Non-associative logics

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Hájek introduced basic logic BL as the logic of continuous t-norms and their residua. Basic logic is a fuzzy logic, i.e. it is complete with respect to linearly ordered models.
Algebraic semantics of BL is the variety of BL algebras. It was proved by Cignoli, Esteva, Godo and Torrens that the variety of BL algebras is generated just by the continuous t-norms on the interval $[0, 1]$ of reals.
The main goal of the talk is to present a non-associative generalization of Hájek's BL logic which has a class $naBL$ of non-associative BL algebras as its algebraic semantics.
Moreover, it is shown that $naBL$ forms a variety generated just by non-associative t-norms. Consequently, the non-associative BL logic is the logic of non-associative t-norms and their residua.

BL-algebras

Definition

An algebra $\mathbf{A} = (A, \vee, \wedge, \cdot, \to, 0, 1)$ of type $\langle 2, 2, 2, 2, 0, 0 \rangle$ is said to be a BL-algebra (basic logic algebra) if

(BL1) $(A, \vee, \wedge, 0, 1)$ is a bounded lattice,

(BL2) $(A, \cdot, 1)$ is a commutative monoid,

(BL3) any $x, y, z \in A$ satisfy $x \cdot y \leq z$ if and only if $x \leq y \to z$ (the so-called adjointness property).

(BL4) the divisibility identity $(x \cdot (x \to y) = x \wedge y)$ and the prelinearity identity $(x \to y \vee y \to x = 1)$ hold.

Definition

A binary operation '$*$' on the interval $[0,1]$ of reals is said to be a t-norm if

(t1) $([0,1],*,1)$ is a commutative monoid,

(t2) '$*$' is continuous in the usual sense,

(t3) If $x,y,z \in [0,1]$ are such that $x \leq y$ then $x \cdot z \leq y \cdot z$.

We can define $x \rightarrow_* y = \max\{a \in [0,1] \mid a * x \leq y\}$ for any t-norm '$*$' and moreover then the algebra $([0,1], \max, \min, *, \rightarrow_*, 0, 1)$ is a BL-algebra. The variety (class) of all BL-algebras is generated just by a t-norms (more precisely, by a BL-algebras derived form a t-norms).

Non-associative residuated structures

Definition

An algebra $\mathbf{A} = (A, \vee, \wedge, \cdot, \to, 0, 1)$ of type $\langle 2, 2, 2, 2, 0, 0 \rangle$ is a non-associative residuated lattice if

(A1) $(A, \vee, \wedge, 0, 1)$ is a bounded lattice,

(A2) $(A, \cdot, 1)$ is a commutative groupoid with the neutral element 1,

(A3) Any $x, y, z \in A$ satisfy $x \cdot y \leq z$ if and only if $x \leq y \to z$ (so-called adjointness property).

Theorem

The class of all non-associative residuated lattices forms an arithmetical variety. Moreover, the variety is 1-regular.

Lemma

If $\mathbf{A} = (A, \vee, \wedge, \cdot, \rightarrow, 0, 1)$ *is a non-associative residuated lattice then for all* $x, y, x_1, x_2 \in A$ *we have*

(i) $x \leq y$ *if and only if* $x \rightarrow y = 1$,

(ii) *If* $x_1 \leq x_2$ *then* $x_1 \cdot y \leq x_2 \cdot y$, $x_2 \rightarrow y \leq x_1 \rightarrow y$ *and* $y \rightarrow x_1 \leq y \rightarrow x_2$,

(iii) $y \cdot (x_1 \vee x_2) = (y \cdot x_1) \vee (y \cdot x_2)$,

(iv) $y \rightarrow (x_1 \wedge x_2) = (y \rightarrow x_1) \wedge (y \rightarrow x_2)$,

(v) $(x_1 \vee x_2) \rightarrow y = (x_1 \rightarrow y) \wedge (x_2 \rightarrow y)$,

(vi) $(x \rightarrow y) \cdot x \leq x, y$,

(vii) $(x \rightarrow y) \rightarrow y \geq x, y$.

In this section we describe congruence kernels (called filters) in the variety of non-associative residuated lattices. In what follows, we need the unary terms

$$\alpha_b^a(x) := (a \cdot b) \to (a \cdot (b \cdot x)),$$

$$\beta_b^a(x) := b \to (a \to ((a \cdot b) \cdot x)).$$

The following lemma justifies their importance in non-associtive residuated lattices.

Lemma

If $\mathbf{A} = (A, \vee, \wedge, \cdot, \to, 0, 1)$ *is a non-associative residuated lattice then for all* $a, b, c \in A$ *we have:*

(i) $(a \cdot b) \cdot \alpha_b^a(x) \leq a \cdot (b \cdot x)$,

(ii) $a \cdot (b \cdot \beta_b^a(x)) \leq (a \cdot b) \cdot x$.

Definition

Let $\mathbf{A} = (A, \vee, \wedge, \cdot, \to, 0, 1)$ be a non-associative residuated lattice. A non-empty subset $F \subseteq A$ is called a **filter** of $\mathbf{A} = (A, \vee, \wedge, \cdot, \to, 0, 1)$ if it satisfies:

(F1) If $x \in F$ and $y \in A$ such that $x \leq y$ then $y \in F$.

(F2) $x \cdot y \in F$ for all $x, y \in F$.

(F2) If $a, b \in A$ and $x \in F$ then $\alpha_b^a(x), \beta_b^a(x) \in F$.

Theorem

Let $\mathbf{A} = (A, \vee, \wedge, \cdot, \to, 0, 1)$ be a non-associative residuated lattice and let F be a filter. Then the relation

$$\Theta(F) = \{\langle x, y\rangle \in A^2 \mid x \to y, y \to x \in F\}$$

is a congruence on $\mathbf{A}$. Moreover, $1/\Theta(F) = F$.

Let us denote by $\mathfrak{R}$ the sub-variety of the variety of non-associative residuated lattices generated just by its linearly ordered members and called it representable.
In what follows we need the following notation: for any $M \subseteq A$ denote $M^{\perp} = \{x \in A \mid x \vee y = 1 \text{ (for all } y \in M)\}$, the so-called *polar* of M. Moreover, we introduce the identities

$$(x \to y) \vee \alpha_b^a(y \to x) = 1 \qquad (\alpha\text{-prelinearity}),$$

$$(x \to y) \vee \beta_b^a(y \to x) = 1 \qquad (\beta\text{-prelinearity}).$$

Lemma

The following conditions are equivalent:

(i) $\mathbf{A} \in \mathfrak{R}$

(ii) $\mathbf{A}$ *is a non-associative residuated lattice with* α*-prelinearity and* β*-prelinearity.*

(iii) $\mathbf{A}$ *is a non-associative residuated lattice with prelinearity and satisfying the quasi-identities*

$$x \vee y = 1 \implies x \vee \alpha_b^a(y) = 1 \text{ and } x \vee \beta_b^a(y) = 1 \quad (P)$$

(iv) $\mathbf{A}$ *is non-associative residuated lattice with prelinearity and, for all* $M \subseteq A$*, the set* $M^{\perp}$ *is a filter of* $\mathbf{A}$*.*

(v) $\mathbf{A}$ *is a subdirect product of linearly ordered non-associative residuated lattices.*

Definition

A non-associative residuated lattice is an *naBL* algebra if it satisfies divisibility, α-prelinearity and β-prelinearity.

Let us have a non-associative t-norm $*$ (commutative, monotone and continuous binary operation on interval $[0,1]$ of reals), then we can define

$$x \to_* y := \max\{a \in [0,1] \mid a * x \leq z\}.$$

Moreover, the structure $([0,1], \min, \max, *, \to_*, 0, 1)$ is an *naBL* algebra. Let us denote the set of all *naBL* algebras construed from a non-associative t-norms by $na\mathcal{T}$. We can state the main theorem.

General finite embedding theorem

Theorem

Let $\mathbf{A}$ *be an algebra and* $\mathcal{K}$ *be a class of the same type algebras. If for any finite partial subalgebra* $\mathbf{X} \subset \mathbf{A}$ *exist* $\mathbf{B} \in \mathcal{K}$ *and embedding* $f : \mathbf{X} \longrightarrow \mathbf{B}$ *then* $\mathbf{A} \in \mathrm{ISP_U}(\mathcal{K})$.

Proof.

Denote $I = \{X; X \subseteq A \text{ and } X \text{ is finite}\}$. Then for any $X \in I$ there are $\mathbf{A_X} \in \mathcal{K}$ and an embedding $\rho_X : \mathbf{A}|_\mathbf{X} \to \mathbf{A_X}$. Denote further $U(X) = \{Y; Y \in I \text{ and } X \subseteq Y\}$ and $V = \{U(X); X \in I\}$. The set V is nontrivial filter on $\mathcal{P}(I)$ and thus there is an ultrafilter U of $\mathcal{P}(I)$ such that $V \subseteq U$. Finally we introduce the embedding

$$\rho : \mathbf{A} \to (\prod_{X \in I} \mathbf{A}_X)/U,$$

which is derived from partial embeddings ρ_X.

General embedding theorem

Theorem

Let us have a class of the finite types algebras $\mathcal{K}$. Then $\mathbf{A} \in \mathrm{ISP_U}(\mathcal{K})$ if and only if any partial subalgebra $\mathbf{X} \subseteq A$ is embeddable to some algebra from $\mathcal{K}$.

The class of all naBL algebras will be denoted by $na\mathcal{BL}$ and denote by $na\mathcal{T}$ the set of all $naBL$ algebras induced just by nat-norms. In what follows we show that $na\mathcal{T}$ is the generating class for the variety $na\mathcal{BL}$.

Theorem

If **A** *is a linearly ordered naBL algebra then* $\mathbf{A} \in \mathrm{ISP_U}(na\mathcal{T})$.

Proof. Let $X \subseteq A$ be any finite set such that $0, 1 \in X$. We put $X \cdot X = \{x \cdot y \mid x \in X,\ y \in X\}$. Clearly, $1 \in X$ yields the inequality $X \subseteq X \cdot X$. Finiteness of X yields finiteness of $X \cdot X$ and thus we may assume that $X \cdot X = \{x_0, \cdots, x_n\}$ and $X = \{y_0, \cdots, y_m\}$ where $0 = x_0 < x_1 < \cdots < x_n = 1$ and $0 = y_0 < y_1 < \cdots < y_m = 1$. Introduce the mapping $f : X \cdot X \longrightarrow [0, 1]$ by $f(x_i) = \frac{i}{n}$. The mapping f is a lattice embedding.

We construct a nat-norm '$*$' which is a hyperbolic paraboloid on every interval $[y_i, y_{i+1}] \times [y_j, y_{j+1}]$ incident with the points
$\langle f(y_i),\ f(y_j), f(y_i \cdot y_j)\rangle$,
$\langle f(y_{i+1}), f(y_j), f(y_{i+1} \cdot y_j)\rangle$,
$\langle f(y_i), f(y_{j+1}), f(y_i \cdot y_{j+1})\rangle$ and
$\langle f(y_{i+1}),\ f(y_{j+1}), f(y_{i+1} \cdot y_{j+1})\rangle$.
Finally we prove that $f : X \to [0, 1]$ is an embedding to the *naBL* algebra induced by '$*$'. General finite embedding theorem finished the proof.

Corollaries

Moreover, the main theorem has some important corollaries:

Corollary

$na\mathcal{BL} = \mathrm{IP_S SP_U}(na\mathcal{T})$.

Denoting for a class $\mathcal{X}$ of algebras of the same type $\mathcal{V}(\mathcal{X})$ the variety generated by $\mathcal{X}$ and $\mathcal{QV}(\mathcal{X})$ the quasivariety generated by $\mathcal{X}$, we have

Corollary

$na\mathcal{BL} = \mathcal{V}(na\mathcal{T}) = \mathcal{QV}(na\mathcal{T})$.

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