The variety generated by all the ordinal sums of perfect MV-chains

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Derived connectives:

$$\begin{aligned}
\varphi \wedge \psi &:= \varphi \& (\varphi \to \psi) \\
\neg \varphi &:= \varphi \to \bot \\
\varphi \vee \psi &:= ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi) \\
\varphi \leftrightarrow \psi &:= (\varphi \to \psi) \& (\psi \to \varphi) \\
\varphi \curlyvee \psi &:= \neg(\neg \varphi \& \neg \psi) \\
\top &:= \neg \bot
\end{aligned}$$

BL is axiomatized as follows

(A1)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(A2)	$(\varphi \& \psi) \to \varphi$
(A3)	$(\varphi \& \psi) \to (\psi \& \varphi)$
(A4)	$(\varphi \& (\varphi \to \psi)) \to (\psi \& (\psi \to \varphi))$
(A5a)	$(\varphi \to (\psi \to \chi)) \to ((\varphi \& \psi) \to \chi)$
(A5b)	$((\varphi \& \psi) \to \chi) \to (\varphi \to (\psi \to \chi))$
(A6)	$((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$
(A7)	$\bot \to \varphi.$

As an inference rule, we have modus ponens

(MP) $$\frac{\varphi \quad \varphi \to \psi}{\psi}$$

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- Product logic is obtained from BL by adding

$$\neg\varphi \vee ((\varphi \to (\varphi \& \psi)) \to \psi)$$

BL-algebras

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- $\langle *, \Rightarrow\rangle$ form a residuated pair, that is

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- Some derived operations:

$$\sim x := x \Rightarrow 0$$
$$x \oplus y := \sim (\sim x * \sim y)$$

They are BL-algebras of the form $\langle [0,1], *, \Rightarrow, \min, \max, 0, 1 \rangle$.

- Standard MV-algebra is denoted by $[0,1]_{Ł}$ and its operations are:

$$x * y = \max(0, x + y - 1) \qquad x \Rightarrow y = \min(1, 1 - x + y) \qquad \sim x = 1 - x$$

Standard MV, Gödel and Product algebras

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- Standard Product-algebra is denoted by $[0,1]_{\Pi}$ and its operations are:

$$x * y = x \cdot y \qquad x \Rightarrow y = \begin{cases} 1 & \text{if } x \leq y \\ \frac{y}{x} & \text{Otherwise} \end{cases} \qquad \sim x = \begin{cases} 0 & \text{if } x > 0 \\ 1 & \text{Otherwise} \end{cases}$$

Definition ([Fer92, BF00])

A *hoop* is a structure $\mathcal{A} = \langle A, *, \Rightarrow, 1 \rangle$ such that $\langle A, *, 1 \rangle$ is a commutative monoid, and $\Rightarrow$ is a binary operation such that

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- *Totally ordered cancellative hoops coincide with unbounded totally ordered Wajsberg hoops, whereas bounded Wajsberg hoops coincide with (the* 0*-free reducts of) MV-algebras.*

Definition ([BDL93])

Let $\mathcal{A}$ be an MV-algebra and let $x \in \mathcal{A}$: with $ord(x)$ we mean the least (positive) natural n such that $x^n = 0$. If there is no such n, then we set $ord(x) = \infty$.

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Let $\mathcal{A}$ be an MV-algebra. The followings are equivalent:

- *$\mathcal{A}$ is a perfect MV-algebra.*
- *$\mathcal{A}$ is isomorphic to the* disconnected rotation *of a cancellative hoop.*

Definition (Chang's MV-algebra, [Cha58])

It is defined as $C = \langle \{a_n : n \in \mathbb{N}\} \cup \{b_n : n \in \mathbb{N}\}, *, \Rightarrow, \sqcap, \sqcup, b_0, a_0 \rangle$.

It holds that $a_0 > a_1 > a_2 \dots$ and $b_0 < b_1 < b_2 \dots$ and $a_i > b_j$ for every $i, j \in \mathbb{N}$.

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As shown in [BDG07], the logic correspondent to this variety is axiomatized as Ł plus $(2\varphi)^2 \leftrightarrow 2(\varphi^2)$: we will call it $Ł_{Chang}$.

A new disjunction connective - 1

Consider the following connective

$$\varphi \veebar \psi := ((\varphi \rightarrow (\varphi \& \psi)) \rightarrow \psi) \wedge ((\psi \rightarrow (\varphi \& \psi)) \rightarrow \varphi)$$

Call $\uplus$ the algebraic operation, over a BL-algebra, corresponding to $\veebar$; we have that

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Corollary

In every MV-algebra the following equations are equivalent

$$(2x)^2 = 2(x^2)$$
$$(\overline{2}x)^2 = \overline{2}(x^2).$$

Where $2x := x \oplus x$ *and* $\overline{2}x := x \uplus x$.

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Corollary

The equation $x \uplus y = 1$ holds in every cancellative hoop.

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Let $\mathcal{A} = \bigoplus_{i \in I} \mathcal{A}_i$ *be a BL-chain. Then*

$$x \uplus y = \begin{cases} x \oplus y, & \text{if } x, y \in \mathcal{A}_i \text{ and } \mathcal{A}_i \text{ is bounded} \\ 1, & \text{if } x, y \in \mathcal{A}_i \text{ and } \mathcal{A}_i \text{ is unbounded} \\ \max(x, y), & \text{otherwise.} \end{cases}$$

for every $x, y \in \mathcal{A}$.

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Theorem

Let $\mathbb{WH}, \mathbb{CH}, ps\mathbb{WH}$ *be, respectively, the varieties of Wajsberg hoops, cancellative hoops, pseudo-perfect Wajsberg hoops. Then we have that*

$$\mathbb{CH} \subset ps\mathbb{WH} \subset \mathbb{WH}$$

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Every BL_{Chang}-chain is isomorphic to an ordinal sum whose first component is a perfect MV-chain and the others are totally ordered pseudo-perfect Wajsberg hoops.
It follows that every ordinal sum of perfect MV-chains is a BL_{Chang}-chain.

. . . and some results

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The variety of BL_{Chang}-algebras contains the ones of product-algebras and Gödel-algebras: however it does not contain the variety of MV-algebras.

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Corollary

The finite model property does not hold, for BL_{Chang}.

Relation with other connected varieties

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- Which is the relation between P_0 and the variety of $\mathrm{BL}_{\mathrm{Chang}}$-algebras ?

  The variety of $\mathrm{BL}_{\mathrm{Chang}}$-algebras is strictly contained in P_0:

  - Every $\mathrm{BL}_{\mathrm{Chang}}$-chain is a perfect BL-chain.
  - There are perfect BL-chains that are not $\mathrm{BL}_{\mathrm{Chang}}$-chains: an example is given by $C \oplus [0,1]_{Ł}$.

Theorem ([EGHM03])

Completeness

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- *Every totally ordered product chain is of the form* $\mathbf{2} \oplus \mathcal{A}$, *where* $\mathcal{A}$ *is a cancellative hoop.*

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Proposition

Product logic is finitely strongly complete w.r.t. $[0,1]_\Pi$ *([EGH96]). As a consequence every countable totally ordered cancellative hoop partially embeds into* $(0,1]_C$*.*

Theorem

Every countable perfect MV-chain partially embeds into $\mathcal{V}$, the disconnected rotation of $(0, 1]_C$.

Completeness - Ł$_{\text{Chang}}$

Theorem

Every countable perfect MV-chain partially embeds into $\mathcal{V}$, the disconnected rotation of $(0,1]_C$.

Corollary

The logic Ł$_{Chang}$ is finitely strongly complete w.r.t. $\mathcal{V}$.

Completeness - $Ł_{\mathrm{Chang}}$

Theorem

Every countable perfect MV-chain partially embeds into $\mathcal{V}$, the disconnected rotation of $(0,1]_C$.

Corollary

The logic $Ł_{Chang}$ is finitely strongly complete w.r.t. $\mathcal{V}$.

Theorem

$Ł_{Chang}$ logic is not strongly complete w.r.t. $\mathcal{V}$.

Completeness - $\mathrm{BL}_{\mathrm{Chang}}$

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Every countable BL_{Chang}-chain partially embeds into $\omega\mathcal{V}$.

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Every countable BL_{Chang}-chain partially embeds into $\omega\mathcal{V}$.

Corollary

BL_{Chang} enjoys the finite strong completeness w.r.t. $\omega\mathcal{V}$. As a consequence, the variety of BL_{Chang}-algebras is generated by the class of all ordinal sums of perfect MV-chains and hence is the smallest variety to contain this class of algebras.

Completeness - BL_{Chang}

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Theorem

BL_{Chang} logic is not strongly complete w.r.t. $\omega\mathcal{V}$.

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APPENDIX

Chang's MV-algebra

Definition

Chang's *MV*-algebra ([Cha58]) is defined as

$$\mathbf{C}_\infty = \langle \{a_n : \ n \in \mathbb{N}\} \cup \{b_n : \ n \in \mathbb{N}\}, *, \Rightarrow, \sqcap, \sqcup, b_0, a_0 \rangle \, .$$

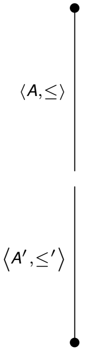
Where for each $n, m \in \mathbb{N}$, it holds that $b_n < a_m$, and, if $n < m$, then $a_m < a_n, \ b_n < b_m$; moreover $a_0 = 1, \ b_0 = 0$ (the top and the bottom element).

The operation $*$ is defined as follows, for each $n, m \in \mathbb{N}$:

$$b_n * b_m = b_0, \ b_n * a_m = b_{\max(0, n-m)}, \ a_n * a_m = a_{n+m}.$$

back

Let $\mathcal{A}$ be a l.o. cancellative hoop. We define an algebra, $\mathcal{A}^*$, called the *disconnected rotation* of $\mathcal{A}$. Let $\mathcal{A} \times \{0\}$ be a disjoint copy of A. For every $a \in A$ we write a' instead of $\langle a, 0\rangle$. Consider $\langle A' = \{a' : a \in A\}, \leq\rangle$ with the inverse order and let $A^* := A \cup A'$. We extend these orderings to an order in A^* by putting $a' < b$ for every $a, b \in A$. Finally, we take the following operations in A^*: $1 := 1_{\mathcal{A}}$, $0 := 1'$, $\sqcap_{\mathcal{A}^*}, \sqcup_{\mathcal{A}^*}$ as the meet and the join with respect to the order over A^*. Moreover,



$$\sim_{\mathcal{A}^*} a := \begin{cases} a' & \text{if } a \in A \\ b & \text{if } a = b' \in A' \end{cases}$$

$$a *_{\mathcal{A}^*} b := \begin{cases} a *_{\mathcal{A}} b & \text{if } a, b \in A \\ \sim_{\mathcal{A}^*} (a \Rightarrow_{\mathcal{A}^*} \sim_{\mathcal{A}^*} b) & \text{if } a \in A, b \in A' \\ \sim_{\mathcal{A}^*} (b \Rightarrow_{\mathcal{A}^*} \sim_{\mathcal{A}^*} a) & \text{if } a \in A', b \in A \\ 0 & \text{if } a, b \in A' \end{cases}$$

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Ordinal Sums

- Let $\langle I, \leq \rangle$ be a totally ordered set with minimum 0. For all $i \in I$, let $\mathcal{A}_i$ be a totally ordered Wajsberg hoop such that for $i \neq j$, $A_i \cap A_j = \{1\}$, and assume that $\mathcal{A}_0$ is bounded.

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- Then $\bigoplus_{i \in I} \mathcal{A}_i$ (the *ordinal sum* of the family $(\mathcal{A}_i)_{i \in I}$) is the structure whose base set is $\bigcup_{i \in I} A_i$, whose bottom is the minimum of $\mathcal{A}_0$, whose top is 1, and whose operations are

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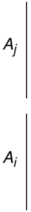

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A_j

A_i

$$x \Rightarrow y = \begin{cases} x \Rightarrow^{\mathcal{A}_i} y & \text{if } x, y \in A_i \\ y & \text{if } \exists i > j (x \in A_i \text{ and } y \in A_j) \\ 1 & \text{if } \exists i < j (x \in A_i \setminus \{1\} \text{ and } y \in A_j) \end{cases}$$

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- As a consequence, if $x \in A_i \setminus \{1\}$, $y \in A_j$ and $i < j$ then $x < y$.
- Note that, since every bounded Wajsberg hoop is the 0-free reduct of an MV-algebra, then the previous definition also works with these structures.

back

Definition

Let $\mathcal{A}$ and $\mathcal{B}$ be two algebras of the same type $\mathcal{F}$. We say that

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- $\mathcal{A}$ is a partial subalgebra of $\mathcal{B}$ if $A \subseteq B$ and for every $f \in \mathcal{F}$ and $\overline{a} \in A^{ar(f)}$

$$f^{\mathcal{A}}(\overline{a}) = \begin{cases} f^{\mathcal{B}}(\overline{a}) & \text{if } f^{\mathcal{B}}(\overline{a}) \in A \\ \text{undefined} & \text{otherwise.} \end{cases}$$

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- $\mathcal{A}$ is partially embeddable into $\mathcal{B}$ when every finite partial subalgebra of $\mathcal{A}$ is embeddable into $\mathcal{B}$.

back

Partial algebra

Definition

Let $\mathcal{A}$ and $\mathcal{B}$ be two algebras of the same type $\mathcal{F}$. We say that

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- $\mathcal{A}$ is partially embeddable into $\mathcal{B}$ when every finite partial subalgebra of $\mathcal{A}$ is embeddable into $\mathcal{B}$.
- A class K of algebras is partially embeddable into an algebra $\mathcal{A}$ if every finite partial subalgebra of a member of K is embeddable into $\mathcal{A}$.

back

First-order logics - syntax and semantics

- We work with (first-order) languages without equality, containing only predicate and constant symbols: as quantifiers we have $\forall$ and $\exists$. The notions of terms and formulas are defined inductively like in classical case.

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- As regards to semantics, given an axiomatic extension L of BL we restrict to L-chains: the first-order version of L is called L$\forall$ (see [Háj98, CH10] for an axiomatization). A first-order $\mathcal{A}$-interpretation ($\mathcal{A}$ being an L-chain) is a structure $\mathbf{M} = \langle M, \{r_P\}_{P\in\mathbf{P}}, \{m_c\}_{c\in\mathbf{C}}\rangle$, where M is a non-empty set, every r_P is a fuzzy $ariety(P)$-ary relation, over M, in which we interpretate the predicate P, and every m_c is an element of M, in which we map the constant c.

First-order logics - syntax and semantics

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- Given a map $v: VAR \to M$, the interpretation of $\|\varphi\|^{\mathcal{A}}_{\mathbf{M},v}$ in this semantics is defined in a Tarskian way: in particular the universally quantified formulas are defined as the infimum (over $\mathcal{A}$) of truth values, whereas those existentially quantified are evaluated as the supremum. Note that these inf and sup could not exist in $\mathcal{A}$: an $\mathcal{A}$-model $\mathbf{M}$ is called *safe* if $\|\varphi\|^{\mathcal{A}}_{\mathbf{M},v}$ is defined for every φ and v.

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- A model is called *witnessed* if the universally (existentially) quantified formulas are evaluated by taking the minimum (maximum) of truth values in place of the infimum (supremum): see [Háj07, CH06, CH10] for details.
- The notions of soundness and completeness are defined by restricting to safe models (even if in some cases it is possible to enlarge the class of models: see [BM09]): see [Háj98, CH10, CH06] for details.

Definition

Let L be an axiomatic extension of BL. With $L\forall^w$ we define the extension of $L\forall$ with the following axioms

$$(C\forall) \qquad (\exists y)(\varphi(y) \to (\forall x)\varphi(x))$$
$$(C\exists) \qquad (\exists y)((\exists x)\varphi(x) \to \varphi(y)).$$

Theorem ([CH06, proposition 6])

Ł∀ coincides with $Ł\forall^w$, that is $Ł\forall \vdash (C\forall),(C\exists)$.

An immediate consequence is:

Corollary

Let L be an axiomatic extension of Ł. Then $L\forall$ coincides with $L\forall^w$.

Theorem ([CH06, theorem 8])

Let L be an axiomatic extension of BL. Then $L\forall^w$ enjoys the strong witnessed completeness with respect to the class K of L-chains, i.e.

$$T \vdash_{L\forall^w} \varphi \quad \textit{iff} \quad \|\varphi\|^{\mathcal{A}}_{\mathbf{M}} = 1,$$

for every theory T, formula φ, algebra $\mathcal{A} \in K$ and witnessed $\mathcal{A}$-model $\mathbf{M}$ such that $\|\psi\|^{\mathcal{A}}_{\mathbf{M}} = 1$ for every $\psi \in T$.

Lemma ([Mon11, lemma 1])

Let L be an axiomatic extension of BL, let $\mathcal{A}$ be an L-chain, let $\mathcal{B}$ be an L-chain such that $A \subseteq B$ and let $\mathbf{M}$ be a witnessed $\mathcal{A}$-structure. Then for every formula φ and evaluation v, we have $\|\varphi\|^{\mathcal{A}}_{\mathbf{M},v} = \|\varphi\|^{\mathcal{B}}_{\mathbf{M},v}$.

First-order logics: results III

Theorem

There is a $Ł_{Chang}$-chain such that $Ł_{Chang}\forall$ is strongly complete w.r.t. it. More in general, every $Ł_{Chang}$-chain that is strongly complete w.r.t $Ł_{Chang}$ is also strongly complete w.r.t. $Ł_{Chang}\forall$.

For $\mathrm{BL}_{\mathrm{Chang}}\forall$, however, the situation is not so good.

Theorem

$BL_{Chang}\forall$ cannot enjoy the completeness w.r.t. a single BL_{Chang}-chain.