)$;
- $\mathcal{R} \wedge_{\rightarrow} \mathcal{S} : X \rightharpoonup Z$: $\mathcal{R} \wedge_{\rightarrow} \mathcal{S}(x, z) = \bigwedge_{y \in Y} \mathcal{R}(x, y) \rightarrow \mathcal{S}(y, z)$;
- $\mathcal{R} \wedge_{\leftarrow} \mathcal{S} : X \rightharpoonup Z$: $\mathcal{R} \wedge_{\leftarrow} \mathcal{S}(x, z) = \bigwedge_{y \in Y} \mathcal{S}(y, z) \rightarrow \mathcal{R}(x, y)$;
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L. Běhounek, M. Daňková: *Relational compositions in Fuzzy Class Theory*, Fuzzy Sets Syst. **160** (2008), 1005-1036.

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$\mathcal{R} \bigwedge_{\leftarrow} \mathcal{S} = (\mathcal{S}_{-} \bigwedge_{\rightarrow} \mathcal{R}_{-})_{-}$.

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- $\mathcal{T} \leq \mathcal{R} \wedge_{\rightarrow} \mathcal{S} \Leftrightarrow \mathcal{R}_{-} \vee_{\otimes} \mathcal{T} \leq \mathcal{S}$;*(right-residual composition)*
- *if* $(\mathcal{R} \vee_{\otimes} \mathcal{S}) \otimes \mathcal{T} = \top_{XZ}$, *then* $\mathcal{R} \vee_{\otimes} (\mathcal{S} \otimes \mathcal{R}_{-} \vee_{\otimes} \mathcal{T}) = \top_{XZ}$; *(weak modular law)*

Definition

Let $(L, \to, \top)$ *be a symmetrical cdeo algebra.*

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- $\mathcal{T} \leq \mathcal{R} \bigwedge_{\rightsquigarrow} \mathcal{S} \Leftrightarrow \mathcal{R}_{-} \bigvee_{\tilde{\otimes}} \mathcal{T} \leq \mathcal{S}$ *(left-residual composition);*

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- *if* $(\mathcal{R} \bigvee_{\tilde{\otimes}} \mathcal{S}) \tilde{\otimes} \mathcal{T} = \top\!\!\top_{XZ}$, *then* $\mathcal{R} \bigvee_{\tilde{\otimes}} (\mathcal{S} \tilde{\otimes} \mathcal{R}_{-} \bigvee_{\tilde{\otimes}} \mathcal{T}) = \top\!\!\top_{XZ}$; *(weak modular law for* $\tilde{\otimes}$*)*

If moreover L is associative, then the followings hold:

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- $(\mathcal{R} \bigvee_{\otimes} \mathcal{S}) \otimes \mathcal{T} = \perp\!\!\!\perp_{XZ} \Leftrightarrow (\mathcal{T} \bigvee \tilde{\otimes} \mathcal{S}_{-}) \tilde{\otimes} \mathcal{R} = \perp\!\!\!\perp_{XY}$.(one equivalence of cycle law)

We need to extend the notion of category and hence we propose the following definition.

Definition

A pseudo bi-category $\mathcal{C} = (\mathcal{O}(\mathcal{C}), \mathcal{M}(\mathcal{C}), \circ, \tilde{\circ}, \mathcal{I}, \tilde{\mathcal{I}})$ *consists in a class of objects, a class of morphisms, two partial composition operations and two families of identities (in the usual sense) where we require only:*

- *right neutrality of the identities* $\mathcal{I}_X$ *with respect to* $\circ$*;*
- *left neutrality of the identities* $\tilde{\mathcal{I}}_X$ *with respect to* $\tilde{\circ}$*;*

Note that the associativity of the compositions is not required.

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A pseudo bi-category $\mathcal{C}$ *is called bi-category if the two partial compositions are associative.*

Definition

We say that a pseudo bi-category is symmetrical if $\forall X \in \mathcal{O}(\mathcal{C})$ $\mathcal{I}_X = \tilde{\mathcal{I}}_X$ *and there exists the opposite operation* $[\cdot]_- : \mathcal{M}(\mathcal{C}) \to \mathcal{M}(\mathcal{C})$, $\alpha \in \mathcal{C}(X,Y) \mapsto \alpha_- \in \mathcal{C}(Y,X)$ *such that* $\forall \alpha, \beta \in \mathcal{M}(\mathcal{C})$*:*

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What about $\mathbf{R}(L)$?

Definition

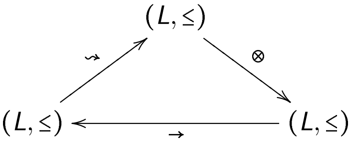
We say that a pseudo bi-category is symmetrical if $\forall X \in \mathcal{O}(\mathcal{C})$ $\mathcal{I}_X = \tilde{\mathcal{I}}_X$ *and there exists the opposite operation* $[\cdot]_- : \mathcal{M}(\mathcal{C}) \to \mathcal{M}(\mathcal{C})$, $\alpha \in \mathcal{C}(X,Y) \mapsto \alpha_- \in \mathcal{C}(Y,X)$ *such that* $\forall \alpha, \beta \in \mathcal{M}(\mathcal{C})$*:*

$$(\alpha \tilde{\circ} \beta)_- = \beta_- \circ \alpha_-.$$

What about $\mathbf{R}(L)$?

Hence, if $(L, \to, \top)$ is a symmetrical cdeo algebra, then the class $\mathbf{R}(L)$ of all L-binary relations between sets with the two compositions $\bigvee_{\otimes}$ and $\bigvee_{\tilde{\otimes}}$ is a symmetrical pseudo bi-category that is a weak and non-commutative generalization of a Dedekind category. Moreover, if $(L, \to, \top)$ is a symmetrical and associative cdeo algebra, $\mathbf{R}(L)$ becomes, of course,a bi-category.

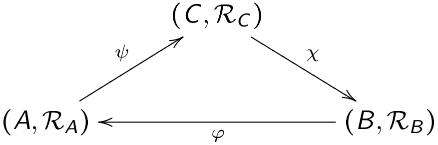
Looking at $\leq$ in a symmetrical cdeo algebra as a (crisp) L-relation and at the operations $\rightarrow$, $\otimes$ and $\rightsquigarrow$ as L-relations, one has:



$$\leq(a, c \rightsquigarrow b) = \top \Leftrightarrow \leq(a \otimes c, b) = \top \Leftrightarrow \leq(c, a \rightarrow b) = \top$$

Definition

Let $(A, \mathcal{R}_A)$, $(B, \mathcal{R}_B)$ and $(C, \mathcal{R}_C)$ be sets equipped with fixed L-relations, where $L = (L, \rightarrow, \top)$ is a symmetrical cdeo algebra. The diagram

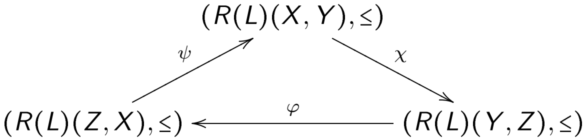


with $\varphi \in \mathbf{R}(C)(A, B)$, i.e. φ is a C-valued binary relation from A to B, $\psi \in \mathbf{R}(B)(A, C)$ and $\chi \in \mathbf{R}(A)(B, C)$ is a weak L-Galois triangle if for all $a \in A, b \in B, c \in C$ the following equivalences hold:

$$\mathcal{R}_A(a, \chi(b, c)) = \top \Leftrightarrow \mathcal{R}_B(\psi(a, c), b) = \top \Leftrightarrow \mathcal{R}_C(c, \varphi(a, b)) = \top.$$

Proposition

Let $L = (L, \to, \top)$ be a symmetrical cdeo algebra. Then for any triple of sets (X, Y, Z) one has a weak L-Galois triangle



where
$\varphi \in \mathbf{R}(\mathbf{R}(L)(X,Y))(\mathbf{R}(Z,X),\mathbf{R}(Y,Z))$,
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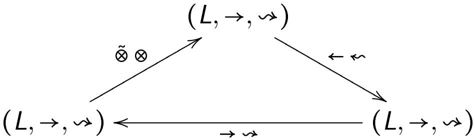
are defined, $\forall \alpha \in \mathbf{R}(L)(Z,X), \beta \in \mathbf{R}(L)(Y,Z), \gamma \in \mathbf{R}(L)(X,Y)$ by:

- $\varphi(\alpha,\beta) = (\beta \bigwedge_{\leftarrow} \alpha)_-$;
- $\psi(\alpha,\gamma) = (\alpha \bigvee_{\otimes} \gamma)_-$;
- $\chi(\beta,\gamma) = (\gamma \bigwedge_{\rightsquigarrow} \beta)_-$.

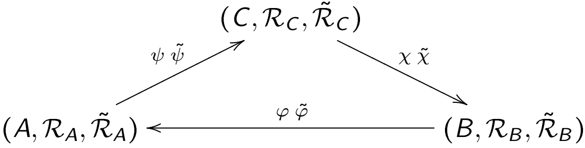
If the symmetrical cdeo algebra $(L, \rightarrow, \top)$ is associative, one has stronger conditions on the triangle



- $a \rightarrow (b \leftarrow c) = (a \tilde{\otimes} c) \rightarrow b$;
- $a \rightsquigarrow (b \leftsquigarrow c) = (a \otimes c) \rightsquigarrow b$;
- $a \rightarrow (b \leftsquigarrow c) = c \rightsquigarrow (a \rightarrow b)$;
- $a \rightsquigarrow (b \leftarrow b) = c \rightarrow (a \rightsquigarrow b)$.

Definition

Let $(A, \mathcal{R}_A, \tilde{\mathcal{R}}_A)$, $(B, \mathcal{R}_B, \tilde{\mathcal{R}}_B)$ and $(C, \mathcal{R}_C, \tilde{\mathcal{R}}_C)$ be the sets equipped with two fixed L-relations, where $L = (L, \rightarrow, \top)$ is a symmetrical cdeo algebra. The diagram



with
$\varphi, \tilde{\varphi} \in \mathbf{R}(C)(A, B)$, $\psi, \tilde{\psi} \in \mathbf{R}(B)(A, C)$ *and* $\chi, \tilde{\chi} \in \mathbf{R}(A)(B, C)$ *is a symmetrical L-Galois triangle if for all* $a \in A, b \in B, c \in C$ *the following equalities hold:*

1. $\mathcal{R}_A(a, \chi(b, c)) = \mathcal{R}_B(\tilde{\psi}(a, c), b)$;
2. $\tilde{\mathcal{R}}_A(a, \tilde{\chi}(b, c)) = \tilde{\mathcal{R}}_B(\psi(a, c), b)$;
3. $\mathcal{R}_A(a, \tilde{\chi}(b, c)) = \tilde{\mathcal{R}}_C(c, \varphi(a, b))$;
4. $\tilde{\mathcal{R}}_A(a, \chi(b, c)) = \mathcal{R}_C(c, \tilde{\varphi}(a, b))$.

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$$\tilde{\mathcal{S}}_X(A,B) = \bigwedge_{x \in X} A(x) \rightsquigarrow B(x), \text{ for all } A, B \in L^X$$

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Proposition

Let $L = (L, \to, \top)$ be a symmetrical cdeo algebra. Then for any triple of sets (X, Y, Z) the diagram

$$
\begin{array}{ccc}
 & (\mathbf{R}(L)(X,Y), \mathcal{S}_{XY}, \tilde{\mathcal{S}}_{XY}) & \\
\overset{\psi\ \tilde{\psi}}{\nearrow} & & \overset{\chi\ \tilde{\chi}}{\searrow} \\
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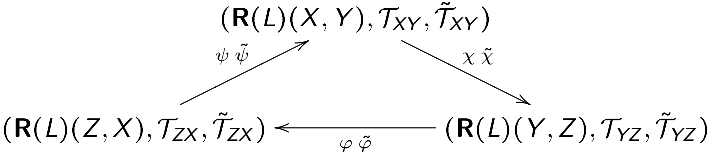

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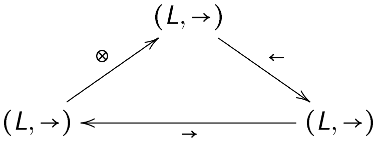

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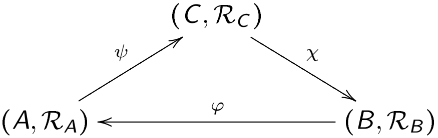

with $\varphi \in \mathbf{R}(C)(A, B)$, $\psi \in \mathbf{R}(B)(A, C)$ and $\chi \in \mathbf{R}(A)(B, C)$ is a strong L-Galois triangle if for all $a \in A, b \in B, c \in C$ the following equalities hold:

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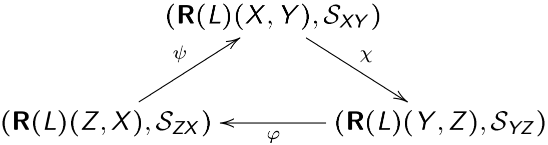

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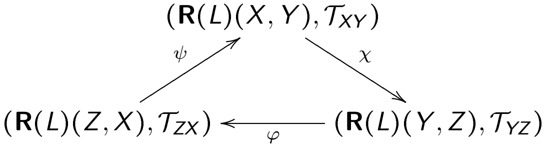

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