(Non-associative) Substructural Fuzzy Logics II

Predicate Logics

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Motivation

- The family of fuzzy logics and their algebraic semantics is ever growing (non divisible, non integral, non commutative, non associative fuzzy logics, fragments, expansions).
- General theories for the algebraic study of non-classical logics (AAL: Blok, Pigozzi, Czelakowski, Font, Jansana, et al, general theory of fuzzy logics: Cintula and Noguera) might be too abstract.
- The working mathematical fuzzy logician needs a general down-to-earth framework (forthcoming Chapter 2 of *Handbook of Mathematical Fuzzy Logic*).

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However: this talk can be seen as elaboration of $\mathrm{SL}^{\ell}\forall$.

Conventions and basic notions

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Assume from now on that L is semilinear substructural logic with language contain the connectives: $\rightarrow, \mathbf{1}$, and $\vee$.

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Basic notions

- Predicate language $\mathcal{P} = \langle \mathbf{P},\ \mathbf{F},\ \mathbf{ar} \rangle$
- Quantifiers: $\forall$ and $\exists$
- $\mathcal{P}$-terms, $\langle \mathcal{L}, \mathcal{P} \rangle$-atomic formulae, $\langle \mathcal{L}, \mathcal{P} \rangle$–formulae
- free and bound occurrences of variables in formulae,
- substitutability of a term for a variable into a formula
- a *theory* T is a tuple $\langle \mathcal{P}, \Gamma \rangle$, where $\mathcal{P}$ is a predicate language and Γ is a set of $\mathcal{P}$-formulae.

Definition (Structure)

A $\mathcal{P}$*-structure* $\mathbf{S}$ is a tuple $\langle \boldsymbol{A}, \mathbf{S} \rangle$, where

- $\boldsymbol{A}$ is an L-algebra,
- $\mathbf{S}$ is a tuple $\langle S, \langle P_{\mathbf{S}} \rangle_{P \in \mathbf{P}}, \langle f_{\mathbf{S}} \rangle_{f \in \mathbf{F}} \rangle$, where
  - S is a non-empty domain,
  - $f_{\mathbf{S}}$ is a function $S^n \to S$ for each $f \in \mathbf{F}$,
  - $P_{\mathbf{S}}$ is a mapping $S^n \to A$ for each $P \in \mathbf{P}$.

Definition (Evaluation)

Let $\mathbf{S} = \langle \boldsymbol{A}, \mathbf{S} \rangle$ be a structure. An $\mathbf{S}$*-evaluation* v is a mapping from the set of object variables into S.

'Tarski style' truth definition

We define the *values* of the terms and *truth values* of the formulae in $\mathcal{P}$-structure $\mathbf{S} = \langle \mathbf{A}, \mathbf{S} \rangle$ for an $\mathbf{S}$-evaluation v as:

$$
\begin{aligned}
\|x\|^{\mathbf{S}}_{v} &= v(x)\,, && \\
\|f(t_1,\dots,t_n)\|^{\mathbf{S}}_{v} &= f_{\mathbf{S}}(\|t_1\|^{\mathbf{S}}_{v},\ \dots,\ \|t_n\|^{\mathbf{S}}_{v})\,, && \text{for } f \in \mathbf{F} \\
\|P(t_1,\dots,t_n)\|^{\mathbf{S}}_{v} &= P_{\mathbf{S}}(\|t_1\|^{\mathbf{S}}_{v},\ \dots,\ \|t_n\|^{\mathbf{S}}_{v})\,, && \text{for } P \in \mathbf{P} \\
\|c(\varphi_1,\dots,\varphi_n)\|^{\mathbf{S}}_{v} &= c^{\mathbf{A}}(\|\varphi_1\|^{\mathbf{S}}_{v},\ \dots,\ \|\varphi_n\|^{\mathbf{S}}_{v})\,, && \text{for } c \in \mathcal{L} \\
\|(\forall x)\varphi\|^{\mathbf{S}}_{v} &= \inf_{\leq^{\mathbf{A}}}\{\|\varphi\|^{\mathbf{S}}_{v[x\to a]} \mid a \in S\}\,, && \\
\|(\exists x)\varphi\|^{\mathbf{S}}_{v} &= \sup_{\leq^{\mathbf{A}}}\{\|\varphi\|^{\mathbf{S}}_{v[x\to a]} \mid a \in S\}\,. &&
\end{aligned}
$$

If the infimum does not exist, $\|(\forall x)\varphi\|^{\mathbf{S}}_{v}$ is undefined.
analogously for $\|(\exists x)\varphi\|^{\mathbf{S}}_{v}$

Definition (Safe structures)

$\mathbf{S}$ is *safe* iff $\|\varphi\|^{\mathbf{S}}_{v}$ is defined for each $\mathcal{P}$-formula φ and each $\mathbf{S}$-evaluation v.

Two (natural) semantical consequence relations

Conventions

- $\mathbf{S} \models \varphi[\mathrm{v}]$ if $\|\varphi\|^{\mathbf{S}}_{\mathrm{v}} \geq \mathbf{1}$.
- $\mathbf{S} \models \varphi$ if $\mathbf{S} \models \varphi[\mathrm{v}]$ for each $\mathbf{S}$-evaluation v.
- $\mathbf{S} \models \Gamma$ if $\mathbf{S} \models \varphi$ for each $\varphi \in \Gamma$.

Definition (Model)

A $\mathcal{P}$-structure $\mathbf{M} = \langle \mathbf{A}, \mathbf{M} \rangle$ is called a (linear) *$\mathcal{P}$-model* of a $\mathcal{P}$-theory T if it is *safe*, $\mathbf{M} \models T$, (and $\boldsymbol{A}$ is linear.)

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Definition (Semantical consequence relation(s))

A $\mathcal{P}$-formula φ is a semantical consequence of a $\mathcal{P}$-theory T w.r.t. the class all/linear models, in symbols $T \models \varphi$ (or $T \models^{\ell} \varphi$) if

$$\mathbf{M} \models \varphi \text{ for each (linear) model } \mathbf{M} \text{ of } T$$

Problem

In general we can only prove that:

$$\models \;\subseteq\; \models^{\ell}$$

E.g. in Gödel logic is is well known that $\varphi \vee \psi \models^{\ell}_{\mathrm{G}} \psi \vee (\forall x)\varphi$ but $\varphi \vee \psi \not\models_{\mathrm{G}} \psi \vee (\forall x)\varphi$

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Remark

Recall that in propositional semilinear logic these two consequence relations coincide.

It is in fact **the** defining feature of these logics!

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Question

What is the *right* first-order fuzzy logic?

Predicate logics $\mathrm{L}\forall^m$ and $\mathrm{L}\forall$ – axiomatic systems

The *minimal predicate logic over* L in $\mathcal{P}$, denoted as $\mathrm{L}\forall^m$:

(P)	the rules resulting from the rules of L by substituting propositional variables by $\langle \mathcal{L}, \mathcal{P} \rangle$-formulae,	
$(\forall 1)$	$\vdash_{\mathrm{L}\forall^m} (\forall x)\varphi(x, \vec{z}) \to \varphi(t, \vec{z})$	t is substitutable for x in φ
$(\exists 1)$	$\vdash_{\mathrm{L}\forall^m} \varphi(t, \vec{z}) \to (\exists x)\varphi(x, \vec{z})$	t is substitutable for x in φ
$(\forall 2)$	$\chi \to \varphi \vdash_{\mathrm{L}\forall^m} \chi \to (\forall x)\varphi$	x is not free in χ
$(\exists 2)$	$\varphi \to \chi \vdash_{\mathrm{L}\forall^m} (\exists x)\varphi \to \chi$	x is not free in χ

The *predicate logic over* L in $\mathcal{P}$, denoted as $\mathrm{L}\forall$, extends $\mathrm{L}\forall^m$ by:

$(\forall 2)^\vee$	$(\chi \to \varphi) \vee \psi \vdash_{\mathrm{L}\forall} (\chi \to (\forall x)\varphi) \vee \psi$	x is not free in χ, ψ
$(\exists 2)^\vee$	$(\varphi \to \chi) \vee \psi \vdash_{\mathrm{L}\forall} ((\exists x)\varphi \to \chi) \vee \psi$	x is not free in χ, ψ

Completeness theorems

Theorem (Completeness theorem for $\mathrm{L}\forall^m$)

Let L *be a logic and* $T \cup \{\varphi\}$ *a* $\mathcal{P}$*-theory. TFAE:*

- $T \vdash_{\mathrm{L}\forall^m} \varphi$,
- $T \models \varphi$,
- *There is a predicate language* $\mathcal{P}' \supseteq \mathcal{P}$ *such that* $\mathbf{M} \models \varphi$ *for each exhaustive, fully named, model* $\mathbf{M}$ *of* $\langle \mathcal{P}', T \rangle$.

Theorem (Completeness theorem for $\mathrm{L}\forall$)

Let L *be a finitary logic and* $T \cup \{\varphi\}$ *a* $\mathcal{P}$*-theory. TFAE:*

- $T \vdash_{\mathrm{L}\forall} \varphi$,
- $T \models^{\ell} \varphi$,
- *There is a predicate language* $\mathcal{P}' \supseteq \mathcal{P}$ *such that* $\mathbf{M} \models \varphi$ *for each exhaustive, fully named, linear model* $\mathbf{M}$ *of* $\langle \mathcal{P}', T \rangle$.

Theorems (for x not free in χ)

If L expands SL, then the $\mathrm{L}\forall^m$ proves:

$\chi \leftrightarrow (\forall x)\chi$

$(\forall x)(\varphi \to \psi) \to ((\forall x)\varphi \to (\forall x)\psi)$

$(\forall x)(\varphi \to \psi) \to ((\exists x)\varphi \to (\exists x)\psi)$

$(\forall x)(\chi \to \varphi) \leftrightarrow (\chi \to (\forall x)\varphi)$

$(\exists x)(\chi \to \varphi) \to (\chi \to (\exists x)\varphi)$

$(\exists x)(\varphi \vee \psi) \leftrightarrow (\exists x)\varphi \vee (\exists x)\psi$

$(\exists x)\chi \leftrightarrow \chi$

$(\forall x)(\forall y)\varphi \leftrightarrow (\forall y)(\forall x)\varphi$

$(\exists x)(\exists y)\varphi \leftrightarrow (\exists y)(\exists x)\varphi$

$(\forall x)(\varphi \to \chi) \leftrightarrow ((\exists x)\varphi \to \chi)$

$(\exists x)(\varphi \to \chi) \to ((\forall x)\varphi \to \chi)$

$(\exists x)(\varphi \mathbin{\&} \chi) \leftrightarrow (\exists x)\varphi \mathbin{\&} \chi$

The logic $\mathrm{L}\forall$ furthermore proves:

$(\forall x)\varphi \vee \chi \leftrightarrow (\forall x)(\varphi \vee \chi)$

$(\exists x)(\varphi \wedge \chi) \leftrightarrow (\exists x)\varphi \wedge \chi$

If L is associative, then $\mathrm{L}\forall^m$ proves:

$\vdash_{\mathrm{L}\forall^m} (\exists x)(\varphi^n) \leftrightarrow ((\exists x)\varphi)^n$

Alternative axiomatizations of $\mathrm{L}\forall^m$ and $\mathrm{L}\forall$

If L expands SL, then $\mathrm{L}\forall^m$ can be axiomatized as:

(P)	the rules resulting from the rules of L by substituting propositional variables by $\langle \mathcal{L}, \mathcal{P} \rangle$-formulae,	
$(\forall 1)$	$\vdash_{\mathrm{L}\forall^m} (\forall x)\varphi(x, \vec{z}) \to \varphi(t, \vec{z})$	t is substitutable for x in φ
$(\exists 1)$	$\vdash_{\mathrm{L}\forall^m} \varphi(t, \vec{z}) \to (\exists x)\varphi(x, \vec{z})$	t is substitutable for x in φ
$(\forall 2')$	$\vdash_{\mathrm{L}\forall^m} (\forall x)(\chi \to \varphi) \to (\chi \to (\forall x)\varphi)$	x is not free in ψ
$(\exists 2')$	$\vdash_{\mathrm{L}\forall^m} (\forall x)(\varphi \to \chi) \to ((\exists x)\varphi \to \chi)$	x is not free in ψ
$(\forall 0)$	$\varphi \vdash_{\mathrm{L}\forall^m} (\forall x)\varphi$	

The logic $\mathrm{L}\forall$ is an extension of $\mathrm{L}\forall^m$ by:

$(\forall 3)$ $\vdash_{\mathrm{L}\forall} (\forall x)(\varphi \vee \psi) \to (\forall x)\varphi \vee \psi$ where x is not free in ψ,

Congruence

Let φ, ψ, δ be sentences, χ is a formula and χ' is obtained from χ by replacing some occurrences of φ by ψ. Then:

$$\vdash \varphi \leftrightarrow \varphi \qquad \varphi \leftrightarrow \psi \vdash \psi \leftrightarrow \varphi \qquad \varphi \leftrightarrow \delta, \delta \leftrightarrow \psi \vdash \varphi \leftrightarrow \psi$$

$$\varphi \leftrightarrow \psi \vdash_{\mathrm{L}\forall^m} \chi \leftrightarrow \chi'$$

Constants theorem

Let $\Sigma \cup \{\varphi(x, \vec{z})\}$ set of formulae and c a constant not occurring in $\Sigma \cup \{\varphi(x, \vec{z})\}$. Then $\Sigma \vdash \varphi(c, \vec{z})$ iff $\Sigma \vdash \varphi(x, \vec{z})$

Deduction theorems

Let L be axiomatic expansion of FL or of $(\mathrm{SL}_w)_\triangle$.

Then, both $\mathrm{L}\forall^m$ and $\mathrm{L}\forall$ enjoy the deduction theorem of L

Syntactic properties $\text{Ł}\forall$

Proof by Cases Property and Semilinearity Property

For each theory T and sentences φ, ψ, and χ holds:

$$\frac{T, \varphi \vdash_{\text{Ł}\forall} \chi \qquad T, \psi \vdash_{\text{Ł}\forall} \chi}{T, \varphi \vee \psi \vdash_{\text{Ł}\forall} \chi} \qquad \frac{T, \varphi \to \psi \vdash_{\text{Ł}\forall} \chi \qquad T, \psi \to \varphi \vdash_{\text{Ł}\forall} \chi}{T \vdash_{\text{Ł}\forall} \chi}$$

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Let L be an axiomatic expansion of FL_e^ℓ.

For each $\mathcal{P}$-theory T, $\mathcal{P}$-formula $\varphi(x)$, and a constant $c \notin \mathcal{P}$:
$T \cup \{\varphi(c)\}$ is a conservative expansion of $T \cup \{(\exists x)\varphi(x)\}$.

Skolemization

Let Σ a class of formulae satisfying some technical restrictions

Definition

A logic $\mathrm{L}\forall$ is Σ-*preSkolem* if $T \cup \{\varphi(c)\}$ is a conservative expansion of $T \cup \{(\exists x)\varphi(x)\}$ for each language $\mathcal{P}$, each $\mathcal{P}$-theory T, each $\mathcal{P}$-formula $\varphi(x) \in \Sigma$ and any constant $c \notin \mathcal{P}$.

Some examples

If L is extension of	then $\mathrm{L}\forall$ is Σ-preSkolem for Σ being
FL_e^ℓ	the class of all formulae
$(\mathrm{SL}_w^\ell)_\triangle$	the class of all formulae starting with $\triangle$

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Theorem (Skolemization for Σ-preSkolem logics)

For any $\mathcal{P}$-formula $\varphi(x,\vec{y}) \in \Sigma$, if a $\mathcal{P}$-theory $T \vdash (\forall\vec{y})(\exists x)\varphi(x,\vec{y})$, then for any functional symbol $f_\varphi \notin \mathcal{P}$ of a proper arity holds:

$$T \cup \{(\forall\vec{y})\varphi(f_\varphi(\vec{y}),\vec{y})\} \text{ is a conservative expansion of } T$$

Witnessed completeness

Definition

A linear $\mathcal{P}$-model $\mathbf{M}$ is *witnessed* if for each $\mathcal{P}$-formula $\varphi(x, \vec{y})$ and for each $\vec{a} \in M$ there are $b_s, b_i \in M$ st.

$$\|(\forall x)\varphi(x, \vec{a})\|^{\mathbf{M}} = \|\varphi(b_i, \vec{a})\|^{\mathbf{M}} \qquad \|(\exists x)\varphi(x, \vec{a})\|^{\mathbf{M}} = \|\varphi(b_s, \vec{a})\|^{\mathbf{M}}.$$

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Definition

The *witnessed* predicate logic $\mathrm{L}\forall^w$ extends $\mathrm{L}\forall$ by:

$$(\exists x)((\exists y)\psi(y, \vec{z}) \to \psi(x, \vec{z})) \qquad (\exists x)(\psi(x, \vec{z}) \to (\forall y)\psi(y, \vec{z}))$$

Theorem

If $\mathrm{L}\forall$ *is preSkolem,* T *a theory and* φ *a formula, TFAE:*

- $T \vdash_{\mathrm{L}\forall^w} \varphi$*.*
- $\mathbf{M} \models \varphi$ *for each* witnessed *linear model* $\mathbf{M}$ *of* T*.*

Thank you for your attention!